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Principio di linearizzazione per problemi a frontiera libera della fluidodinamica

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Abstract

We investigate the motion of spatially periodic surface waves. An infinite layer of incompressible viscous Newtonian fluid is bounded below by a plane and at the top by a free surface. Hydrodynamical forces are acting on the fluid, whose motion is supposed to be spatially periodic. On the bottom surface we impose Dirichlet boundary condition with no incoming or outgoing flux, while the motion of the free surface, where capillarity is assumed, is driven by classical dynamic and kinematic condition.

We obtain three results:

- A linearization principle stating that, if linearizing the problem near a stationary solution gives a stable linear system, then the nonlinear problem is stable (and thus well posed and globally solvable) for initial data which are sufficiently near the stationary solution.
- The rest state is linearly stable, and thus sufficiently small initial data give global, unique, regular and exponentially decaying solutions for the nonlinear problem.
- Whatever the size of the initial data, a unique and regular solution to the nonlinear problem exists for a sufficiently small time.

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Contents

Chapter 1

Introduction

This thesis deals with a free boundary problem for fluid dynamics. These kind of problems go back as far as the analysis of Newton on the shape of a rotating fluid, and have been considered by many great physicists and mathematicians.

The problem in its general form deals with a fluid contained in a time-varying domain Ω_t, whose boundary has a "free" part $\Gamma_t \subseteq \partial\Omega_t$, which is moving according to some dynamic and kinematic conditions driven by the fluid itself and/or external factors. The fluid can be subjected to hydrodynamical forces, or more generally by electromagnetic or thermal effects, bringing respectively to MHD and NSF models. The instances in nature of this kind of situation are endless: from water waves, models for earthquakes and lava motion in geophysics, to galaxy's shape and helioseismology in astrophysics.

We will consider a somewhat simplified situation, where temperature and electromagnetic fields play no rôle, and the fluid is viscous and incompressible. For more elaborate models, we refer to [44] for bibliographics references. Looking at normal dynamical balance on the free surface, we see that

$$\mathbb{T}\boldsymbol{n} = \mathbb{F}\boldsymbol{n},$$

where $\boldsymbol{n}$ is the exterior normal to Γ_t, $\mathbb{T}$ is the stress tensor of the fluid and $\mathbb{F}$ is the tensor of the external forces on the free boundary. For a viscous incompressible fluid, with velocity field $\boldsymbol{v}$, pressure p and viscosity coefficient ν, classical continuum mechanics asserts that the stress tensor is given by

$$\mathbb{T} = -pI + \nu\mathbb{D},$$

where I is the identity matrix δ_{ij} and $\mathbb{D}$ is the doubled symmetric rate of strain tensor, which, for Newtonian fluids, is given by

$$\mathbb{D} = \mathbb{D}(\boldsymbol{v}) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T = \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i}\right)_{i,j=1,2,3},$$

i.e., the doubled symmetric part of the derivative of the velocity field.

The external stress $\mathbb{F}$ can be caused by a variety of phenomena. There can be, for example, another fluid with possibly different viscosity on the other side of Γ_t, in which case we speak of "two phase problems", which are considered in [1], [8], [9], [10], [17], [37], [38]. Another frequent situation (which may holds also for two phase problems) is that the free surface itself produces a normal force to counter its deformation, through the phenomenon of capillarity. Given a surface tension coefficient $\sigma > 0$, the capillarity force acts in a direction normal to the free surface, and is given by

$$\mathbb{F} = -\sigma H_t \delta_{ij},$$

where H_t is the doubled mean curvature of the surface Γ_t. The sign is given according to standard differential geometry, where a convex body has positive mean curvature at any point of its boundary; therefore, the capillarity force tries to "flatten" the free surface. Finally, an external pressure p_{ext} defined in the whole space give rise to an external stress on the surface of the form

$$\mathbb{F} = -p_{\text{ext}} \delta_{ij}.$$

All these external stresses contributes in confining the fluid, which is an important factor for the well posedness of the problem e.g., in rotating fluids. In absence of any external stress on the surface, one may also consider self–gravitational force as a confinement factor, as is done in [27], [29], [14], [30], [31].

While dynamic conditions may vary from problem to problem, kinematic conditions on the boundary are in most cases the same. If $V_{\boldsymbol{n}}$ denotes the normal velocity of the free boundary, it must hold

$$\boldsymbol{v} \cdot \boldsymbol{n} = V_{\boldsymbol{n}}, \tag{1.1}$$

on Γ_t, which expresses the fact that the free surface consists for all $t > 0$ of the same fluid particles, which do not leave it and are not incident on it from inside Ω_t. On the rest of the boundary (if there is any) one can assume for example Dirichlet boundary conditions, with or without incoming flow. This is the case, for example, of flows through an inclined plane, or rotating fluids in a bucket. Neumann–type boundary conditions can also be assumed, as well as mixed ones. When there is no fixed boundary, we speak of an isolated liquid mass. This problem has been treated in [23], [25], [14] when no surface tension is present, and in [13], [24], [29], [20], [26], [27] for capillary fluids.

Our specific problem deals with periodic surface waves. We consider a layer of fluid, bounded at its bottom by a fixed surface, and at the top by a free boundary. Understanding the properties of this kind of motion has many applications, e.g. in seismology, where seismic waves are the result of earthquakes or explosion, or water waves in the ocean. Without periodicity assumption the problem has been treated in [3], [36], [42], [40] for a heavy fluid without capillarity. When surface tension is present, it has been treated in [4], [5], [40], [39], [41] without periodicity assumption, and in [16] for periodic motions.

– *The problem.*

We now describe the problem more precisely. A viscous, incompressible fluid, with associated velocity field $\boldsymbol{v}$ and pressure p, fills at any time $t \geq 0$ a domain Ω_t, where it satisfies the incompressible Navier–Stokes equations with external force $\boldsymbol{f}$ and viscosity ν. The density of the fluid is supposed to be 1. We suppose that this domain can be described as $\Omega_t := \{(x_1, x_2, x_3) : 0 \leq x_3 \leq \phi(x', t)\}$, where $x' = (x_1, x_2)$ and ϕ is a sufficiently regular function whose graph in $\mathbb{R}^3$ is the free boundary of the fluid, Γ_t, with exterior normal $\boldsymbol{n}$. We suppose that the velocity field, the pressure and the free boundary function ϕ are periodic for every $t \geq 0$, with periodic cell Σ being a fixed rectangle in $\mathbb{R}^2$.

On the bottom part of the boundary we impose Dirichlet boundary conditions $\boldsymbol{v}((x', 0), t) = \boldsymbol{\alpha}(x', t)$, for some sufficiently smooth, Σ-periodic $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, 0)$, with zero normal component (i.e., no incoming or outgoing flux is assumed). We suppose that on the free boundary capillarity is acting, and thus we impose the stress balance condition $\mathbb{T}(\boldsymbol{v}, p)\boldsymbol{n} = -\sigma H_t \boldsymbol{n}$, where $\sigma > 0$ is the surface tension coefficient. Finally, the kinematic condition (1.1) is assumed. Given a suitable Σ-periodic initial velocity field $\boldsymbol{v}_0$ at time $t = 0$, defined in a Σ-periodic domain Ω_0, whose boundary Γ_0 is the graph of $\phi_0 = \phi(\cdot, 0)$, one is thus lead to the following evolution problem:

$$\begin{cases} \boldsymbol{v}_{,t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla \cdot \mathbb{T}(\boldsymbol{v}, p) = \boldsymbol{f} & \text{on } \Omega_t, \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Omega_t, \\ \mathbb{T}(\boldsymbol{v}, p)\boldsymbol{n} = -\sigma H_t \boldsymbol{n} & \text{on } \Gamma_t, \\ V_{\boldsymbol{n}} = \boldsymbol{v} \cdot \boldsymbol{n} & \text{on } \Gamma_t, \\ \boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x) & \text{in } \Omega_0 \\ \boldsymbol{v}((x', 0), t) = \boldsymbol{\alpha}(x', t) & \text{on } \Sigma, \text{ for } t \geq 0, \end{cases} \tag{1.2}$$

where the underscript comma in $\boldsymbol{v}_{,t}$ denotes the partial derivative w.r.t. t (we will always assume such a notation).

Denoting by Π_0 the orthogonal projection on the tangent space to Γ_0, this system is coupled with the natural compatibility conditions

$$\begin{cases} \nabla \cdot \boldsymbol{v}_0 = 0 & \text{in } \Omega_0, \\ \boldsymbol{v}_0(x',0) = \boldsymbol{\alpha}(x',0) & \text{on } \Sigma, \\ \Pi_0 \mathbb{D}(\boldsymbol{v}_0)\boldsymbol{n}_0 = 0 & \text{on } \Gamma_0, \end{cases}$$

where $\boldsymbol{n}_0$ is the exterior normal to Γ_0.

We obtain a linearization principle for this problem. When $\boldsymbol{f}$ and $\boldsymbol{\alpha}$ are independent of time, we consider a stationary solution $(\boldsymbol{v}_b, p_b)$, in some domain $\Omega_b := \{(x', x_3) : 0 \leq x_3 \leq \phi_b(x')\}$, of

$$\begin{cases} (\boldsymbol{v}_b \cdot \nabla)\boldsymbol{v}_b - \nabla \cdot T(\boldsymbol{v}_b, p_b) = \boldsymbol{f} & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{v}_b = 0 & \text{in } \Omega_b, \\ T(\boldsymbol{v}_b, p_b)\boldsymbol{n}_b = -\sigma H_b \boldsymbol{n}_b & \text{on } \mathcal{G}, \\ \boldsymbol{v}_b \cdot \boldsymbol{n}_b = 0 & \text{on } \mathcal{G}, \\ \boldsymbol{v}_b(x',0,t) = \boldsymbol{\alpha}(x') & \text{on } \Sigma, \text{ for } t \geq 0, \end{cases} \tag{1.3}$$

where $\mathcal{G}$ is the surface defined by $x_3 = \phi_b(x')$, $\boldsymbol{n}_b$ its exterior normal and H_b its doubled mean curvature. We then linearize system (1.2) near this stationary solution, and prove that if the linearized system is exponentially stable (in a suitable sense), then for any initial data $\boldsymbol{v}_0$, Ω_0 which is sufficiently near to the stationary solution, there exists a unique global in time solution to (1.2), which exponentially converges to the stationary solution of (1.3). We apply this principle to prove the exponential stability of the rest state for periodic motion, and finally prove a local (in time) solvability theorem for problem (1.2), with arbitrarily large initial data.

– *The setting.*

Looking back at (1.2) the first feature of this system is that the domain in which the velocity field and the pressure are defined is varying with time. This is of course typical of free boundary problems for hydrodynamics, and the first step to address the latters is to transform the corresponding system in a fixed domain. There are mainly two methods to do this. The first one is to consider the Lagrangian formulation, given by the change of variables

$$x = \xi + \int_0^t \boldsymbol{u}(\xi, s)ds =: X(\xi, t), \qquad \xi \in \Omega_0$$

where $\boldsymbol{u}$ is the velocity field expressed in Lagrangian coordinates: $\boldsymbol{u}(\xi, t) = \boldsymbol{v}(X(\xi, t))$, where $\boldsymbol{v}$ is the velocity field in the usual Eulerian coordinates. This reduces, at each time $t \geq 0$, the differential equation in Ω_t to one defined in Ω_0. Most of the works cited above use this approach, with minor additional arguments.

In [4], [3] however, a second method is applied, where all the Ω_t are considered as perturbations of the domain Ω_b corresponding to the rest state. Thus, a time depending diffeomorphism Φ_t is chosen in such a way that all the Ω_t (Ω_0 included) are given as $\Omega_t = \Phi_t(\Omega_b)$, where $\Omega_b \neq \Omega_0$ in general. This diffeomorphism transforms the system in a more complicated one, but it has the advantage that studying the existence for large times can be settled once and for all in Ω_b. This kind of approach is called nowadays Hanzawa transformation, although this technique was frequently used well before the work of Hanzawa [11] on the Stefan problem. At some point, a form of it is used in all the results we know of concerning stability and existence of global solutions for free boundary problems in fluid dynamics.

It is worth noting that, for surface waves, this choice of coordinates seems more natural, due to the simple topological restrictions these problems pose. Even more, the presence of capillarity suggests that the free boundary will be more regular than what a “pure” Lagrangian approach suggests, as will be apparent in the following discussion on regularity. We will thus adopt this approach, and henceforth any norm we consider will be computed in Ω_b, through a suitable Hanzawa transformation, the particular from of which will be defined in chapter 3.

Let us now discuss briefly the regularity framework we choose. Existence and regularity for free boundary problems for the Navier–Stokes equations are usually set in three different kind of spaces: Hölder spaces ([23], [13]), anisotropic Sobolev–Slobodetskii spaces $W_p^{2,1}$ with high summability $p > 3$ ([25], [14], [1]), or L^2 anisotropic Sobolev–Slobodetskii spaces $W_2^{r,\frac{r}{2}}$ with noninteger order of differentiability $r > 2$ (see [3], [4], [24] to cite only the first works).

Our choice is the latter, for a variety of reason. Working in Hölder spaces involves as usual a great deal technical difficulties, while the $W_p^{2,1}$ setting suits well into the no–capillarity case, but, in principle, requires additional analysis to deal with the boundary condition $\mathbb{T}(\boldsymbol{v}, p)\boldsymbol{n} = -\sigma H \boldsymbol{n}$ when $\sigma > 0$: this indeed gives rise to an elliptic equation for the free boundary, which, at least in Lagrangian coordinates, is not well defined due to the low *a-priori* regularity of the velocity field. Moreover, the L^2 setting for the spaces $W_2^{r,\frac{r}{2}}$ makes them highly compatible with Fourier and Laplace transform methods in dealing with the corresponding linearized problem, and the Hilbert space structure is handy in giving spectral conditions to ensure stability of the stationary solution.

– *The result.*

Using Hanzawa transformation, the linearization of (1.2) around the stationary solution is a lower order perturbation of

$$\begin{cases} \boldsymbol{v}_{,t} - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} & \text{in } \Omega_b \\ \nabla\cdot\boldsymbol{v} = h & \text{in } \Omega_b \\ \mathbb{T}(\boldsymbol{v},p) - \sigma\Delta_{\mathcal{G}}\rho\boldsymbol{n}_b = \boldsymbol{d} & \text{on } \mathcal{G} \\ \rho_{,t} + \boldsymbol{v}_b\cdot\nabla\rho - \gamma\boldsymbol{n}_b\cdot\boldsymbol{v} = g & \text{on } \mathcal{G}, \\ \boldsymbol{v}\lfloor_\Sigma = 0, \qquad \boldsymbol{v}\lfloor_{t=0} = \boldsymbol{v}_0, \qquad \rho\lfloor_{t=0} = \rho_0, \end{cases} \tag{1.4}$$

with suitable compatibility conditions, where $\rho = \phi - \phi_b$ is the perturbation of the free boundary from the stationary boundary ϕ_b, $\Delta_{\mathcal{G}}$ is the Laplace–Beltrami operator on $\mathcal{G}$ and $\gamma = \sqrt{1+|\nabla\phi_b|^2}$ is the surface area factor. Indeed, the initial problem (1.2) gives, through Hanzawa transformation, a lower order perturbation of the previous system, with nonlinear right hand sides depending on $\boldsymbol{v}$, p and ρ. For the purpose of this informal discussion we will omit the complete form of the system, which will be computed in chapter 3.

We now look for relationships between the regularity of the variables $\boldsymbol{v}$ and ρ, neglecting the pressure and the nonhomogeneous terms. Suppose the velocity belongs to $W_2^{r,\frac{r}{2}}(\Omega_T)$, where $\Omega_T = \Omega_b \times [0,T)$ and

$$W_2^{r,\frac{r}{2}}(\Omega_T) = L^2(0,T;W_2^r(\Omega_b)) \cap W_2^{\frac{r}{2}}(0,T;L^2(\Omega_b)).$$

Looking at the third equation in (1.4), it involves $\mathbb{D}(\boldsymbol{v})$, (or else, $\nabla\boldsymbol{v}$) on the free boundary, and a second order elliptic operator for ρ arising from the curvature term. If $\mathcal{G}_T = \mathcal{G}\times[0,T)$, by trace theorems one has

$$\mathbb{D}(\boldsymbol{v})\lfloor_{\mathcal{G}} \in W_2^{r-\frac{3}{2},\frac{r}{2}-\frac{3}{4}}(\mathcal{G}_T),$$

and the elliptic operator $\Delta_{\mathcal{G}}$ suggests that ρ will gain two derivatives with respect to $\mathbb{D}(\boldsymbol{v})$. More precisely, we will at least have

$$\rho \in L^2(0,T;W_2^{r+\frac{1}{2}}(\mathcal{G})). \tag{1.5}$$

Now, while $\boldsymbol{v} \in W_2^{r,\frac{r}{2}}(\Omega_T)$ gives a uniform estimate on $\|\boldsymbol{v}(\cdot,t)\|_{W_2^{r-1}(\Omega_b)}$ in time, (1.5) alone is not enough to have such an estimate for $\|\rho(\cdot,t)\|_{W_2^{r-\frac{1}{2}}(\mathcal{G})}$, since we lack informations on the time derivative of ρ. However, looking at the fourth equation in (1.4) and using trace theorems, one actually gets

$$\rho_{,t} \in L^2(0,T;W_2^{r-\frac{1}{2}}(\mathcal{G})), \tag{1.6}$$

which, using (1.5) and interpolation, is enough to ensure an even stronger uniform estimate in time, namely

$$\sup_{t<T} \|\rho(\cdot,t)\|_{W_2^r(\mathcal{G})} < +\infty.$$

Notice that capillarity is essential to gain spatial regularity of the free surface. Otherwise, the only information one has from the fourth equation arise from

$$\sup_{t<T} \|\rho\|_{W_2^{r-\frac{1}{2}}(\mathcal{G})} \le \|\rho_0\|_{W_2^{r-\frac{1}{2}}(\mathcal{G})} + \sqrt{T} \left(\int_0^T \|\rho_{,t}\|^2_{W_2^{r-\frac{1}{2}}(\mathcal{G})} dt \right)^{\frac{1}{2}},$$

which can be bounded assuming $W_2^{r-\frac{1}{2}}$ regularity of the initial perturbation and the natural regularity condition (1.6). Therefore, if one wants to look for solution of (1.2) with $\boldsymbol{v} \in W_2^{r,\frac{r}{2}}(\Omega_T)$, the natural regularity conditions on the initial data seem to be

$$\boldsymbol{v}_0 \in W_2^{r-1}(\Omega_b) \quad \text{and} \quad \rho_0 \in W_2^r(\mathcal{G}).$$

Notice that these initial conditions are optimal for the linear system in the sense of extension theorems, (see theorem theorem 2.2.1 and (4.87) in this regard). It is worth noting that in [4], [5], [40], [39], [41], a different regularity is required.

A simplified statement of our main theorem is the following. See theorem 5.2.1 for the full result.

Theorem 1.0.1 *Let $r \in (\frac{5}{2}, 3)$. Suppose that any Σ-periodic solution $\boldsymbol{v}$, q, ρ of the homogeneous linearized system corresponding to* (1.2) *around the stationary solution $\boldsymbol{v}_b$, p_b ϕ_b is exponentially stable, i.e., there exists $\gamma > 0$ and c such that*

$$\|\boldsymbol{v}(\cdot,t) - \boldsymbol{v}_b\|_{W_2^{r-1}(\Omega_b)} + \|\rho(\cdot,t)\|_{W_2^r(\Sigma)} \le ce^{-\gamma t}(\|\boldsymbol{v}_0 - \boldsymbol{v}_b\|_{W_2^{r-1}(\Omega_b)} + \|\rho_0\|_{W_2^r(\Sigma)}).$$

Then there exists $\delta > 0$, $\gamma' > 0$ and c' such that if

$$\|\boldsymbol{v}_0 - \boldsymbol{v}_b\|_{W_2^{r-1}(\Omega_b)} + \|\rho_0\|_{W_2^r(\Sigma)} < \delta,$$

the nonlinear problem (1.2) *has a unique Σ-periodic solution with optimal regularity, which exponentially converges to the stationary solution, i.e.:*

$$\|\boldsymbol{v}(\cdot,t) - \boldsymbol{v}_b\|_{W_2^{r-1}(\Omega_b)} + \|\rho(\cdot,t)\|_{W_2^r(\Sigma)} < c'e^{-\gamma' t}(\|\boldsymbol{v}_0 - \boldsymbol{v}_b\|_{W_2^{r-1}(\Omega_b)} + \|\rho_0\|_{W_2^r(\Sigma)}).$$

We will use the linearization principle above to prove exponential stability of the rest state $\Omega_b = \{0 \le x_3 \le h\}$, $\boldsymbol{v}_b = 0$, $p_b = p_{atm} + g(h - x_3)$, $\boldsymbol{f} = (0, 0, -g)$, where p_{atm} is the atmospheric pressure, g is the acceleration of gravity, and h is the height of the fluid. We analyse the spectrum of the associated linearized problem, proving that indeed it is exponentially stable in the sense of the previous theorem. Thus we obtain the following

Theorem 1.0.2 *Let $r \in (\frac{5}{2}, 3)$. There exists $\delta > 0$, $\gamma' > 0$ and c' such that if*

$$\|\boldsymbol{v}_0\|_{W_2^{r-1}(\Omega_b)} + \|\phi_0 - h\|_{W_2^r(\Sigma)} < \delta,$$

the nonlinear problem (1.2) for $\boldsymbol{f} = \boldsymbol{\alpha} = 0$ has a unique Σ-periodic solution with optimal regularity, which exponentially converges to the rest state, i.e.

$$\|\boldsymbol{v}(\cdot, t)\|_{W_2^{r-1}(\Omega_b)} + \|\phi(\cdot, t) - h\|_{W_2^r(\Sigma)} < c' e^{-\gamma' t}\delta.$$

Exponential stability results for the rest state (without periodicity assumptions) are addressed in [4] for $3 < r < 7/2$ and in [39] for $5/2 < r < 3$ and $\phi_0 - h \in W_2^{r+\frac{1}{2}}(\Sigma)$. In [18] exponential stability is proved for $r = 1$ regardless of the size of the initial data, provided a global in time and smooth solution exists. In [16] the periodic case is studied, and exponential stability of the rest state is proved for $r = 3$. Both these two works employ energy methods.

Although in most of the literature some kind of linearization around the rest state is used, a general linearization principle is, to the best of our knowledge, still unproved. Consequently, exponential stability has been obtained only in special cases ($r = 1, 2$, near the rest state) where an energy inequality can be used.

– *Future developements.*

Some natural questions regarding the stated results are left open. One may wonder, for example, what is the rôle that the change of variables has on the stability of the stationary solution. At lest formally, different choices of the Hanzawa transformation lead to different linearization, which in principle can have different spectral properties. The physics of the problem, however, suggests that if one choice gives rise to a stable linearization, then the same must hold for every other choice. One is thus lead to the following general question:

How does the particular choice of coordinate transformation to write down the linearized system affects the stability of the stationary solution we are linearizing at?

Clearly one may also ask what is the relationship between the stability of a linearization using a Hanzawa transformation and the one obtained using some form of a Lagrangian formulation. Of course one can try to reduce the problem of the stability of a stationary solution to the positivity of some energy functional's second variation, which conceivably is independent of the coordinates chosen. In this direction, see the works of Solonnikov on rotating fluids [33], [34].

On a more refined level, the rate of convergence to the stationary solution (whenever stability occurs) of a global solution on the nonlinear problem depends on the spectrum of the linearization. This, again, seems to depend on the particular choice of variables used to solve the nonlinear problem. Another natural question can thus be the following

How does the particular choice of coordinate transformation to write down the linearized system affects the spectrum of the latter?

Another kind of problem arises considering the situation where there is no surface tension. In this case the Hanzawa approach seems to fail due to a too strong nonlinearity in the equation arising from the kinematic condition. Moreover, one expects to observe an asymptotic decay which is polynomial in time, instead of the claimed exponential decay in the capillary case; in this regard, see [5]. Therefore we have the following question

What is the asymptotic behaviour of the solutions of the nonlinear problem with no capillarity, near a stable stationary solution?

Moreover, it is worth noting that the terms "linearization principle" as we used it is not entirely accurate. A linearization principle in its full strength would give information also in the unstable case, stating that if the linearization has positive eigenvalues, then the nonlinear problem is unstable. Thus one can ask

Does a full linearization principle hold for periodic surface waves?

Finally, the study of the stability of time periodic surface waves seems particularly important in applications. One expects that the spectrum of the monodromy operator will enclose all the relevant informations on the stability of a time periodic solution of the free boundary problem, but concrete examples are at the moment missing. Therefore another problem is

Does a linearization principle in the time periodic case hold?

We expect that, for the last two questions, the methods developed in [12], [34] can be successfully applied.

– *Contents.*

In chapter 2 we provide background material on anisotropic Sobolev–Slobodetskii space and Laplace transform. Most of the theorems are classical and well known, except maybe the ones in section 3, which deals with anisotropic Sobolev–Slobodetskii spaces for small time. These are introduced in order to have a scale invariant norm with respect to time, and will we proof the less known properties of these spaces.

In chapter 3 we describe our choice of Hanzawa transformation, which differs from the usual one, and is taken to simplify the subsequent work. The regularity properties of this transformation are not optimal, but this won't affect the study of the nonlinear problem.
Then, changing coordinates to reduce system (1.2) to one defined in the fixed domain Ω_b, we derive an explicit form for the linear part of the system near a stationary solution, and compute the higher order, nonlinear terms.

In chapter 4 we deal with the linearized problem. We first prove existence and optimal regularity estimates for the complex parameter model problems in the half–space obtained localising the Laplace transformed time dependent linear problem. These can be explicitly solved via Fourier transform methods, and Parceval identity provides optimal parameter dependant estimates.
In section 2 we construct the solution of the complex parameter linear problem via a Shauder localisation method, and prove its uniqueness through a coercive inequality.
In the last section we reduce the original time dependent problem to one with homogenous initial data, and obtain a solution via the previous results and Laplace transform.

In chapter 5 we prove the abstract linearization principle, the exponential stability of the rest state, and a local (in time) existence theorem. We first estimate the nonlinear terms in the suitable spaces, using the explicit expression computed in chapter 3.
To prove the linearization principle, we construct the solution as a sum of two addends: the first one solves a linear problem with a relatively large initial data, and the second a nonlinear one with a relatively small one. The stability hypothesis guarantees that the "linear" part is decaying exponentially, and the now quadratic behaviour of the a-priori estimate for the nonlinear part implies its decay as well. Thus the solution of the full nonlinear problem can be constructed for any fixed (and large) T, with decaying norm and we can iteratively repeat the construction to obtain a global solution.
In section 3, the nonlinear stability of the rest state is deduced from the analysis of the spectrum of the corresponding linearized problem, which is easily done due to the simple form of the linearization.

The final section of this chapter is devoted to the local existence theorem for arbitrarily large initial data. The result is substantially known (although not in the periodic case), since other techniques works as well in this case. The purpose of this section is to show that, even if our Hanzawa transformation is far from optimal, we nonetheless obtain sufficiently sharp estimates for the nonlinear terms to obtain the result.

Notation.

We won't use a different notation for spaces of scalar or vector valued functions, since all the properties we will use of these spaces are unaffected by the number of components of their elements.
Whenever this causes no confusion, we will use Einstein's convention on summation over repeated indexes, and use the comma notation for partial differentiation, where

$$\frac{\partial f}{\partial t} = f_{,t}, \qquad \frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} = f_{,x_1^{\alpha_1} \dots x_N^{\alpha_N}},$$

for any multiindex $(\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ with length $k = \alpha_1 + \dots + \alpha_N$. Differential operators will be denoted as ∇, D or with the standard partial differential notation, and Δ will denote the Laplace operator in $\mathbb{R}^N$. Vectors will usually be written in bolded fonts, while their components will be written with the same unbolded symbol, with the relative coordinate as an upper index, i.e. $\boldsymbol{v} = (v^1, v^2, v^3)$ for a vector $\boldsymbol{v} \in \mathbb{R}^3$.
A dot will indicate for the standard scalar product between vectors, while multiplication of a vector and a tensor will be denoted by simple juxtaposition; for example, if $\mathbb{M} = (M_{ij})$ and $\boldsymbol{N} = (N^i)$

$$\mathbb{M}\boldsymbol{N} = \left(M_{ij} N^j\right)_i, \qquad \boldsymbol{N}\mathbb{M} = \left(N^i M_{ij}\right)_j.$$

Whenever more factors are present (especially when differential operators are involved), for ease of reading we won't specify the order of multiplication, since it is usually readily recovered from the context.
In chain of inequalities, a constant c will keep the same symbol, even if it changes its value from line to line, when it does so in a way independent of the quantities involved in the inequality. Whenever a specific dependance on the data is relevant, we will write it between brackets, e.g. $c(T)$ denotes a constant which depends on T, but not on the quantities involved in the inequality it appears in. The remaining notation is standard, or will be specified when introduced.

Chapter 2

Preliminaries

In this chapter we develop some of the basic tools we will need to deal with problem (1.2). All our results are settled in anisotropic Sobolev–Slobodetskii spaces, whose theory has been developed thoughtfully by the Russian school in the fifties. These spaces have had an enormous impact in the study of PDE's, and in particular the anisotropic case has been found especially fruitful in the study of parabolic problems. For a general survey of this vast subject see the book by Besov, Il'in and Nikol'skii [6], or the work of Triebel [43].

In the first section we recall the definition and basic properties of isotropic Sobolev–Slobodetskii spaces $W_2^l(\Omega)$, where l stands for the order of (weak) differentiability. These where introduced by Sobolev in [22] for integer l and generalised by Slobodetskii in [21] for not necessarily integer values of l.

In the second section we introduce the theory of anisotropic Sobolev–Slobodetskii spaces as constructed by Slobodetskii in [21] and developed my many other Russian mathematicians in the following years (see [15] for example). For our purpose, i.e. dealing with second order parabolic systems, we require spaces whose function have spacial derivatives of twice the order of time derivative. Most of the results presented have natural analogues for summability exponents $p \neq 2$.

In the third section we describe a modified norm for anisotropic Sobolev–Slobodetskii space, introduced by Solonnikov in [19]. This modified norm has many nice scaling properties in time, and are especially useful when dealing with parabolic problems for small time.

One of the many powerful application of anisotropic Sobolev–Slobodetskii space, is described in the last section. Agranovich and Vishik in [2] introduced a general strategy to deal with parabolic systems through Laplace transform. Since this is the approach we will employ in the study of the linear problem, we briefly describe the main tools constructed in [2].

Most of the results stated in this chapter are classical and can be found in the above mentioned works. Sometimes we will need refinements or variations of classical theorems, and whenever a handy reference is not available we will provide a sketch of the proof.

2.1 Isotropic Sobolev–Slobodetskii spaces

If l is a nonnegative integer the isotropic Sobolev–Slobodetskii space on a bounded domain $\Omega \subset \mathbb{R}^N$ coincides with the usual Sobolev space, i.e. the set of functions $u : \Omega \to \mathbb{R}$ with finite norm

$$\|u\|^2_{W^l_2(\Omega)} = \sum_{|j|\le l} \int_\Omega |D^j u(x)|^2 dx,$$

where $D^j u$ is the j-th distributional derivative. Here j is a multiindex $j = (j_1, \dots, j_N)$ and $|j|$ its length $j_1 + \dots + j_N$. When $l = [l] + \{l\}$, where $\{l\} \in (0, 1)$ is the fractional part of l, the norm is

$$\|u\|^2_{W^l_2(\Omega)} := \|u\|^2_{W^{[l]}_2(\Omega)} + \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{N+2\{l\}}} dxdy. \tag{2.1}$$

We will denote by $\| \ \|_{\mathring{W}^l_2(\Omega)}$ the principal part of the previous norms, i.e.

$$\|u\|^2_{\mathring{W}^l_2(\Omega)} = \begin{cases} \|D^l u\|^2_{L^2(\Omega)} & \text{if } l \text{ is an integer,} \\ \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{N+2\{l\}}} dxdy & \text{otherwise.} \end{cases} \tag{2.2}$$

This norm is derived from a natural inner product defined as

$$\begin{aligned} (u, v)_{W^l_2(\Omega)} := & \sum_{|j|\le[l]} (D^j u, D^j v)_{L^2(\Omega)} \\ & + \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{\big(D^j u(x) - D^j u(y)\big)\big(D^j v(x) - D^j v(y)\big)}{|x-y|^{N+2\{l\}}} dxdy, \end{aligned}$$

with which $W^l_2(\Omega)$ is a Hilbert space. Its dual will be denoted by $W^{-l}_2(\Omega)$, thus giving a meaning to the symbol $W^l_2(\Omega)$ for all real l.

If the boundary of Ω is smooth enough (of class $\mathcal{C}^1$ suffices) any $u \in W^l_2(\Omega)$ can be extended to the whole $\mathbb{R}^N$ with preservation of class and controlled norm. More precisely, there exists a continuation operator $C : W^l_2(\Omega) \to W^l_2(\mathbb{R}^N)$ with the following properties:

1. $C(u)\lfloor_\Omega = u$;
2. $C(u)$ has compact support in $\mathbb{R}^N$;
3. $\|C(u)\|_{W_2^l(\mathbb{R}^N)} \leq c\|u\|_{W_2^l(\Omega)}$.

Using this continuation property, equivalent norms (which for simplicity we'll still denote with $\|u\|_{W_2^l(\Omega)}$) can be defined using the finite difference operator $\Delta_z u(x) = u(x+z) - u(x)$. Two examples are the followings

$$\|u\|^2_{W_2^l(\Omega)} \simeq \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]} \int_{\{|z|\leq 1\}} \|\Delta_z D^j C(u)(x)\|^2_{L^2(\Omega)} \frac{dz}{|z|^{N+2\lambda}},$$

$$\|u\|^2_{W_2^l(\Omega)} \simeq \|u\|^2_{L^2(\Omega)} + \sum_{|j|=[l]} \int_{\{|z|\leq 1\}} \|\Delta_z^k C(u)(x)\|^2_{L^2(\Omega)} \frac{dz}{|z|^{N+2l}}, \tag{2.3}$$

for any integer $k > l$, where $\Delta_z^k v$ is the k-times iterated finite difference operator, whose explicit expression is

$$\Delta_z^k v(x) = \Delta_z \Delta_z^{k-1} v(x) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} v(x+jz).$$

We recall some embedding properties of Sobolev–Slobodetskii spaces.

Theorem 2.1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and $u \in W_2^l(\Omega)$. Then*

$$\|u\|_{L^{\frac{2N}{N-2l}}(\Omega)} \leq c\|u\|_{W_2^l(\Omega)}, \qquad \text{if } l < \frac{N}{2}, \tag{2.4}$$

$$\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{W_2^l(\Omega)}, \qquad \text{if } l > \frac{N}{2}. \tag{2.5}$$

There are much more refined results of this type (especially in the $l \geq N/2$ case, and we refer to [43] for them. All the constant in the embedding theorem above, as well as in the equivalence inequalities of the various previous norms depend on Ω, mainly because the continuation operator, which allows to reduce the inequalities involving $W_2^l(\Omega)$ to similar inequalities involving $W_2^l(\mathbb{R}^N)$, depends on the geometry of Ω. It will be important to have scale invariant inequalities in time, and thus we will consider more closely the continuation operator in the one dimensional case. For our purposes it suffice to analyse only the case $l \in (0,1)$, and we have the following result.

Theorem 2.1.2 *Let $l \in (0,1)$ and $u \in W_2^l([0,T])$, equipped with the standard norm* (2.1)*. There is an extension $C_T(u)$ of u to $[0,+\infty)$ such that*

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{|C_T(u)(x)-C_T(u)(y)|^2}{|x-y|^{1+2l}}dxdy \\ \leq c_l\left(\int_0^T\int_0^T\frac{|u(x)-u(y)|^2}{|x-y|^{1+2l}}dxdy+\frac{1}{T^{2l}}\int_0^T|u(x)|^2dx\right)\end{aligned} \tag{2.6}$$

for a constant c_l depending only on l and not on u or T.

Proof. The inequality follows from a scaling argument. If c_l is the constant for which (2.6) holds for $T=1$ and for a fixed extension operator C_1, then given any function $u \in W_2^l([0,T])$ we define

$$u_T(x) = u(Tx) \in W_2^l([0,1]), \qquad C_T(u) = C_1(u_T)(\frac{x}{T}).$$

A change of variables thus gives (2.6). □

It will be useful to study the structure of $W_2^l(\Omega)$ as an algebra of function. To this end we will prove the following theorem (see [33] for a a refined statement using Besov spaces).

Proposition 2.1.3 *For arbitrary functions u, v given in a smooth domain $\Omega \subset \mathbb{R}^N$ it holds*

1.
$$\|uv\|_{L^2(\Omega)} \leq c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^{\frac{N}{2}-l}(\Omega)},$$
for any $0 \leq l \leq N/2$.

2.
$$\begin{aligned}\|uv\|_{W_2^l(\Omega)} &\leq c\|u\|_{W_2^l(\Omega)}(\|v\|_{W_2^{\frac{N}{2}}(\Omega)} + \|v\|_{L^\infty(\Omega)}) \\ &\leq c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^s(\Omega)},\end{aligned} \tag{2.7}$$
for any $0 \leq l \leq N/2 < s$.

3.
$$\begin{aligned}\|uv\|_{W_2^l(\Omega)} &\leq c\big(\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^s(\Omega)} + \|v\|_{W_2^l(\Omega)}\|u\|_{W_2^s(\Omega)}\big) \\ &\leq c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^l(\Omega)}.\end{aligned} \tag{2.8}$$
for any $l > s > N/2$.

Proof. The proof of the first statement is just an application of Hölder inequality with exponents $N/(N-2l)$ and N/l and the embedding theorem 2.4. To prove the other two estimates we can suppose, using the continuation operator above, that u and v are defined in the whole of $\mathbb{R}^N$. It is easy to prove by induction that

$$\Delta_z^k(uv)(x) = \sum_{j=0}^{k} \binom{k}{j} \Delta_z^j u(x) \Delta_z^{k-j} v(x+jz). \tag{2.9}$$

If one then choose $k > 2l$ in the norm (2.3), each of the resulting term in the above formula has a factor in which the finite difference operator is applied at least $k/2 > l$ times. To prove (2.8) it suffice to notice that

$$\|\Delta_z^j u(x)\Delta_z^{k-j} v(x+jz)\|_{L^2(\mathbb{R}^N)} \leq \begin{cases} c_k \|u\|_{L^\infty(\mathbb{R}^N)} \|\Delta_z^{k-j} v\|_{L^2(\mathbb{R}^N)} & \text{if } j \leq \frac{k}{2}, \\ c_k \|v\|_{L^\infty(\mathbb{R}^N)} \|\Delta_z^{j} u\|_{L^2(\mathbb{R}^N)} & \text{if } j > \frac{k}{2}), \end{cases}$$

which, plugged into (2.3), gives for any $l \geq 0$

$$\|uv\|_{W_2^l(\mathbb{R}^N)} \leq c_k(\|u\|_{L^\infty(\mathbb{R}^N)}\|v\|_{W_2^l(\mathbb{R}^N)} + \|v\|_{L^\infty(\mathbb{R}^N)}\|u\|_{W_2^l(\mathbb{R}^N)}); \tag{2.10}$$

The embedding inequality (2.5) thus concludes the proof of (2.8), since $l > N/2$. Suppose now that $l < N/2$ to prove the inequality at point 2. We treat as above the terms in (2.9) corresponding to $j > k/2$: due to the embedding inequality (2.5), for those terms an estimate of the form

$$\int_{\{|z|<1\}} \|\Delta_z^j u(x)\Delta_z^{k-j} v(x+jz)\|^2_{L^2(\mathbb{R}^N)} \frac{dz}{|z|^{N+2l}} \leq c\|v\|_{W_2^s(\mathbb{R}^N)}\|u\|_{W_2^l(\mathbb{R}^N)},$$

holds true.

For $j \leq k/2$ we use Hölder inequality instead:

$$\|\Delta_z^j u(x)\Delta_z^{k-j} v(x+jz)\|_{L^2(\mathbb{R}^N)} \leq c_k \|u\|_{L^{\frac{2N}{N-2l}}(\mathbb{R}^N)} \|\Delta_z^{k-j} v\|_{L^{\frac{N}{l}}(\mathbb{R}^N)}.$$

Using the embedding (2.4) we get

$$\begin{aligned} \int_{\{|z|<1\}} &\|\Delta_z^j u(x)\Delta_z^{k-j} v(x+jz)\|^2_{L^2(\mathbb{R}^N)} \frac{dz}{|z|^{N+2l}} \\ &\leq c_k \|u\|^2_{W_2^l(\mathbb{R}^N)} \int_{|z|<1} \|\Delta_z^{k-j} v\|^2_{L^{\frac{N}{l}}(\mathbb{R}^N)} \frac{dz}{|z|^{N+2l}}, \end{aligned} \tag{2.11}$$

and for $s > N/2$ we have

$$
\begin{aligned}
\int_{\{|z|<1\}} \|\Delta_z^{k-j} v\|^2_{L^{\frac{N}{l}}(\mathbb{R}^N)} \frac{dz}{|z|^{N+2l}} &= \int_{|z|<1} \frac{\|\Delta_z^{k-j} v\|^2_{L^{\frac{N}{l}}(\mathbb{R}^N)}}{|z|^{\frac{2l}{N}(N+2s)}} \frac{dz}{|z|^{N-\frac{2l}{N}2s}} \\
&\leq c_{l,s} \left[\int_{|z|<1} \|\Delta_z^{k-j} v\|^{\frac{N}{l}}_{L^{\frac{N}{l}}(\mathbb{R}^N)} \frac{dz}{|z|^{N+2s}} \right]^{\frac{2l}{N}} \\
&\leq c_{l,s} \|v\|^{2(1-\frac{2l}{N})}_{L^\infty(\mathbb{R}^N)} \|v\|^{2\frac{2l}{N}}_{W_2^s(\mathbb{R}^N)}
\end{aligned}
$$

where on the second line we applied Hölder inequality with exponent $p = N/2l$ on the first factor. Using this estimate in (2.11), applying Young inequality and the embedding inequality (2.5) gives

$$
\int_{\{|z|<1\}} \|\Delta_z^j u(x) \Delta_z^{k-j} v(x+jz)\|^2_{L^2(\mathbb{R}^N)} \frac{dz}{|z|^{N+2l}} \leq c \|u\|^2_{W_2^l(\mathbb{R}^N)} \|v\|^2_{W_2^s(\mathbb{R}^N)},
$$

and the conclusion. □

In the particular case where u and v depend on two disjoint sets of variables one can prove a refined version of the previous theorem. With $\mathbb{R}^N_+$ we denote the half-space defined by the condition $x_1 > 0$.

Lemma 2.1.4 *Let $N \geq 2$ and l not an integer. If $u(x) = u(x_1)$ and $v(x) = v(x_2, \ldots, x_N)$ then*

$$
\|uv\|_{W_2^l(\mathbb{R}^N_+)} \leq c(\|u\|_{L^2(\mathbb{R}_+)} \|v\|_{W_2^l(\mathbb{R}^{N-1})} + \|v\|_{L^2(\mathbb{R}^{N-1})} \|u\|_{W_2^l(\mathbb{R}_+)}).
$$

Proof. The claim follows from the fact that the $W_2^l(\mathbb{R}^N_+)$ norm is equivalent, for l not an integer, to

$$
\|f\|^2_{B^l_{2,2}(\mathbb{R}^N_+)} = \|f\|^2_{L^2(\mathbb{R}^N_+)} + \sum_{i=1}^N \int_0^{+\infty} \|\Delta^k_{he_i} f\|^2_{L^2(\mathbb{R}^N_+)} \frac{dh}{h^{1+2l}},
$$

where e_i is the vector whose j-th component is δ_{ij}, and k is any integer greater than l. Indeed it suffice to notice that for $i < 1$

$$
\|\Delta^k_{he_i} uv\|^2_{L^2(\mathbb{R}^N_+)} = \begin{cases} \|v\|^2_{L^2(\mathbb{R}^{N-1})} \|\Delta^k_{he_1} v\|^2_{L^2(\mathbb{R}_+)} & \text{for } i = 1, \\ \|u\|^2_{L^2(\mathbb{R}_+)} \|\Delta^k_{he_i} v\|^2_{L^2(\mathbb{R}^{N-1})} & \text{for } i > 1, \end{cases}
$$

and this, together with $\|uv\|_{L^2(\mathbb{R}^N_+)} = \|u\|_{L^2(\mathbb{R}_+)} \|v\|_{L^2(\mathbb{R}^{N-1})}$, gives the desired inequality. □

Another equivalent norm in the case $\Omega = \mathbb{R}^N$ is the following:

$$\|v\|_{H^l(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (1 + |\xi|^2)^l |\widetilde{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \tag{2.12}$$

where $\widetilde{u}$ is the Fourier transform of u:

$$\widetilde{u}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi \cdot x} u(x) dx.$$

This norm can be used to define fractional order Sobolev–Slobodetskii spaces in an arbitrary smooth domain or submanifold. For smooth domains Ω, it suffices define $H_2^l(\Omega)$ as the set of restrictions of functions $u \in H_2^l(\mathbb{R}^N)$ to Ω. One can then prove that this set coincides with $W_2^l(\Omega)$, and that fixing a continuation operator C the $\|u\|_{W_2^l(\Omega)}$ norm is equivalent to $\|C(u)\|_{H_2^l(\mathbb{R}^N)}$. For smooth bounded d-dimensional submanifolds M of $\mathbb{R}^N$, one needs to fix a partition of unity $\{\phi_j\}_{j \in J}$ subordinated to a finite open covering $\{U_j\}_{j \in J}$ of M on which smooth local charts $\{\Phi_j : U_j \cap M \to \mathbb{R}^d\}_{j \in J}$ are defined. Then one defines

$$\|u\|_{H_2^l(M)} = \left[\sum_j \|(\phi_j \cdot u) \circ \Phi^{-1}\|^2_{H_2^l(\mathbb{R}^d)} \right]^{\frac{1}{2}}.$$

This norm turns again out to be equivalent to many others, but the procedure described here to construct Sobolev–Slobodetskii spaces on submanifolds is especially useful when considering the relationships between $W_2^l(\Omega)$ and $W_2^s(\partial\Omega)$. Indeed, let $\mathbb{R}^N_+ := \{(x_1, \ldots, x_N) : x_n > 0\}$. Using partitions of unity and local charts, one is able to reduce such a problem to the study of the relationships between $W_2^l(\mathbb{R}^N)$ and $W_2^s(\mathbb{R}^N_+)$. To this end we recall the following well known restriction and extension theorems:

Theorem 2.1.5 *For each $l > \frac{1}{2}$, there exists:*

1. *A continuous restriction operator $R : W_2^l(\mathbb{R}^N_+) \to W_2^{l-\frac{1}{2}}(\mathbb{R}^{N-1})$, which agrees with the usual restriction on smooth functions.*

2. *A continuous extension operator $E : W_2^{l-\frac{1}{2}}(\mathbb{R}^{N-1}) \to W_2^l(\mathbb{R}^N_+)$ such that $R \circ E = E \circ R = I$ on smooth functions.*

Using this theorem and the procedure described above, one then obtains extension and restriction operators between $W_2^l(\Omega)$ and $W_2^{l-\frac{1}{2}}(\partial\Omega)$. Notice that smoothness of $\partial\Omega$ is essential here, and Ω being locally near $\partial\Omega$ the epigraph of a Lipschitz function suffices. For higher values of l one can construct a more general extension operator, prescribing also the normal derivatives on the boundary.

Theorem 2.1.6 *For $l - k > \frac{1}{2}$, there exists a continuous extension operator*

$$E : \Pi_{j=0}^{k} W_2^{l-j-\frac{1}{2}}(\partial\Omega) \to W_2^l(\Omega),$$

such that

$$\frac{\partial^j}{\partial \boldsymbol{n}^j} E(\phi_1, \ldots, \phi_k) = \phi_j \qquad \text{on } \partial\Omega.$$

Clearly also this theorem can be stated for general smooth bounded submanifolds with boundary of $\mathbb{R}^N$.

Another application of the norm (2.12) is an easy proof of the so-called *interpolation inequality*: If $\eta > 0$, $k \geq 0$, then Young inequality with exponents $\frac{k+\eta}{\eta}$ and $\frac{k+\eta}{k}$ gives for any $s > 0$

$$s^{\frac{\eta}{k+\eta}}(1 + |\xi|^2)^{\frac{k}{k+\eta}} \leq c\big(s + (1 + |\xi|^2)\big),$$

and thus

$$s^\eta (1 + |\xi|^2)^k \leq c\big(s^{k+\eta} + (1 + |\xi|^2)^{k+\eta}\big), \tag{2.13}$$

which, used into (2.12) gives

$$s^\eta \|u\|^2_{W_2^k(U)} \leq c(\|u\|^2_{W_2^{k+\eta}(U)} + s^{k+\eta}\|u\|^2_{L^2(U)}), \tag{2.14}$$

for $U = \mathbb{R}^N$. Again, the inequality holds true for arbitrary smooth bounded $\Omega = U$, or for U being a bounded smooth submanifold of $\mathbb{R}^N$. Reading this inequality for large s and dividing by s^η gives a precise quantitative version of the statement that, if $h > k$, the W_2^k can be controlled by a small part of the W_2^h norm plus a large part of the L^2 norm, i.e. the interpolation inequality

$$\|u\|^2_{W_2^k(U)} \leq \varepsilon \|u\|^2_{W_2^h(U)} + c(\varepsilon)\|u\|^2_{L^2(U)},$$

valid for $\varepsilon > 0$, $h > k$.

Another useful interpolation inequality is used to deal with the necessary condition $l > \frac{1}{2}$ in theorem 2.1.5. Indeed the continuity estimate for the restriction operator fails to be true in the limit case $l = \frac{1}{2}$, i.e. there exists no constant c such that the inequality $\|u\|_{L^2(\mathbb{R}^{N-1})} \leq c\|u\|_{W_2^{1/2}(\mathbb{R}^N_+)}$ holds true. For example, let $u(x, y, z) = \phi(r)r^{-1/2}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and $\phi(r) = 1$ for $r \leq 1$ and $\phi(r) = 0$ for $r \geq 2$. Clearly $u \in W_2^1(\mathbb{R}^3)$, and thus by the restriction theorem $u(x, y, 0) \in W_2^{1/2}(\mathbb{R}^2)$. However $u(x, 0, 0) = \phi(|x|)|x|^{-1/2} \notin L^2(\mathbb{R})$. On the other hand it is clear that the $L^2(\mathbb{R}^{N-1})$ norm of $u\lfloor_{\mathbb{R}^{N-1}}$ is controlled by its $W_2^\eta(\mathbb{R}^{N-1})$-norm for any $\eta > 0$, and thus by $\|u\|_{W_2^{\eta+\frac{1}{2}}(\mathbb{R}^N)}$. A quantitative version of this control is given by the following interpolation inequality, which again can be stated for an arbitrary smooth bounded $\Omega \subset \mathbb{R}^N$: for any $s > 0$ it holds

$$s^\eta \|u\|^2_{L^2(\partial\Omega)} \leq c(\|u\|^2_{W_2^{\eta+\frac{1}{2}}(\Omega)} + s^{\eta+\frac{1}{2}} \|u\|^2_{L^2(\Omega)}). \tag{2.15}$$

To prove this inequality, we denote by $\widehat{u}$ the Fourier transform w.r.t. the first $N-1$ variables, and $\xi' = (\xi_1, \dots, \xi_{N-1})$. By the inversion formula for the Fourier transform

$$\widehat{u}(\xi', x_N) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\xi_N x_N} \widetilde{u}(\xi', \xi_N) d\xi_N \tag{2.16}$$

and thus

$$\begin{aligned}
|\widehat{u}(\xi', 0)|^2 &\leq \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} |\widetilde{u}(\xi', \xi_N)| d\xi_N \right]^2 \\
&\leq \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} (1 + s + |\xi_N|^2)^{\eta+\frac{1}{2}} |\widetilde{u}(\xi', \xi_N)|^2 d\xi_N \int_{-\infty}^{+\infty} \frac{d\xi_N}{(1 + s + |\xi_N|^2)^{\eta+\frac{1}{2}}} \\
&\leq \frac{c}{(1+s)^\eta} \int_{-\infty}^{+\infty} (1 + s + |\xi_N|^2)^{\eta+\frac{1}{2}} |\widetilde{u}(\xi)|^2 d\xi_N.
\end{aligned}$$

Integrating in ξ' and multiplying by s^η gives, through Parceval identity,

$$s^\eta \|u\|^2_{L^2(\mathbb{R}^{N-1})} \leq c \int_{\mathbb{R}^N} (1+s+|\xi|^2)^{\eta+\frac{1}{2}} |\widetilde{u}|^2 d\xi \leq c(\|u\|^2_{W_2^{\eta+\frac{1}{2}}(\mathbb{R}^N)} + s^{\eta+\frac{1}{2}} \|u\|^2_{L^2(\mathbb{R}^N)}).$$

We finally recall the definition of the Sobolev–Slobodetskii spaces of functions $u(x')$ defined on $\mathbb{R}^2$, periodic in $x' \in \Sigma$, with finite $W_2^l(\Sigma)$ norm. To simplify notations we will suppose $\Sigma = (0, 2\pi)^2$. The space $W_2^l(\Sigma)$ is the (in general, proper) subspace of $W_2^l(\Sigma)$ formed by the restrictions to Σ of Σ-periodic functions. Its norm is given by

$$\|u\|_{W_2^l(\Sigma)} := \sum_{\xi' \in \mathbb{Z}^2} (1 + |\xi'|^2)^l |u_{\xi'}|^2, \tag{2.17}$$

with $u_{\xi'}$ being the ξ'-th Fourier coefficient of u:

$$u_{\xi'} := \int_\Sigma u(x') e^{-i\xi' \cdot x'} dx'. \tag{2.18}$$

Notice that for $l < 0$ the norm above is still equivalent to the norm of the dual of $W_2^l(\Sigma)$. Given $0 < h \leq +\infty$, the Sobolev–Slobodetskii space of periodic (in $x' \in \Sigma$) functions $u = u(x', x_3)$ with $(x', x_3) \in \mathbb{R}^2 \times [0, h)$, is defined as

$$W_2^l(\Sigma \times [0, h)) := L^2(0, L; W_2^l(\Sigma)) \cap W_2^l(0, h; L^2(\Sigma)),$$

with norm

$$\|u\|^2_{W_2^l(\Sigma\times[0,h))} := \int_0^h \|u(\cdot,x_3)\|^2_{W_2^l(\Sigma)}dx_3 + \int_\Sigma \|u(x',\cdot)\|^2_{W_2^l(0,h)}dx'.$$

These norms are equivalent to the usual $W_2^l(\Sigma)$ and $W_2^l(\Sigma\times[0,h))$ norms on the subspace of the restrictions to Σ of Σ-periodic function, and thus the subscript # will often be omitted. Moreover, will will use the notation $\Sigma^\infty = \Sigma\times[0,+\infty)$.

For all these spaces classical trace, extension, interpolation and embedding theorems hold true, see [43] for a comprehensive treatment of the subject.

2.2 Anisotropic Sobolev–Slobodetskii spaces

The anisotropic Sobolev–Slobodetskii space is defined as the set of functions $u = u(x,t)$, defined in $Q_T := \Omega\times[0,T)$, $0 < T \le +\infty$ such that

$$u \in W_2^{l,\frac{l}{2}}(Q_T) := L^2(0,T;W_2^l(\Omega)) \cap W_2^{\frac{l}{2}}(0,T;L^2(\Omega)),$$

with norm

$$\|u\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} := \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_0^T \sum_{0\le j\le\left[\frac{l}{2}\right]} \|D_t^j u(\cdot,t)\|_{L^2(\Omega)}dt$$
$$+ \int_0^T \frac{dh}{h^{1+2\left\{\frac{l}{2}\right\}}} \int_h^T \|\Delta_{-h} D_t^{\left[\frac{l}{2}\right]} u(\cdot,t)\|^2_{L^2(\Omega)}dx.$$

An equivalent norm, which will still be denoted with the same symbol, is

$$\|u\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} := \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)}dx. \tag{2.19}$$

Applying (2.1.3) in each variables, gives that for any smooth function v,

$$\|uv\|_{W_2^{l,\frac{l}{2}}(Q_T)} \le c_v \|u\|_{W_2^{l,\frac{l}{2}}(Q_T)}, \quad \forall u \in W_2^{l,\frac{l}{2}}(Q_T), \tag{2.20}$$

where c_v can, for example, be taken as

$$\sup_{0\le t\le T} \|v(\cdot,t)\|_{W_2^\theta(\Omega)} + \sup_\Omega \|v(x,\cdot)\|_{W_2^{\theta'}(0,T)},$$

with $\theta = \max\{\frac{N}{2}+\varepsilon, l\}$, $\theta' = \max\{\frac{1}{2}+\varepsilon, l\}$, $\varepsilon > 0$ arbitrary.

For smooth bounded Ω, a continuous continuation operator exists, with analogue properties to the one related to the isotropic spaces. Moreover, when $Q_T = \mathbb{R}^{N+1}$ one can use the norm

$$\|u\|_{H^{l,\frac{l}{2}}(\mathbb{R}^{N+1})} = \int_{\mathbb{R}^{N+1}} (1+|s|+|\xi|^2)^l |\widetilde{u}(\xi,s)|^2 d\xi ds, \tag{2.21}$$

instead, and can proceed as in the isotropic case, defining an equivalent norm on $W_2^{l,\frac{l}{2}}(Q_T)$ using the $H^{l,\frac{l}{2}}(\mathbb{R}^{N+1})$ norm via the continuation operator. All these spaces can be defined (through local maps and partitions of unity) for a smooth submanifold $\mathcal{G}$ of $\mathbb{R}^N$ in which case we will use the notation $G_T = \mathcal{G} \times [0,T)$. Standard theory ensures that these are Hilbert space with respect to the natural inner product, whose dual will still be denoted by $W_2^{-l,-\frac{l}{2}}$.

Deriving with respect to spatial variables or time variables has a different effect on the space one falls in, since it is easy to prove that

$$\|D_x^k D_t^h u\|_{W_2^{l-k-2h,\frac{l-k-2h}{2}}(Q_T)} \leq c\|u\|_{W_2^{l,\frac{l}{2}}(Q_T)}. \tag{2.22}$$

Regarding the extension and restriction operator analogues, in the anisotropic case one has to distinguish between considering the spatial restriction on $\partial\Omega \times [0,T)$ and the time restriction on $\Omega \times \{0\}$, as the following theorem shows.

Theorem 2.2.1 *Let $u \in W_2^{l,\frac{l}{2}}(Q_T)$, $0 < T \leq +\infty$, with Ω being a bounded smooth domain.*

1. *If $l > \frac{1}{2}$, there exist a continuous space restriction operator*

$$R_x : W_2^{l,\frac{l}{2}}(Q_T) \to W_2^{l-\frac{1}{2},\frac{l}{2}-\frac{1}{4}}(\partial\Omega \times [0,T)).$$

2. *If $l > 1$ there exists a continuous time restriction operator*

$$R_t : W_2^{l,\frac{l}{2}}(Q_T) \to W_2^{l-1}(\Omega \times \{0\}).$$

3. *If $l - k > \frac{1}{2}$, there exists a continuous space extension operator*

$$E_x : \Pi_{j=0}^k W_2^{l-j-\frac{1}{2},\frac{l}{2}-\frac{j}{2}-\frac{1}{4}}(\partial\Omega \times [0,T)) \to W_2^{l,\frac{l}{2}}(Q_T),$$

such that

$$\frac{\partial^j}{\partial \boldsymbol{n}^j} E_x(\phi_1,\ldots,\phi_k) = \phi_j \qquad \text{on } \partial\Omega.$$

4. If $l-k>1$, there exists a continuous time extension operator

$$E_t : \Pi_{j=0}^{k} W_2^{l-2j-1}(\Omega) \to W_2^{l,\frac{l}{2}}(Q_T),$$

such that

$$\frac{\partial^j}{\partial t^j} E_t(\psi_1, \dots, \psi_k) = \psi_j \qquad \text{on } \Omega \times \{0\}.$$

All the constant in this theorem depend on Ω, and, more importantly, on T; the next section will give a somewhat deeper discussion on this latter dependance.

It will be useful to define the following auxiliary spaces

$$W_2^{l,0}(Q_T) := L^2(0,T; W_2^l(\Omega)), \qquad W_2^{0,\frac{l}{2}}(Q_T) := W_2^{\frac{l}{2}}(0,T; L^2(\Omega)),$$

with the natural norms corresponding to the two addends in (2.19). Clearly an equivalent norm can be defined when $Q_T = \mathbb{R}^N \times \mathbb{R}$ as

$$\|u\|_{H^{l,0}(\mathbb{R}^N)} = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^N} (1+|\xi|^2)^l |\widetilde{u}(\xi,s)|^2 d\xi ds,$$

$$\|u\|_{H^{0,\frac{l}{2}}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \int_{-\infty}^{+\infty} (1+|s|)^l |\widetilde{u}(\xi,s)|^2 ds d\xi.$$

We will need the following proposition.

Lemma 2.2.2 *Let $l > \frac{1}{2}$ and $\rho = \rho(x,t)$ be a function such that $\rho \in W_2^{l+\frac{1}{2},\frac{\eta}{2}+\frac{1}{4}}(\mathbb{R}^{N+1})$, $\rho_{,t} \in W_2^{l-\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(\mathbb{R}^{N+1})$. Then*

$$\|\rho(\cdot,0)\|_{W_2^l(\mathbb{R}^N)} \le \frac{1}{2\pi}(\|\rho\|_{W_2^{l+\frac{1}{2},0}(\mathbb{R}^{N+1})} + \|\rho_{,t}\|_{W_2^{l-\frac{1}{2},0}(\mathbb{R}^N)}),$$

for a constant independent of ρ.

Proof. Consider the spatial Fourier transform $\widehat{\rho}$. By the inversion formula (2.16), we have

$$\begin{aligned}
|\widehat{\rho}(\xi,0)|^2 &\le \frac{1}{4\pi^2}\left[\int_{-\infty}^{+\infty} |\widetilde{\rho}(\xi,s)| ds\right]^2 \\
&\le \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} (1+s^2+|\xi|^2)|\widetilde{\rho}(\xi,s)|^2 ds \int_{-\infty}^{+\infty} \frac{ds}{1+s^2+|\xi|^2} \\
&\le \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{1+s^2+|\xi|^2}{(1+|\xi|^2)^{\frac{1}{2}}} |\widetilde{\rho}(\xi,s)|^2 ds \int_{-\infty}^{+\infty} \frac{dr}{1+r^2}.
\end{aligned}$$

Therefore, multiplying by $(1+|\xi|^2)^l$ and integrating in $\xi \in \mathbb{R}^N$ we get

$$\|\rho\|^2_{W_2^l(\mathbb{R}^N)} \le c \int_{\mathbb{R}^{N+1}} (1+|\xi|^2)^{l+\frac{1}{2}} |\widetilde{\rho}(\xi,s)|^2 + (1+|\xi|^2)^{l-\frac{1}{2}} |s\widetilde{\rho}(\xi,s)|^2 d\xi ds,$$

and thus, being $\widetilde{\rho_{,t}}(s) = is\widetilde{\rho}(s)$, the claim. □

Again one can state analogous theorems for arbitrary smooth domains Ω, or for bounded smooth submanifold with or without boundary in $\mathbb{R}^N$. For future reference we state the following proposition.

Theorem 2.2.3 *Let $\mathcal{G}$ be a smooth bounded submanifold of $\mathbb{R}^N$, and $T > 0$. For any $l > \frac{1}{2}$ it holds the estimate*

$$\sup_{0 \le t < T} \|\rho(\cdot,t)\|_{W_2^l(\mathcal{G})} \le c(T) (\|\rho\|_{W_2^{l+\frac{1}{2},0}(\mathcal{G}\times[0,T))} + \|\rho_{,t}\|_{W_2^{l-\frac{1}{2}}(\mathcal{G}\times[0,T))}),$$

for a constant $c(T)$ independent of $\rho : \mathcal{G} \times [0,T) \to \mathbb{R}$, with $C(+\infty) < +\infty$.

It is worth noting that the constant $c(T)$ blows up as $T \to 0$, as well as the continuity constant of the restriction operator $R_{,t}$ in theorem 2.2.1 (which also depend on T). This is one of the difficulties one has to deal with in using these theorems to study existence of nonlinear equations for small time.

We will also make use of weighted anisotropic Sobolev–Slobodetskii spaces: given a function (or vector field) f defined in Q_∞ we denote by f^* its extension to zero for $t < 0$, and set

$$W_{2,\gamma}^{l,\frac{l}{2}}(Q_\infty) := \{f : e^{-\gamma t} f^* \in W_2^{l,\frac{l}{2}}(\Omega \times \mathbb{R})\}, \tag{2.23}$$

normed with

$$\|f\|_{W_{2,\gamma}^{l,\frac{l}{2}}(Q_\infty)} = \|e^{-\gamma t} f^*\|_{W_2^{l,\frac{l}{2}}(\Omega\times\mathbb{R})},$$

and similarly for functions defined on $G_\infty = \mathcal{G} \times [0,+\infty)$ where $\mathcal{G}$ is a regular submanifold of $\mathbb{R}^3$. Given $f \in W_2^{l,\frac{l}{2}}(Q_\infty)$, in the case $\{\frac{l}{2}\} \ne \frac{1}{2}$ there is a simple criterion to check if $f \in W_{2,\gamma}^{l,\frac{l}{2}}(Q_\infty)$.

Theorem 2.2.4 *Suppose that l is not an integer. A function $f \in W_2^{l,\frac{l}{2}}(Q_\infty)$ belongs to $W_{2,\gamma}^{l,\frac{l}{2}}(Q_\infty)$ if and only if*

1. *$e^{-\gamma t} f \in W_2^{l,\frac{l}{2}}(Q_\infty)$.*
2. *$\frac{\partial^k f}{\partial t^k}\lfloor_{t=0} = 0$ for almost all $x \in \Omega$ and every integer k with $0 \le k < \frac{l}{2} - \frac{1}{2}$.*

If both conditions are satisfied, then

$$c_1 \|f\|_{W_{2,\gamma}^{l,\frac{l}{2}}(Q_\infty)} \le \|e^{-\gamma t} f\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \le \|f\|_{W_{2,\gamma}^{l,\frac{l}{2}}(Q_\infty)}.$$

for c_1 independent of f.

2.3 Sobolev–Slobodetskii spaces for small T

Neglecting the role of the exponential weight, one can seek for analogous results in $W_2^{l,\frac{l}{2}}(Q_T)$ of theorem 2.2.4. Finding the exact scaling with respect to T is particularly important when studying existence for small time of solutions to nonlinear parabolic equations. The following theorem suggests that in some cases a modification of the standard norm of $W_2^{l,\frac{l}{2}}$ is needed.

Theorem 2.3.1 *Suppose that l is not an integer, $f \in W_2^{l,\frac{l}{2}}(Q_T)$, and let $Q_{-\infty,T} := \Omega \times (-\infty, T)$, $T \leq +\infty$. Then, $f^* \in W_2^{l,\frac{l}{2}}(Q_{-\infty,T})$ if and only if $\frac{\partial^k f}{\partial t^k}\lfloor_{t=0}= 0$ for almost all $x \in \Omega$ and every integer k with $0 \leq k < \frac{l}{2} - \frac{1}{2}$. Moreover, there are constants c_1 and c_2 which do not depend on f or T such that if $1 > \{\frac{l}{2}\} > \frac{1}{2}$, then*

$$c_1\|f^*\|_{W_2^{l,\frac{l}{2}}(Q_{-\infty,T})} \leq \|f\|_{W_2^{l,\frac{l}{2}}(Q_T)} \leq c_2\|f^*\|_{W_2^{l,\frac{l}{2}}(Q_{-\infty,T})}, \tag{2.24}$$

while in the case $\{\frac{l}{2}\} < \frac{1}{2}$, it holds

$$\begin{aligned} c_1\|f^*\|_{W_2^{l,\frac{l}{2}}(Q_{-\infty,T})} &\leq \left[\|f\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} + \frac{1}{T^{2\{\frac{l}{2}\}}}\int_0^T \|D_t^{[\frac{l}{2}]}f(x,\cdot)\|^2_{L^2(\Omega)}dt\right]^{\frac{1}{2}} \\ &\leq c_2\|f^*\|_{W_2^{l,\frac{l}{2}}(Q_{-\infty,T})}. \end{aligned} \tag{2.25}$$

Proof. We restrict the proof to the case $\frac{l}{2} \in (0,1) \setminus \{\frac{1}{2}\}$, since the general case easily follows. Moreover it is clearly sufficient to prove the statement in the spaces $W_2^{\frac{l}{2}}(0,T)$, since the mixed norm analogous can be obtained integrating the corresponding inequalities over Ω. An easy calculation shows that

$$\|f^*\|^2_{W_2^{\frac{l}{2}}((-\infty,T])} = 2\|f\|^2_{W_2^{\frac{l}{2}}([0,T])} + 2\int_0^T \frac{|f(t)|^2}{t^l}dt,$$

which is finite iff the second integral on the right is finite. This implies $f(0) = 0$ in the case $l > 1$, and we will show the opposite implication, by proving the inequalities (2.24) for $l > 1$ and $f(0) = 0$, and (2.25) for $l < 1$. From the formula above, it suffice to prove only the first inequality in (2.24), (2.25). We will be done as soon as we prove the following inequalities for any function f:

$$\int_0^T \frac{|f(t) - f(0)|^2}{t^l}dt \leq c\int_0^T\int_0^T \frac{|f(t) - f(s)|^2}{|t-s|^{1+l}}dtds, \tag{2.26}$$

valid for $l > 1$, and

$$\int_0^T \frac{|f(t)|^2}{t^l} dt \le c \int_0^T \int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dt ds + \frac{1}{T^l} \int_0^T |f(t)|^2 dt, \qquad (2.27)$$

valid for $0 < l < 1$ To prove (2.26) in the case $l > 1$, write

$$f(t) - f(0) = ⨍_0^t \big(f(t) - f(s)\big) ds + ⨍_0^t \big(f(s) - f(0)\big) ds.$$

Multiplying by $|t-s|^{\frac{1+l}{2}}$ inside the first integral and applying Hölder inequality with exponents 2 on both, we obtain

$$\begin{aligned} |f(t)-f(0)|^2 &\le ⨍_0^t \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} ds ⨍_0^t |t-s|^{1+l} ds + ⨍_0^t |f(s)-f(0)|^2 ds \\ &\le \frac{t^l}{2+l} \int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} ds + \frac{1}{t} \int_0^t |f(s)-f(0)|^2 ds. \end{aligned}$$

Multiplying by t^{-l} and integrating in t gives

$$\begin{aligned} &\int_0^T \frac{|f(t)-f(0)|^2}{t^l} dt \\ &\quad \le \frac{1}{2+l} \int_0^T \int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dt ds + \int_0^T |f(s)-f(0)|^2 ds \int_s^T \frac{dt}{t^{1+l}} \\ &\quad \le \frac{1}{2+l} \int_0^T \int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dt ds + \frac{1}{l} \int_0^T |f(s)-f(0)|^2 \frac{ds}{s^l} \end{aligned}$$

and thus, if $l > 1$, we can bring the last term to the left, obtaining

$$\int_0^T \frac{|f(t)-f(0)|^2}{t^l} dt \le \frac{l}{(2+l)(l-1)} \int_0^T \int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dt ds.$$

Let us consider (2.27). We first prove it in the case $T = +\infty$, i.e. without the additional term. To this end we proceed as above, but this time we use

$$f(t) = ⨍_0^{at} \big(f(t) - f(s)\big) ds + ⨍_0^{at} f(s) ds,$$

for a positive parameter a to be determined later on. This gives, as before

$$|f(t)|^2 \le c_a t^l \int_0^{+\infty} \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} ds + \frac{1}{at} \int_0^{at} |f(s)|^2 ds,$$

and

$$\begin{aligned}\int_0^{+\infty} \frac{|f(t)|^2}{t^l} dt \\ &\leq c_a \int_0^{+\infty} \int_0^{+\infty} \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dtds + \frac{1}{a} \int_0^{+\infty} |f(s)|^2 ds \int_{\frac{s}{a}}^{+\infty} \frac{dt}{t^{1+l}} \\ &\leq c_a \int_0^{+\infty} \int_0^{+\infty} \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dtds + \frac{1}{la^{1-l}} \int_0^{+\infty} \frac{|f(s)|^2}{s^l} ds.\end{aligned}$$

Since $l < 1$, we can choose a large enough so that $la^{1-l} > 1$, and thus bring the last term to the left, obtaining, for $T = +\infty$

$$\int_0^{+\infty} \frac{|f(t)|^2}{t^l} dt \leq c \int_0^{+\infty} \int_0^{+\infty} \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dtds. \tag{2.28}$$

Inequality (2.27) for $T < +\infty$ is obtained through theorem 2.1.2, applying the inequality for $T = +\infty$ to the extension $C(f)$ of f. The estimates (2.6) and (2.28) then give (2.27). □

Inequalities such as (2.26) and (2.27) are commonly referred to as "fractional Hardy inequalities", from the classical Hardy inequality

$$\int_0^T \frac{|f(t)|^2}{t^2} dt \leq 4 \int_0^T |f'(t)|^2 dt,$$

valid for smooth f's vanishing at $t = 0$.

Some remarks will be useful

Remark 2.3.2

1. The additional term

$$\frac{1}{T^{2\{\frac{l}{2}\}}} \int_0^T \|D_t^{[\frac{l}{2}]} f(x,\cdot)\|^2_{L^2(\Omega)} dt$$

in (2.25) *is a necessary one in the case* $\{\frac{l}{2}\} < \frac{1}{2}$ *and* T *small. Indeed, for example in the* $l < 1$ *case, one cannot hope in a bound of the form*

$$\int_0^T \frac{|f(t)|^2}{t^l} dt \leq c \int_0^T \int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+l}} dtds + \int_0^T |f(t)|^2 dt,$$

with c *independent of* T*, since* $f \equiv 1$ *would give*

$$\frac{T^{l+1}}{l+1} \leq cT,$$

which fails for $T \to 0$.

2. *This feature is caused by the non completely local nature of the Sobolev–Slobodetskii norm. Consider for example the equality*
$$\|f\|_{W_2^k(\Omega)} = \|f\|_{W_2^k(\Omega')},$$
valid for all the functions f having support contained in $\Omega' \subset \Omega$. This is clearly true for k integer, since we have a completely local way of defining the norm. However for fractional l with $\{l\} < \frac{1}{2}$ and supp $f \subset \Omega' \subset \Omega$ *it holds only*
$$\|f\|_{W_2^l(\Omega')} \leq \|f\|_{W_2^l(\Omega)} \leq c(\Omega)\|f\|_{W_2^l(\Omega')},$$
with a constant $c(\Omega)$ that, generally speaking, can blow up when Ω shrinks to zero in some direction. This feature must be taken into account, e.g., when localising functions via partitions of unity.

3. *Another drawback of this behaviour is that when looking at anisotropic Sobolev–Slobodetskii space for small times, it is useful to use a modified norm, which scales well in the case l is not an integer, namely*
$$\|f\|^2_{\widehat{W}_2^{l,\frac{l}{2}}(Q_T)} = \begin{cases} \|f\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} & \text{if } \{\frac{l}{2}\} > \frac{1}{2}, \\ \|f\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} + \dfrac{1}{T^{2\{\frac{l}{2}\}}} \displaystyle\int_0^T \|D_t^{[\frac{l}{2}]} f(x,\cdot)\|^2_{L^2(\Omega)} dt & \text{if } \{\frac{l}{2}\} < \frac{1}{2}. \end{cases}$$

The space $H_2^{l,\frac{l}{2}}(Q_T)$, l not an integer, is defined as the set $W_2^{l,\frac{l}{2}}(Q_T)$ equipped with the norm
$$\|f\|^2_{H_2^{l,\frac{l}{2}}(Q_T)} = \|f\|^2_{\widehat{W}_2^{l,\frac{l}{2}}(Q_T)} + \sum_{0 \leq 2k < l-1} \sup_{0 \leq t \leq T} \|D_t^k f(\cdot,t)\|^2_{W_2^{l-1-2k}(\Omega)}.$$
For fixed T, this norm is clearly equivalent to the standard one, by (2.22) and theorem 2.2.1. It will be useful to define the auxiliary norms
$$\|f\|^2_{H_2^{l,0}(Q_T)} := \|f\|^2_{W_2^{l,0}(Q_T)} + \sum_{0 \leq 2k < l-1} \sup_{0 \leq t \leq T} \|D_t^k f(\cdot,t)\|^2_{W_2^{l-1-2k}(\Omega)},$$
$$\|f\|^2_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)} = \begin{cases} \|f\|^2_{W_2^{0,\frac{l}{2}}(Q_T)} & \text{if } \{\frac{l}{2}\} > \frac{1}{2}, \\ \|f\|^2_{W_2^{0,\frac{l}{2}}(Q_T)} + \frac{1}{T^{2\{\frac{l}{2}\}}}\|D_t^{[\frac{l}{2}]} f\|^2_{L^2(Q_T)} & \text{if } \{\frac{l}{2}\} < \frac{1}{2}, \end{cases}$$
so that
$$\|f\|^2_{H_2^{l,\frac{l}{2}}(Q_T)} = \|f\|^2_{H_2^{l,0}(Q_T)} + \|f\|^2_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)}.$$
The main feature of this modified norm is that, while the continuity constants in theorem 2.2.1 depend on T, analogous theorems holds with respect to the norm $H_2^{l,\frac{l}{2}}$ with constant independent of T.

Theorem 2.3.3 *Let $T \leq 1$, l not an integer and Ω a bounded smooth domain or submanifold of $\mathbb{R}^N$.*

1. *For $l > 1$, the restriction operator satisfies*

$$\|f(\cdot,0)\|_{W_2^{l-2j-1}(\Omega)} \leq c\|f\|_{H_2^{l,\frac{l}{2}}(Q_T)}, \qquad 0 \leq 2j < l-1.$$

with constant independent of T.

2. *For any $\phi_j \in W_2^{l-2j-1}(\Omega)$, there exists $f \in H_2^{l,\frac{l}{2}}(Q_T)$ such that $D^k_{,t} f\lfloor_{t=0} = \phi_j$ for all j such that $0 \leq 2j < l-1$ and*

$$\|f\|_{H_2^{l,\frac{l}{2}}(Q_T)} \leq c \sum_{0 \leq 2j < l-1} \|\phi_j\|_{W_2^{l-2j-1}(\Omega)},$$

with constant independent of T.

3. *There exists a continuation operator $C : H_2^{l,\frac{l}{2}}(Q_T) \to W_2^{l,\frac{l}{2}}(Q_\infty)$ such that*

$$c_1\|C(f)\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \leq \|f\|_{H_2^{l,\frac{l}{2}}(Q_T)} \leq c_2\|C(f)\|_{W_2^{l,\frac{l}{2}}(Q_\infty)},$$

with constants independent of T.

Proof. The first claim is obvious from the definition of $H_2^{l,\frac{l}{2}}$. We now prove the following claim: if $f \in W_2^{l,\frac{l}{2}}(Q_\infty)$, then

$$\|f\|_{H_2^{l,\frac{l}{2}}(Q_T)} \leq c\|f\|_{W_2^{l,\frac{l}{2}}(Q_\infty)}, \tag{2.29}$$

with constant independent of T. The inequality

$$\sum_{0 \leq 2k < l-1} \sup_{0 \leq t \leq T} \|D_t^k f(\cdot,t)\|_{W_2^{l-1-2k}(\Omega)} \leq c\|f\|_{W_2^{l,\frac{l}{2}}(Q_\infty)},$$

follows from standard restriction estimates for unbounded intervals. Since $\|f\|_{W_2^{l,\frac{l}{2}}(Q_T)} \leq \|f\|_{W_2^{l,\frac{l}{2}}(Q_\infty)}$, it suffice to consider the case $\{\frac{l}{2}\} < \frac{1}{2}$. Inequality (2.28) gives in this case, for $T \leq 1$,

$$\begin{aligned}\frac{1}{T^{2\{\frac{l}{2}\}}} \int_0^T \|D_t^{[\frac{l}{2}]} f\|^2_{L^2(\Omega)} &\leq \int_\Omega \int_0^{+\infty} \frac{|D_t^{[\frac{l}{2}]} f|^2}{t^{2\{\frac{l}{2}\}}} dt dx \\ &\leq c \int_\Omega \|f\|^2_{\mathring{W}_2^{\frac{l}{2}}(0,+\infty)} dx \leq c\|f\|^2_{W_2^{l,\frac{l}{2}}(Q_\infty)},\end{aligned}$$

with constant independent of T, which concludes the proof of (2.29). Now point 2 follows from standard extension theorems for unbounded intervals, and it remains to prove point 3. To this end, let

$$\phi_j(x) = D^j_{,t} f(x,t)\lfloor_{t=T} \in W_2^{l-2j-1}(\Omega), \qquad 0 \le 2j < l-1.$$

By theorem 2.2.1, there exists $F \in W_2^{l,\frac{l}{2}}(Q_{T,\infty})$ (where $Q_{T,\infty} := \Omega \times [T,+\infty)$) such that $D^j_{,t} F\lfloor_{t=T} = \phi_j$ and

$$\|F\|_{W_2^{l,\frac{l}{2}}(Q_{T,\infty})} \le c \sum_{0\le 2j<l-1} \|\phi_j\|_{W_2^{l-2j-1}(\Omega)} \le c\|f\|_{H_2^{l,\frac{l}{2}}(Q_T)},$$

with constant independent of T. We claim that the function $C(f) = f$ for $t \le T$ and $C(f) = F$ for $t > T$ satisfies

$$\|C(f)\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \le c\|f\|_{H_2^{l,\frac{l}{2}}(Q_T)}. \tag{2.30}$$

This, together with (2.29), will prove point 3. To prove (2.30), notice that clearly

$$\|C(f)\|^2_{W_2^{l,0}(Q_\infty)} + \sum_{0\le j\le[\frac{l}{2}]} \int_\Omega \|C(f)\|^2_{W_2^j(\mathbb{R}_+)} \le c(\|f\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} + \|F\|^2_{W_2^{l,\frac{l}{2}}(Q_{T,\infty})})$$
$$\le c\|f\|^2_{H_2^{l,\frac{l}{2}}(Q_T)},$$

by the local nature of the norms involved and $D^j_{,t}F\lfloor_{t=T} = D^j f\lfloor_{t=T}$ for all $0 \le 2j \le l-1$. Letting $k = \left[\frac{l}{2}\right]$, it remains to estimate

$$\int_\Omega \int_0^{+\infty} \int_0^{+\infty} \frac{|D^k_t C(f)(x,t) - D^k_t C(f)(x,s)|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} dtdsdx.$$

We can omit the dependance on $x \in \Omega$ and integrate at the end. It holds

$$\int_0^{+\infty}\int_0^{+\infty} \frac{|D^k_t C(f)(t) - D^k_t C(f)(s)|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} dtds = \int_0^T\int_0^T \frac{|D^k_t f(t) - D^k_t f(s)|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} dtds$$
$$+ \int_T^{+\infty}\int_T^{+\infty} \frac{|D^k_t F(t) - D^k_t F(s)|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} dtds + 2\int_T^{+\infty}\int_0^T \frac{|D^k_t f(t) - D^k_t F(s)|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} dsdt$$

and we just have to estimate the last integral, since the first two are bounded by $\|f\|^2_{H_2^{l,\frac{l}{2}}(Q_T)}$. In the case $\{\frac{l}{2}\} > \frac{1}{2}$ we add and subtract ϕ_k and bound it with

$$\int_T^{+\infty} ds \int_0^T \frac{|D_t^k f(t) - \phi_k|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} dt + \int_0^T dt \int_T^{+\infty} \frac{|D_t^k F(s) - \phi_k|^2}{|t-s|^{1+2\{\frac{l}{2}\}}} ds$$
$$\leq c \int_0^T \frac{|D_t^k f(t) - \phi_k|^2}{|t-T|^{2\{\frac{l}{2}\}}} dt + \int_T^{+\infty} \frac{|D_t^k F(s) - \phi_k|^2}{|T-s|^{2\{\frac{l}{2}\}}} ds$$

and applying (2.26), (2.28) we conclude. In the $\{\frac{l}{2}\} > \frac{1}{2}$ case we proceed in a similar way, splitting $|D_t^k f(t) - D_t^k F(s)|$ as $|D_t^k f(t)| + |D_t^k F(s)|$ and employing (2.27) instead. Integrating these inequalities in $x \in \Omega$ gives (2.30), and concludes the proof.

□

Point 3 of the previous theorem is a useful instrument to obtain restriction and interpolation inequalities in the $H_2^{l,\frac{l}{2}}$ setting with constants independent of T when $T \ll 1$. We omit a complete discussion, and will prove the relevant inequalities when needed.

2.4 The Laplace transform and applications

Given a function $f : [0, +\infty) \to \mathbb{C}$, its Laplace transform is defined as

$$Lf(z) = \int_0^{+\infty} e^{-zt} f(t) dt.$$

It is easy to see that if for some $\gamma \geq 0$, $e^{-\gamma t} f$ is integrable, then $Lf(z)$ is well defined an holomorphic in the semiplane $\operatorname{Re} z > \gamma$. Moreover the Laplace transform is intimately connected with the Fourier transform, since

$$Lf(\sigma + i\tau) = \int_{\mathbb{R}} e^{-i\tau t}(e^{-\sigma t} f(t)) dt = \mathcal{F}(e^{-\sigma t} f(t))(\tau),$$

where $\mathcal{F}$ denotes the Fourier transform.

Given $\gamma \geq 0$, the space $E_\gamma^{l,\frac{l}{2}}$ is the set of functions $u : \Omega \times \{\operatorname{Re}\lambda \geq \gamma\} \to \mathbb{C}$, such that

1. For all z with $\operatorname{Re} z > \gamma$ and almost all z with $\operatorname{Re} z = \gamma$, $u(\cdot, z) \in W_2^l(\Omega)$.
2. For almost all $x \in \Omega$, $u(x, \cdot)$ is holomorphic in $\{\operatorname{Re} z > \gamma\}$ and

$$\|u(x,\cdot)\|^2_{E_\gamma^{\frac{l}{2}}} := \sup_{\sigma > \gamma} \int_{\sigma - i\infty}^{\sigma + i\infty} |u(x,z)|^2 |z|^l dz < +\infty,$$

and $\|D_x^k u(x,\cdot)\|_{E_\gamma^0} < +\infty$ for all integer k such that $0 \leq k \leq [l]$.

As a norm, we take

$$\|u\|^2_{E^{l,\frac{l}{2}}_\gamma} = \sup_{\sigma>\gamma} \int_{\sigma-i\infty}^{\sigma+i\infty} \|u(\cdot,z)\|^2_{W^l_2(\Omega)} + |z|^l \|u(\cdot,z)\|^2_{L^2(\Omega)} dz,$$

which makes $E^{l,\frac{l}{2}}_\gamma$ a Banach space. Here the integral in the norm is to be meant as the limit for $N \to +\infty$ of the integral between $-N$ and N.

The main result of this section states that the Laplace transform is a bicontinuous mapping between $W^{l,\frac{l}{2}}_{2,\gamma}(Q_\infty)$ and $E^{l,\frac{l}{2}}_\gamma$.

Theorem 2.4.1 *Let $\gamma \geq 0$. The Laplace transform with respect to time is a bicontinuous mapping between $W^{l,\frac{l}{2}}_{2,\gamma}(Q_\infty)$ and $E^{l,\frac{l}{2}}_\gamma$. It has the following properties:*

1. *For any integer $k = k_1 + \dots + k_N$ and almost any $x \in \Omega$,*

$$L\left(\frac{\partial^k f}{\partial x_1^{k_1} \dots \partial x_N^{k_N}}\right)(x,z) = \frac{\partial^k L(f)}{\partial x_1^{k_1} \dots \partial x_N^{k_N}}(x,z),$$

$$L\left(\frac{\partial^k f}{\partial t^k}\right)(x,z) = z^k L(f)(x,z).$$

2. *If $f_T(x,t) := f(x,t+T)$, then*

$$Lf_T(x,z) = e^{Tz} Lf_T(x,z).$$

3. *For almost any $x \in \Omega$ and $\sigma \geq \gamma$*

$$\|e^{-\sigma t} f(x,t)\|_{L^2(\mathbb{R}_+)} = \frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} |L(f)(x,z)|^2 dz.$$

4. *The inverse of the Laplace transform on $E^{l,\frac{l}{2}}_\gamma$ is*

$$M(u)(x,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} u(x,z) e^{zt} dz,$$

for $\sigma > \gamma$.

We will now briefly describe how the Laplace transform fits into the general strategy to solve linear parabolic problems settled down by Agranovich and Vishik in [2].

Consider the evolution system

$$\frac{\partial \boldsymbol{U}}{\partial t} - \mathcal{A}\boldsymbol{U} = \boldsymbol{F},$$

where $\boldsymbol{U}$ and $\boldsymbol{F}$ are suitable vectors of functions, and $\mathcal{A}$ is differential (w.r.t. spatial variables) operator with at most second order derivatives. This system is usually coupled with a series of boundary conditions, which we will write in the form $\mathcal{B}\boldsymbol{U} = \boldsymbol{G}$ for some differential operator $\mathcal{B}$, and we suppose homogeneous initial data $\boldsymbol{U}(0) = 0$. One seek for solution of this system such that each component $\boldsymbol{U}^i$ of $\boldsymbol{U}$ belongs to the proper anisotropic Sobolev–Slobodetskii space $W_2^{l_i,\frac{l_i}{2}}$. Of course this is not always reasonable due to the structure of $\mathcal{A}$, and even if formally this seems to be possible, a number of compatibility conditions (depending, in general, also on the regularity required) the at the initial time must be imposed. We won't go into the details of the various well posedness criteria for parabolic systems, and refer to [2] for a precise characterisation of the parabolicity of the system

$$\begin{cases} \frac{\partial \boldsymbol{U}}{\partial t} - \mathcal{A}\boldsymbol{U} = \boldsymbol{F} & t > 0, \\ \mathcal{B}\boldsymbol{U} = \boldsymbol{G} & t > 0, \\ \boldsymbol{U}(0) = 0 \end{cases} \tag{2.31}$$

Since $\boldsymbol{U}(0) = 0$, we can think of a solution as having components in suitable $W_{2,\gamma}^{l_i,\frac{l_i}{2}}$ spaces, i.e. considering it as extended as 0 for $t < 0$ preserving regularity and with proper growth conditions at $+\infty$. Applying the Laplace transform to (2.31) gives the complex parameter dependent problem

$$\begin{cases} z\boldsymbol{u} - \mathcal{A}\boldsymbol{u} = \boldsymbol{f}, \\ \mathcal{B}\boldsymbol{u} = \boldsymbol{g} \end{cases}$$

where

$$\boldsymbol{u} = L\boldsymbol{U}, \qquad \boldsymbol{f} = L\boldsymbol{F}, \qquad \boldsymbol{g} = L\boldsymbol{G}.$$

As soon as one can prove that for sufficiently large γ, this complex parameter dependent problem has a unique solution in the corresponding product of $E_\gamma^{l,\frac{l}{2}}$ spaces (depending on the component), one can invert the Laplace transform to obtain a solution of (2.31). Consider the z dependent norms

$$\|\boldsymbol{u}\|_\lambda = \sum_i \|\boldsymbol{u}^i\|_{W_2^{l_i,\frac{l_i}{2}}} + |z|^{l_i}\|\boldsymbol{u}^i\|_{L^2},$$

and similarly for $\boldsymbol{f}$ and $\boldsymbol{g}$. To obtain meaningful parabolic estimates one usually seeks for λ *independent* two-sided inequalities of the form

$$c_1\|\boldsymbol{u}\|_z \leq \|\boldsymbol{f}\|_z + \|\boldsymbol{g}\|_z \leq c_2\|\boldsymbol{u}\|_z.$$

holding for any $\operatorname{Re}\lambda \geq \gamma$ since the latters, integrated over $\operatorname{Re} z = \gamma$, translate by theorem 2.4.1 into a two-sided estimate in terms of the $W_{2,\gamma}^{l_i,\frac{l_i}{2}}$ norms of $\boldsymbol{U}$, $\boldsymbol{F}$ and $\boldsymbol{G}$.

Chapter 3

The linearized problem

In this chapter we introduce a change of variables in order to reduce problem (1.2) to a problem in a fixed domain. There are mainly two ways to do this that has been proved fruitful in the literature. One is employing a Lagrangian description, given by using coordinates

$$x = \xi + \int_0^t \boldsymbol{u}(\xi, s)ds =: X(\xi, t), \qquad \xi \in \Omega_0,$$

where $\boldsymbol{u}$ is the velocity field in Lagrangian coordinates, linked to the velocity field $\boldsymbol{v}$ in Eulerian coordinates by the relation

$$\boldsymbol{u}(\xi, t) = \boldsymbol{v}(X(\xi, t), t).$$

Thus, for example, the free surface is defined as

$$\Gamma_{,t} := \{\xi + \int_0^t \boldsymbol{u}(\xi, s)ds : \xi \in \Gamma_0\}.$$

Using this coordinate system makes it difficult to recover the regularising effect that the equations can have on the free surface. As we will see, in the case $\sigma > 0$ one expects a regularising effect on $\Gamma_{,t}$ due to the presence of an elliptic operator arising from the curvature term. Thus it seems that the Lagrangian description fits more to the case where an *a-priori* regularising effect is not apparent, i.e. in the case $\sigma = 0$. This approach has been successfully applied by Solonnikov, etc etc in ...

Another approach is to use the so called Hanzawa transformation, which goes back to [11]. This consists in fixing an arbitrary smooth domain Ω_b, sufficiently near (or, eventually, coincident) to Ω_0 in such a way that for $0 \le t \le T$, all the $\Omega_{,t}$ can be considered normal perturbations of Ω_b. Thus the new variable y is defined by

$$x = y + \boldsymbol{N}\varphi(y, t), \qquad y \in \Omega_b,$$

for a suitable choice of $\varphi(y,t)$, where $\boldsymbol{N}$ is the exterior normal to Ω_b. This transformation has obviously a wide variety of choices for φ, and it allows a more precise study of the regularity of the free surface. It has been applied to free-boundary problems for the Navier–Stokes equation by many authors, see ...

Since we work with capillary fluids we will employ the latter transformation, and the particular choice we make will be described in the first section. We then proceed in section 2 in studying the linearization of problem (1.2) (with respect to this choice of variables) near a stationary solution. Explicit calculations of the various linear and nonlinear terms will also be done.

3.1 The Hanzawa transformation

We recall that the domain where the fluid is constrained is denoted by $\Omega_t = \{x' \in \Sigma, 0 \le x_3 \le \phi(x',t)\}$, with Σ being a rectangle in $\mathbb{R}^2$, $x' = (x_1, x_2)$ and ϕ is a sufficiently regular function whose graph in $\mathbb{R}^3$ is the free boundary of the fluid, $\Gamma_{,t}$, with normal $\boldsymbol{n}$. We suppose that the velocity field $\boldsymbol{v}$ and the pressure p are periodic for every $t \ge 0$, with periodic cell Σ. Furthermore we prescribe the velocity at the bottom of the layer, letting $\boldsymbol{v}(x',0,t) = \boldsymbol{\alpha}(x',t)$ for every $t \ge 0$. With these notations we obtain system (1.2).

We now fix a sufficiently smooth domain Ω_b, defined, for some Σ-periodic smooth ϕ_b, as $\{(x',x_3) : x' \in \Sigma,\ 0 < x_3 < \phi_b(x)\}$. Moreover $\mathcal{G}$ will be the graph of ϕ_b over Σ, $\boldsymbol{N} = (-\nabla'\phi_b, 1)/\sqrt{1+|\nabla'\phi_b|^2}$ its normal, $\Pi_b(\boldsymbol{V}) = \boldsymbol{V} - (\boldsymbol{N}\cdot\boldsymbol{V})\boldsymbol{N}$ the projection of $\boldsymbol{V}$ on the tangent space of $\mathcal{G}$, and $\Pi(\boldsymbol{V}) = \boldsymbol{V} - (\boldsymbol{n}\cdot\boldsymbol{V})\boldsymbol{n}$ the projection on the tangent space to the graph of ϕ. Letting $\rho = \phi - \phi_b$, we rewrite problem (1.2) in terms of the new variable $y \in \Omega_b$ defined as

$$x = e_\rho(y) = y + \theta(y)\rho(y',t)\boldsymbol{e}_3, \tag{3.1}$$

where $y' = (y_1, y_2) \in \Sigma$ and θ is a C^∞ cutoff function with suitable regularity. Although inessential, we will assume, to simplify some calculations, that $\theta = \theta(y_3)$ and $\theta(s) = 0$ for $s < h$ and $\theta = 1$ for $s > 2h$, with

$$\inf_{\Sigma} \phi_b > 3h > 0.$$

We suppose that, for some $l \ge \frac{1}{2}$,

$$\sup_{t<T} \|\rho(\cdot,t)\|_{W_2^{l+\frac{3}{2}}(\Sigma)} \ll 1,$$

so that the transformation (3.1) is at least $C^{1,\alpha}$ and is it invertible due to the smallness of $\sup_\Sigma |\rho|$. Moreover, we will henceforth write $\rho^*(y,t) = \theta(y)\rho(y',t)$.

This change of variable transforms Ω_b to Ω_t, and we will denote by $\mathcal{L} = \mathcal{L}(y, \rho)$ the Jacoby matrix of this transformation:

$$\mathcal{L}(y) = \left(\frac{\partial x_i}{\partial y_j}\right)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \theta\rho_{,y_1} & \theta\rho_{,y_2} & 1+\theta'\rho \end{pmatrix}. \tag{3.2}$$

Furthermore, we will set $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ so that $\widehat{\mathcal{L}} = \mathrm{cof}(\mathcal{L})^T$. One has

$$L = 1 + \theta'\rho, \quad \mathcal{L}^{-T} = \begin{pmatrix} 1 & 0 & \frac{-\theta\rho_{,y_1}}{1+\theta'\rho} \\ 0 & 1 & \frac{-\theta\rho_{,y_2}}{1+\theta'\rho} \\ 0 & 0 & \frac{1}{1+\theta'\rho} \end{pmatrix}, \qquad I - \mathcal{L}^{-T} = \frac{1}{L}\nabla\rho^* \otimes \boldsymbol{e}_3 \tag{3.3}$$

Moreover, the transformation e_ρ converts the operator ∇_x to $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_y$, since $\mathcal{L}^{-T}_{ij} = \frac{\partial y_j}{\partial x_i}$ and by the chain rule $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_j}\frac{\partial y_j}{\partial x_i}$. To shorten the notation we will henceforth write ∇ for ∇_y. We now rewrite system (1.2) in the new variables (y, t). For the term $\boldsymbol{v}_{,t}$ we have

$$\boldsymbol{v}_{,t}(x(y), t) = \nabla_x \boldsymbol{v}\frac{\partial y}{\partial t} + \boldsymbol{v}_{,t} = \rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3 \cdot \nabla)\boldsymbol{v} + \boldsymbol{v}_{,t}.$$

The nonlinear term $(\boldsymbol{v} \cdot \nabla_x)\boldsymbol{v}$ con be rewritten as $(\mathcal{L}^{-1}\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and all the other differential operators are substituted with the rule $\nabla_x \to \widetilde{\nabla}$. Regarding the divergence it is important to notice that

$$\widetilde{\nabla} \cdot \boldsymbol{v} = \frac{1}{L}(\widehat{\mathcal{L}}^T\nabla) \cdot \boldsymbol{v} = 0 \quad \Leftrightarrow \quad (\widehat{\mathcal{L}}^T\nabla) \cdot \boldsymbol{v} = 0,$$

and

$$(\widehat{\mathcal{L}}^T\nabla) \cdot \boldsymbol{v} = \nabla \cdot (\widehat{\mathcal{L}}\boldsymbol{v}),$$

since, as is well known, the cofactor matrix has divergence free rows. The equation for the divergence can thus be written as

$$\nabla \cdot \boldsymbol{v} = \big((I - \widehat{\mathcal{L}}^T)\nabla\big) \cdot \boldsymbol{v} = \nabla \cdot (I - \widehat{\mathcal{L}})\boldsymbol{v}.$$

We now consider the curvature term. Recall that given a surface with normal $\boldsymbol{n}$, smoothly extended in a neighbourhood of the surface, the doubled mean curvature H is defined as $\nabla \cdot \boldsymbol{n}$. We then define $H_s = \nabla_x \cdot \boldsymbol{n}_s$ where $\boldsymbol{n}_s$ is the (upward) normal to the surface with cartesian equation $y_3 = \phi_b(y') + s\rho(y')$. Letting $g_s = 1 + |\nabla(\phi_b + s\rho)|^2$ and $g_b = g_0$ we have

$$H_1 = H_0 + \frac{d}{ds}H_s\lfloor_{s=0} + \int_0^1 (1-s)\frac{d^2}{ds^2}H_s ds.$$

We will use summation convention on repeated indexes, and for any multiindex $\boldsymbol{k} = (k_1, k_2)$, $k_1 + k_2 = k$, and any function $f = f(x_1, x_2)$, we will set

$$f_{,\boldsymbol{k}} = \frac{\partial^k f}{\partial x_1^{k_1} \partial x_2^{k_2}}. \tag{3.4}$$

Since

$$\boldsymbol{n}_s = \frac{(-\nabla'(\phi_b + s\rho), 1)}{\sqrt{g_s}},$$

we have, with summation on the indexes α, $\beta = 1, 2$

$$\begin{aligned} \frac{d}{ds} H_s \lfloor_{s=0} &= \nabla \cdot \frac{1}{\sqrt{g_b}} \big((-\nabla'\rho, 0) - \frac{\nabla'\phi_b \cdot \nabla'\rho}{g_b} (-\nabla'\phi_b, 1) \big) \\ &= -\frac{1}{g_b} \partial_\alpha \big(\delta_{\alpha\beta} \sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \big) \rho_{,\beta} \\ &\quad - \frac{1}{g_b^{\frac{5}{2}}} \nabla'\phi_b \cdot \nabla'|\nabla'\phi_b|^2 \nabla\phi_b \cdot \nabla'\rho + \frac{\nabla'|\nabla'\phi_b|^2 \cdot \nabla'\rho}{g_b^{\frac{3}{2}}} \\ &= -\frac{1}{\sqrt{g_b}} \Delta_{\mathcal{G}} \rho + \boldsymbol{b} \cdot \nabla'\rho := L\rho. \end{aligned} \tag{3.5}$$

Here $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on the surface $\mathcal{G}$, and $\boldsymbol{b}$ is a smooth field depending on ϕ_b. A lengthy but straightforward calculation shows that

$$\frac{d^2}{ds^2} H_s = \rho_{,\alpha}\rho_{,\beta} \sum_{m=1}^{3} \frac{p_{\alpha\beta m}}{g_s^{m+\frac{1}{2}}} + \rho_{,\alpha}\rho_{,\beta\gamma} \sum_{m=1}^{3} \frac{q_{\alpha\beta\gamma m}}{g_s^{m+\frac{1}{2}}}, \tag{3.6}$$

where $p_{\alpha\beta m}$ and $q_{\alpha\beta\gamma m}$ are suitable polynomials in the variables s, $\nabla'\rho$, $\nabla'\phi_b$, $\nabla^2\phi_b$ (and hence not depending on second or higher derivatives in ρ).

Letting then $H_0 = H_b$ the system (1.2) becomes, in the new variables:

$$\begin{cases} \boldsymbol{v}_{,t} - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{l}_1(\boldsymbol{v}, p, \rho) + \boldsymbol{l}_0(\boldsymbol{v}, \rho) & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{v} = l_2(\boldsymbol{v}, \rho) & \text{in } \Omega_b, \\ \nu\Pi_{\mathcal{G}}\mathbb{D}(\boldsymbol{v})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{v}, \rho) & \text{on } \mathcal{G}, \\ -p + \nu\boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{v})\boldsymbol{N} + \sigma L\rho = l_4(\boldsymbol{v}, \rho) - \sigma H_b(y) & \text{on } \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b \cdot \boldsymbol{v}(y', \phi_b(y')) - v^3(y', \phi_b(y')) + \nabla'\rho \cdot \boldsymbol{v}(y', \phi_b(y')) = 0 & \text{on } \Sigma, \\ \boldsymbol{v}(y, 0) = \widetilde{\boldsymbol{v}}_0(y), \quad \text{in } \Omega_b, \qquad \rho(y, 0) = \rho_0(y), \quad \text{on } \mathcal{G}, \\ \boldsymbol{v}(y', t) = \boldsymbol{\alpha}(y') \text{ for } t \geq 0,\ y' \in \Sigma, \end{cases} \tag{3.7}$$

where $\widetilde{\boldsymbol{v}}_0 = \boldsymbol{v}_0 \circ e_{\rho_0}$, L is a second order elliptic operator with lower order terms, whose principal part is $-\Delta_{\mathcal{G}}$, H_b is the doubled mean curvature of the graph of ϕ_b and the $\boldsymbol{l}_i$ are the following nonlinear operators

$$\begin{aligned}
&\boldsymbol{l}_0(\boldsymbol{v}, \rho) = -(\mathcal{L}^{-1}\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \\
&\boldsymbol{l}_1(\boldsymbol{v}, p, \rho) = \nu(\widetilde{\Delta} - \Delta)\boldsymbol{v} + (\nabla - \widetilde{\nabla})p + \rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3 \cdot \nabla)\boldsymbol{v}, \\
&l_2(\boldsymbol{v}, \rho) = \big((I - \widehat{\mathcal{L}}^T)\nabla\big) \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{G}(\boldsymbol{v}, \rho), \quad \boldsymbol{G} = (I - \widehat{\mathcal{L}})\boldsymbol{v}, \\
&\boldsymbol{l}_3(\boldsymbol{v}, \rho) = \nu\Pi_{\mathcal{G}}\big(\Pi_{\mathcal{G}}\mathbb{D}(\boldsymbol{v})\boldsymbol{N}(y) - \Pi\widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n}(e_\rho(y))\big), \\
&l_4(\boldsymbol{v}, \rho) = \nu(\boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{v})\boldsymbol{N} - \boldsymbol{n} \cdot \widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n}) - \sigma \int_0^1 (1-s)\frac{d^2 H_s}{ds^2} ds.
\end{aligned} \tag{3.8}$$

By $\widetilde{\mathbb{D}}$ we mean the doubled transformed rate of strain tensor: $\widetilde{\mathbb{D}}(\boldsymbol{u}) = (\widetilde{\nabla}\boldsymbol{u}) + (\widetilde{\nabla}\boldsymbol{u})^T$. The equation for $\rho_{,t}$ can be equivalently written with variables in $\mathcal{G}$ instead of Σ, simply letting $\rho(y', \phi_b(y')) = \rho(y')$, and we will do so in the following.

From (3.5) we have an explicit expression for L. We will keep the full linear operator instead of its principal part in all that follows. The reason for this is apparent from the following lemma, which would not hold otherwise.

Lemma 3.1.1 *For sufficiently large real s, (depending on ϕ_b and $\boldsymbol{v}_b$), the bilinear form*

$$B_s(\rho) = \int_\Sigma L\rho(s\rho + \nabla'\rho \cdot \boldsymbol{v}_b)dx',$$

is positive definite.

Proof. A straightforward calculation shows that, summing for $\alpha, \beta = 1, 2$,

$$L\rho = -\frac{1}{g_b}\partial_\alpha\left[\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\right] - \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\partial_\alpha\frac{1}{g_b}.$$

We integrate by parts one derivative in the Laplace–Beltrami operator: by periodicity there is no boundary term and by the previous formula the terms in $\partial_\alpha(1/g_b)$ cancel out, giving

$$\int_\Sigma L\rho \cdot s\rho dx' = s\int_\Sigma \frac{1}{g_b}\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\rho_{,\alpha}dx'.$$

From Swartz inequality one immediately obtains

$$\frac{1}{g_b}\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\rho_{,\alpha} = \frac{|\nabla'\rho|^2(1+|\nabla'\phi_b|^2) - (\nabla'\rho\cdot\nabla'\phi_b)^2}{g_b^{\frac{3}{2}}} \geq \frac{|\nabla'\rho|^2}{g_b^{\frac{3}{2}}},$$

and thus

$$\int_\Sigma L\rho \cdot s\rho dx' \geq s \int_\Sigma \frac{|\nabla'\rho|^2}{g_b^{\frac{3}{2}}} dx' \geq cs \int_\Sigma |\nabla'\rho|^2 dx'.$$

Let us look at the remaining term: again integrating by parts one derivative in the Laplace–Beltrami operator, we get

$$\int_\Sigma L\rho \nabla'\rho \cdot \boldsymbol{v} dx' = \int_\Sigma \frac{1}{g_b} \left(\delta_{\alpha\beta}\sqrt{g} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \rho_{,\beta} \left(\rho_{,\alpha\gamma} v_b^\gamma + \rho_{,\gamma} v_{b,\alpha}^\gamma \right) dx'.$$

Clearly

$$\frac{1}{g_b} \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \rho_{,\beta}\rho_{,\gamma} v_{b,\alpha}^\gamma \geq -c'|\nabla'\rho|^2,$$

with a constant depending on ϕ_b and $\boldsymbol{v}_b$. It remains to estimate

$$\int_\Sigma \frac{1}{g_b} \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \rho_{,\beta}\rho_{,\alpha\gamma} v_b^\gamma dx',$$

but since this expression is symmetric in α and β, integrating by parts on the term $\rho_{,\alpha\gamma}$ with respect to x_γ gives

$$2\int_\Sigma \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \frac{\rho_{,\beta}\rho_{,\alpha\gamma} v_b^\gamma}{g_b} dx' =$$
$$= -\int_\Sigma \rho_{,\alpha}\rho_{,\beta}\partial_\gamma \left[\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \frac{v_b^\gamma}{g_b} \right] dx' \geq -c' \int_\Sigma |\nabla'\rho|^2.$$

The claim now easily follows, since gathering together the previous estimates gives

$$B_s(\rho) \geq (cs - c') \int_\Sigma |\nabla'\rho|^2 dx'. \tag{3.9}$$

□

3.2 The linearization

Let us now look at some stationary, sufficiently smooth solution $(\boldsymbol{v}_b, p_b, \phi_b)$ of (1.3), supposing that in the Hanzawa transformation Ω_b is given by ϕ_b. Since we are concerned with stability properties, we suppose that ϕ is sufficiently near (in a sense to be made precise later) to ϕ_b, and thus $\rho = \phi - \phi_b$ is small. In order to linearize the problem near $(\boldsymbol{v}_b, p_b, 0)$, we notice that all the nonlinear terms except $\boldsymbol{l}_0$ are actually linear in the variables $\boldsymbol{v}$ and p;

therefore, letting, $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_b$, $q = p - p_b$ and subtracting the corresponding system for $(\boldsymbol{u}_b, p_b, 0)$, we get a system of the form

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q - \boldsymbol{\Phi}_1(\boldsymbol{u},\rho) = \widetilde{\boldsymbol{l}}_0(\boldsymbol{u},\rho) + \widetilde{\boldsymbol{l}}_1(\boldsymbol{u},\rho,q) & \text{in } \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = \widetilde{l}_2(\boldsymbol{u},\rho) = \nabla\cdot\boldsymbol{G}(\boldsymbol{u},\rho) & \text{in } \Omega_b, \\ \nu\Pi_b\mathbb{D}(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho) = \widetilde{\boldsymbol{l}}_3(\boldsymbol{u},\rho) & \text{on } \mathcal{G}, \\ -q + \nu\boldsymbol{N}\cdot\mathbb{D}(\boldsymbol{u})\boldsymbol{N} + \sigma L\rho - \Phi_4(\rho) = \widetilde{l}_4(\boldsymbol{u},\rho) & \text{on } \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = \widetilde{l}_5(\boldsymbol{u},\rho) & \text{on } \mathcal{G}, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y) \quad \text{in } \Omega_b, \qquad \rho(y',0) = \rho_0(y') \quad \text{on } \Sigma, \\ \boldsymbol{u}(y',t) = 0 \qquad \forall y' \in \Sigma, \quad t \geq 0, \end{cases} \tag{3.10}$$

where the Φ_i's are the first variation of the system w.r.t. perturbations of the form $(\boldsymbol{v}_b + s\boldsymbol{u}, p_b + sq, s\rho)$, and thus, in general, depend on $\boldsymbol{v}_b$, p_b and ϕ_b. We will calculate explicitly the Φ_i and $\widetilde{\boldsymbol{l}}_i$ in the following.

Problem (3.10) is obviously subjected to a set of compatibility conditions, which generally speaking depends on the required smoothness of the solution. We will consider the following ones:

$$\begin{cases} \nabla\cdot\boldsymbol{u}_0 - \Phi_2(\rho_0) = \widetilde{l}_2(\boldsymbol{u}_0,\rho_0), \\ \nu\Pi_b\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_0) = \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_0,\rho_0), \\ \int_\Sigma \rho_0 dy' = 0. \end{cases} \tag{3.11}$$

The first two conditions are the simplest compatibility conditions at the initial time, while the third is the preservation of mass for the perturbation, and a straightforward calculation shows that this in turns implies

$$\int_\Sigma \rho(y',t)dy' \equiv 0, \qquad \int_{\Omega_b} \widetilde{l}_2(\boldsymbol{u},\rho^*)dy + \int_\Sigma \widetilde{l}_5(\boldsymbol{u}(y',\phi_b(y'),\rho(y',t))dy' \equiv 0,$$

identically for $t \geq 0$, for any solution of (3.10).

We now proceed to explicitly calculate the linear and nonlinear terms of this system.

First note that the exact equation for $\rho_{,t}$ is

$$\rho_{,t} + \nabla'\rho\cdot\boldsymbol{v}_b + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 = \nabla'\rho\cdot\boldsymbol{u}, \tag{3.12}$$

and therefore

$$\widetilde{l}_5(\boldsymbol{u},\rho) = \nabla'\rho\cdot\boldsymbol{u}. \tag{3.13}$$

From the explicit matrix given in (3.3), we have

$$\delta\mathcal{L}^{-T} = \begin{pmatrix} 0 & 0 & -\theta\rho_{,y_1} \\ 0 & 0 & -\theta\rho_{,y_2} \\ 0 & 0 & -\theta'\rho \end{pmatrix} := -\nabla\rho^*\otimes\boldsymbol{e}_3, \qquad \delta\mathcal{L}^{-1} = -\boldsymbol{e}_3\otimes\nabla\rho^*, \tag{3.14}$$

and

$$\mathcal{L}^{-T} - I - \delta\mathcal{L}^{-T} = \nabla\rho^* \frac{\theta'\rho}{1+\theta'\rho} \otimes \boldsymbol{e}_3. \tag{3.15}$$

For the first equation, notice that $\boldsymbol{l}_1$ is linear in the arguments $\boldsymbol{v}$ and p. Thus it suffice to compute the linearization of $\boldsymbol{l}_1(\boldsymbol{v}_b, p_b, \rho)$ with respect to ρ. Calling $\delta\boldsymbol{l}_1(\boldsymbol{v}_b, p_b, \rho)$ this linearization we have

$$\delta\boldsymbol{l}_1(\boldsymbol{v}_b, p_b, \rho) = \nu\delta\mathcal{L}^{-T}\nabla\cdot\nabla\boldsymbol{v}_b + \nu\nabla\cdot\delta\mathcal{L}^{-T}\nabla\boldsymbol{v}_b - \nabla\rho^*\frac{\partial p_b}{\partial y_3} + \delta\rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3\cdot\nabla)\boldsymbol{v}_b.$$

For the last term, we have that

$$\rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3\cdot\nabla)\boldsymbol{v}_b = \frac{\rho^*_{,t}}{1+\theta'\rho}\frac{\partial\boldsymbol{v}_b}{\partial y_3} = \rho^*_{,t}\frac{\partial\boldsymbol{v}_b}{\partial y_3} - \frac{\theta'\rho\rho^*_{,t}}{1+\theta'\rho}\frac{\partial\boldsymbol{v}_b}{\partial y_3}.$$

Therefore the linear part is

$$\nu\delta\mathcal{L}^{-T}\nabla\cdot\nabla\boldsymbol{v}_b + \nu\nabla\cdot\delta\mathcal{L}^{-T}\nabla\boldsymbol{v}_b - \nabla\rho^*\frac{\partial p_b}{\partial y_3} - \rho^*_{,t}\frac{\partial\boldsymbol{v}_b}{\partial y_3}, \tag{3.16}$$

and the nonlinear one is

$$\begin{aligned}\widetilde{\boldsymbol{l}}_1(\boldsymbol{u}, q, \rho) = \boldsymbol{l}_1(\boldsymbol{u}, q, \rho) &+ \nu\big(\mathcal{L}^{-T} - I - \delta\mathcal{L}^{-T}\big)\nabla\cdot\nabla\boldsymbol{v}_b \\ &+ \nu\nabla\cdot\big(\mathcal{L}^{-T} - I - \delta\mathcal{L}^{-T}\big)\nabla\boldsymbol{v}_b + \big(\mathcal{L}^{-T} - I - \delta\mathcal{L}^{-T}\big)\nabla p_b \\ &- \frac{\theta'\rho\rho^*_{,t}}{1+\theta'\rho}\frac{\partial\boldsymbol{v}_b}{\partial y_3}.\end{aligned} \tag{3.17}$$

For $\widetilde{\boldsymbol{l}}_0$ we have

$$\boldsymbol{l}_0(\boldsymbol{v}, \rho) - \boldsymbol{l}_0(\boldsymbol{v}_b, 0) = (\delta\mathcal{L}^{-1}\boldsymbol{v}_b\cdot\nabla)\boldsymbol{v}_b + (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_b + (\boldsymbol{v}_b\cdot\nabla)\boldsymbol{u} + \widetilde{\boldsymbol{l}}_0(\boldsymbol{u}, \rho),$$

which adds a further linear term to (3.16), giving

$$\begin{aligned}\boldsymbol{\Phi}_1(\boldsymbol{u}, \rho) =& \nu\delta\mathcal{L}^{-T}\nabla\cdot\nabla\boldsymbol{v}_b + \nu\nabla\cdot\delta\mathcal{L}^{-T}\nabla\boldsymbol{v}_b - \nabla\rho^*\frac{\partial p_b}{\partial y_3} - \theta\rho_{,t}\frac{\partial\boldsymbol{v}_b}{\partial y_3} + \\ &+ \delta\mathcal{L}^{-1}\boldsymbol{v}_b\cdot\nabla\boldsymbol{v}_b + \boldsymbol{u}\cdot\nabla\boldsymbol{v}_b + \boldsymbol{v}_b\cdot\nabla\boldsymbol{u}.\end{aligned} \tag{3.18}$$

The nonlinear term is

$$\begin{aligned}\widetilde{\boldsymbol{l}}_0(\boldsymbol{u}, \rho) &= (\mathcal{L}^{-1} - I - \delta\mathcal{L}^{-1})\boldsymbol{v}_b\cdot\nabla\boldsymbol{u} + (\mathcal{L}^{-1} - I - \delta\mathcal{L}^{-1})\boldsymbol{u}\cdot\nabla\boldsymbol{v}_b + \\ &+ (\mathcal{L}^{-1} - I - \delta\mathcal{L}^{-1})\boldsymbol{v}_b\cdot\nabla\boldsymbol{v}_b + (\mathcal{L}^{-1} - I - \delta\mathcal{L}^{-1})\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}.\end{aligned} \tag{3.19}$$

Regarding the divergence, notice that

$$\widehat{\mathcal{L}} = \begin{pmatrix} 1+\theta'\rho & 0 & 0 \\ 0 & 1+\theta'\rho & 0 \\ -\theta\rho_{y_1} & -\theta\rho_{y_2} & 1 \end{pmatrix} = I(1+\theta'\rho) - \boldsymbol{e}_3\otimes\nabla\rho^*, \tag{3.20}$$

and thus $I - \widehat{\mathcal{L}}^T$ is linear in ρ; therefore

$$\nabla \cdot \boldsymbol{u} = \nabla\rho^* \cdot \frac{\partial \boldsymbol{v}_b}{\partial y_3} - \theta'\rho\nabla \cdot \boldsymbol{v}_b + \nabla\rho^* \cdot \frac{\partial \boldsymbol{u}}{\partial y_3} - \theta'\rho\nabla \cdot \boldsymbol{u},$$

giving

$$\Phi_2(\rho) = \nabla\rho^* \cdot \frac{\partial \boldsymbol{v}_b}{\partial y_3} - \theta'\rho\nabla \cdot \boldsymbol{v}_b = \nabla \cdot (I - \widehat{\mathcal{L}})\boldsymbol{v}_b, \tag{3.21}$$

$$\widetilde{l}_2(\boldsymbol{u}, \rho) = \nabla\rho^* \cdot \frac{\partial \boldsymbol{u}}{\partial y_3} - \theta'\rho\nabla \cdot \boldsymbol{u}, \qquad \boldsymbol{G}(\boldsymbol{u}, \rho) = (\nabla\rho^* \cdot \boldsymbol{u})\boldsymbol{e}_3 - \theta'\rho\boldsymbol{u}. \tag{3.22}$$

Notice that

$$(I - \widehat{\mathcal{L}})\boldsymbol{v}_b = \boldsymbol{G}(\boldsymbol{u}, \rho) = 0, \tag{3.23}$$

in a neighbourhood of Σ, since θ identically vanishes for sufficiently small x_3. We now look at the equation for $T(\boldsymbol{u}, p)$ on the boundary. For the tangential part, we first observe that if $\boldsymbol{N} \cdot \boldsymbol{n} \neq 0$ (which is certainly true by assumption),

$$\Pi\widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n} = 0 \quad \Leftrightarrow \quad \Pi_b\Pi\widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n} = 0.$$

Therefore, taking into account the linearity w.r.t. $\boldsymbol{v}$, the tangential part of the equation can be written as

$$\Pi_b\mathbb{D}(\boldsymbol{u})\boldsymbol{N} = \Pi_b\big(\Pi_b\mathbb{D}(\boldsymbol{u})\boldsymbol{N} - \Pi\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n} - \Pi\widetilde{\mathbb{D}}(\boldsymbol{v}_b)\boldsymbol{n}\big),$$

The linearized part is given by

$$\boldsymbol{\Phi}_3(\rho) = \nu\Pi_b\delta\big(\Pi\widetilde{\mathbb{D}}(\boldsymbol{v}_b)\boldsymbol{n}\big) = \nu\Pi_b\big(\delta\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N} + \mathbb{D}(\boldsymbol{v}_b)\delta\boldsymbol{N} - (\boldsymbol{N}\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N})\delta\boldsymbol{N}\big), \tag{3.24}$$

since $\delta\boldsymbol{N} \cdot \boldsymbol{N} = 0$ and

$$0 = \delta\boldsymbol{N} \cdot \frac{1}{\nu}(p_b + \sigma H_b)\boldsymbol{N} = \delta\boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N} = \boldsymbol{N}\mathbb{D}(\boldsymbol{v}_b)\delta\boldsymbol{N}, \tag{3.25}$$

by the symmetry of $\mathbb{D}(\boldsymbol{v}_b)$. The nonlinear term is then the sum of two terms

$$\begin{aligned}\widetilde{\boldsymbol{l}}_3(\boldsymbol{u}, \rho) =& \nu\Pi_b\big(\Pi_b\mathbb{D}(\boldsymbol{u})\boldsymbol{N} - \Pi\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n}\big) + \\ &+ \nu\Pi_b\big(\Pi\widetilde{\mathbb{D}}(\boldsymbol{v}_b)\boldsymbol{n} - \Pi_b\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N} - \delta(\Pi\widetilde{\mathbb{D}}(\boldsymbol{v}_b)\boldsymbol{n})\big).\end{aligned} \tag{3.26}$$

One can compute $\delta\boldsymbol{N}$ and $\delta\mathbb{D}$ explicitly:

$$\delta\boldsymbol{N} = \Pi_b\frac{(-\nabla'\rho, 0)}{\sqrt{1 + |\nabla'\phi_b|^2}}, \qquad \delta\mathbb{D}(\boldsymbol{v}_b) = -\nabla\rho^* \otimes \nabla v_{b3} - \nabla v_{b3} \otimes \nabla\rho^*. \tag{3.27}$$

For the computation of Φ_4 we have that the first variation w.r.t. $s\rho$ of the equation for $\mathbb{T}(\boldsymbol{v}_b, p_b)$ is given by

$$-q\boldsymbol{N} + p_b\delta\boldsymbol{N} + \nu\delta\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N} + \nu\mathbb{D}(\boldsymbol{u})\boldsymbol{N} + \nu\mathbb{D}(\boldsymbol{v}_b)\delta\boldsymbol{N} = \sigma\delta H\boldsymbol{N} + \sigma H_b\delta\boldsymbol{N}.$$

Since the first variation of the curvature term has already been calculated in (3.5), using (3.25) we get that the projection on $\boldsymbol{N}$ produces the linear term

$$\Phi_4(\rho) = -\nu\boldsymbol{N}\delta\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N}, \tag{3.28}$$

while (3.5) gives

$$L\rho = -\frac{1}{\sqrt{g}}\Delta_{\mathcal{G}}\rho + \boldsymbol{b}\cdot\nabla\rho.$$

The nonlinear one is given computing the difference of the equations for p and p_b, as

$$\begin{aligned}\widetilde{l}_4(\boldsymbol{u},\rho) =&\nu\big(\boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n} - \boldsymbol{N}\mathbb{D}(\boldsymbol{u})\boldsymbol{N} + \boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{v}_b)\boldsymbol{n} - \boldsymbol{N}\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N} - \boldsymbol{N}\delta\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N}\big)\\ &- \sigma\int_0^1 (1-s)\frac{d^2}{ds^2}H_s ds.\end{aligned} \tag{3.29}$$

Chapter 4

The linear problem

In this chapter we will study the optimal regularity properties of the linearized problem

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q - \boldsymbol{\Phi}_1(\boldsymbol{u},\rho) = \boldsymbol{f} & \text{in } \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = h = \nabla\cdot\boldsymbol{F} & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \widetilde{\boldsymbol{\Phi}}(\rho) = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = g & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma, \text{ for all } t\geq 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \rho(x',0) = \rho_0(x'), & \text{for } x\in\Omega_b,\ x'\in\Sigma, \end{cases} \tag{4.1}$$

with suitable regularity conditions on the right hand terms and compatibility conditions on $\boldsymbol{u}_0$, ρ_0. Here $\widetilde{\boldsymbol{\Phi}}$ is defined as

$$\Pi_b\widetilde{\boldsymbol{\Phi}}(\rho) = \boldsymbol{\Phi}_3(\rho), \qquad \widetilde{\boldsymbol{\Phi}}(\rho)\cdot\boldsymbol{N} = \Phi_4(\rho),$$

and the Φ_i are given in (3.18),(3.21), (3.24) and (3.28), while L is given in (3.5). The plan is to perform the Laplace transform technique outlined in section 2.3, and thus to consider the associated complex parameter dependent

$$\begin{cases} \lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q - \widehat{\boldsymbol{\Phi}}_1(\boldsymbol{u},\rho) = \boldsymbol{f} & \text{in } \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = h = \nabla\cdot\boldsymbol{F} & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \widetilde{\boldsymbol{\Phi}}(\rho) = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \lambda\rho + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = g & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma, \end{cases}$$

where $\widehat{\boldsymbol{\Phi}}_1$ is given as in (3.18) substituting the term $\rho^*_{,t}$ with $\lambda\rho^*$. Clearly $\widehat{\boldsymbol{\Phi}}_1$, Φ_2 and $\widetilde{\boldsymbol{\Phi}}$ are lower order perturbations, and thus, via interpolation arguments, we will be done as soon as we prove existence and optimal regularity of the unperturbed linear complex parameter dependant problem

$$\begin{cases} \lambda \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = h = \nabla \cdot \boldsymbol{F} & \text{in } \Omega, \\ \mathbb{T}(\boldsymbol{u}, p)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \lambda\rho(x') + \nabla'\phi(x') \cdot \boldsymbol{u} - u^3 + \boldsymbol{v}_b \cdot \nabla'\rho(x') = g & \text{on } \mathcal{G}, \\ \boldsymbol{u}(y', 0) = 0. \end{cases} \tag{4.2}$$

In the first section we prove solvability and coercive estimates (in weighted isotropic Sobolev–Slobodevskii spaces) for the model problems in the half-space associated to (4.2). In particular, we deal with the model problems obtained by localisation near the boundaries $\mathcal{G}$ and Σ respectively. These two problems are explicitly solvable through partial Fourier transform, and the corresponding coercive estimates are obtained.

In the second section we prove existence and uniqueness of the solution, constructing an "almost solution" of (4.2). This is done via a Shauder approach, gluing together the known solution in the half-space to obtain a solution of (4.2) with perturbed right hand sides. If one writes problem (4.2) as

$$\mathcal{A}_\lambda \boldsymbol{U} = \mathcal{F}_0, \tag{4.3}$$

this procedure defines two linear mappings $R(\mathcal{F})$, $B(\mathcal{F})$ for any right hand side $\mathcal{F}$, such that

$$\mathcal{A}_\lambda R(\mathcal{F}) = (I + B)(\mathcal{F}).$$

Using the coercive estimates for the model problems and interpolation, we will show that for any sufficiently large $\operatorname{Re}\lambda$, B is a contraction operator in the suitable space of right hand sides, and thus $(I + B)$ is invertible. The solution of (4.3) will then be given by $\boldsymbol{U} = R(I + B)^{-1}(\mathcal{F}_0)$. Uniqueness of the solution is given through the coercivity of a bilinear form naturally associated to the homogeneous case of problem (4.2). This coercivity result is the reason we employ the full linear operator L given in (3.5) instead of its principal part $-g_b^{-\frac{1}{2}}\Delta_{\mathcal{G}}$.

In the final section recover the solution of the perturbed complex-parameter dependant problem via a standard iteration scheme, and perform (inverse) Laplace transform to obtain a solution of (4.1). We then prove two different types of optimal regularity estimate for the solution, one for large times T and one for small times using the norms $H_2^{l,\frac{l}{2}}$.

4.1 Model problems in the half-space

In this section we study the model problems in the half space arising from looking at problem (4.2) near the boundary $\mathcal{G}$ and Σ respectively. Therefore, we will be concerned with two such model problems, both of which will be proved to be uniquely solvable with optimal parameter–dependent regularity estimates.

The first one has been treated in [35] and is defined as

$$\begin{cases} \lambda \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \mathbb{R}^3_+, \\ \nabla\cdot\boldsymbol{u} = h := \nabla\cdot\boldsymbol{F} + h' & \text{in } \mathbb{R}^3_+, \\ \nu\big(u^3_{x_j} + u^j_{x_3}\big) = d^j, \quad j = 1,2 & \text{in } \mathbb{R}^2, \\ -q + 2\nu u^3_{x_3} - \sigma\Delta'\rho = d^3 & \text{in } \mathbb{R}^2, \\ \lambda\rho + \boldsymbol{V}\cdot\nabla'\rho + u^3 = g & \text{in } \mathbb{R}^2, \end{cases} \tag{4.4}$$

where $\mathbb{R}^2 \subset \mathbb{R}^3_+$ as $\{x_3 = 0\}$ and primed variables and differential operators are to be meant in $\mathbb{R}^2$.

We set $\Sigma^\infty = \Sigma \times [0, +\infty)$, and consider first an auxiliary problem.

Theorem 4.1.1 *Let $l \geq 0$, and $\boldsymbol{V}' = (V_1, V_2)$ a constant vector. For sufficiently large* $\operatorname{Re}\lambda$ *(depending also on $|\boldsymbol{V}|$), there exists a unique Σ-periodic solution $(\boldsymbol{u}, q, \rho)$ of*

$$\begin{cases} \lambda \boldsymbol{u} + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = 0 & \text{in } \mathbb{R}^3_+, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \mathbb{R}^3_+, \\ \nu\big(u^3_{x_j} + u^j_{x_3}\big) = d^j, \quad j = 1,2 & \text{in } \mathbb{R}^2, \\ -q + 2\nu u^3_{x_3} - \sigma\Delta'\rho = d^3, & \text{in } \mathbb{R}^2, \\ \lambda\rho + \boldsymbol{V}'\cdot\nabla'\rho + u^3 = g, & \text{in } \mathbb{R}^2, \end{cases} \tag{4.5}$$

such that $\boldsymbol{u} \to 0$ and $q \to c$ for $x_3 \to +\infty$. It satisfies the estimates

$$\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Sigma^\infty)} + \|\nabla q\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Sigma^\infty)} + \|q(0)\|^2_{W_2^{l+\frac12}(\Sigma)}$$
$$+ |\lambda|^{l+\frac12}\|q(0)\|^2_{L^2(\Sigma)} + \|\rho\|^2_{W_2^{l+\frac52}(\Sigma)} + \|\lambda\rho\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|\lambda\rho\|^2_{W_2^1(\Sigma)}$$
$$\leq c\big(\|\boldsymbol{d}\|^2_{W_2^{l+\frac12}(\Sigma)}|\lambda|^{l+\frac12}\|\boldsymbol{d}\|^2_{L^2(\Sigma)} + \|g\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\Sigma)}\big).$$

$$\int_{\{x_3=s\}} q\,dS = -\int_\Sigma d^3 dS = \overline{d}, \qquad \forall s \geq 0 \tag{4.6}$$

$$\|q - \overline{d}\|^2_{L^2(\Sigma^\infty)} \leq c(\|\boldsymbol{d}\|^2_{W_2^{-\frac12}(\Sigma)} + \|g\|^2_{W_2^{\frac12}(\Sigma)}). \tag{4.7}$$

Proof. The proof of the first estimate is identical to the one in [35], using Fourier series instead of Fourier transforms. We recall it briefly: for any $\xi \in \mathbb{Z}^2$, let $\boldsymbol{u}_\xi$, p_ξ, ρ_ξ be the ξ-th Fourier coefficient with respect to (x_1, x_2) of $\boldsymbol{u}$, p and ρ respectively, as in (2.18). System (4.5) is then reduced to

$$\begin{cases} \nu\left(r_1^2 - \dfrac{d^2}{dx_3^2}\right)u_\xi^j + i\xi_j q_{\xi'} = 0, & \text{for } j = 1,2,\ x_3 > 0 \\ \nu\left(r_1^2 - \dfrac{d^2}{dx_3^2}\right)u_\xi^3 + \dfrac{dq_\xi}{dx_3} = 0 & \text{for } x_3 > 0, \\ i\xi_1 u_\xi^1 + i\xi_2 u_\xi + \dfrac{du_\xi^3}{dx_3} = 0 & \text{for } x_3 > 0, \\ \nu\left(\dfrac{du^j}{dx_3} + i\xi_j u_\xi^3\right) = d_\xi^j & \text{for } j = 1,2,\ x_3 = 0, \\ -q_\xi + 2\nu\dfrac{du_\xi^3}{dx_3} + \sigma|\xi|^2\rho_\xi = d_\xi^3 & \text{for } x_3 = 0, \\ \lambda_1\rho_\xi + u_\xi^3 = g_\xi & \text{for } x_3 = 0, \\ \boldsymbol{u}_\xi \to 0, q_\xi \to c & \text{for } x_3 \to +\infty, \end{cases}$$

where $r_1 = r_1(\lambda, \xi) = \sqrt{\lambda_1\nu^{-1} + |\xi|^2}$, $-\pi \le \operatorname{Arg}(r_1) < \pi$, $\lambda_1 = \lambda + i\boldsymbol{V}' \cdot \xi$. This system of ODE with parameter ξ and λ can be explicitly solved for $\operatorname{Re}\lambda > 0$ as

$$\begin{aligned} u_\xi^i = &-\frac{1-\delta_{i3}}{\nu r_1}e_0(x_3)d_\xi^i + \frac{e_0(x_3)}{\nu^2 r_1(r_1+|\xi|)P_1}\sum_{j=1}^3 U_{ij}d_\xi^j + \frac{e_1(x_3)}{\nu^2(r_1+|\xi|)P_1}\sum_{j=1}^3 V_{ij}d_\xi^j \\ &- \frac{\sigma|\xi|^2(e_0(x_3)U_{i3} + r_1 e_1(x_3)V_{i3})}{\nu^2\lambda_1 r_1(r_1+|\xi|)P_1}g_\xi, \qquad i = 1,2,3, \end{aligned}$$

$$q_\xi = \frac{r_1\lambda_1}{\nu P_1}\left[\left(2\nu + \frac{\sigma|\xi|^2}{r_1\lambda_1}\right)(i\xi_1 d_\xi^1 + i\xi_2 d_\xi^2) - \nu\left(r_1 + \frac{|\xi|^2}{r_1}\right)(d_\xi^3 - \frac{\sigma}{\lambda_1}g_\xi)\right]e^{-|\xi|x_3},$$

$$\rho_\xi = (g_\xi - u_\xi^3)/\lambda_1,$$

where

$$e_0(x_3) = e^{-r_1x_3}, \qquad e_1(x_3) = \frac{e^{-r_1x_3} - e^{-|\xi|x_3}}{r_1 - |\xi|},$$

$$P = (r_1^2+|\xi|^2)^2 - 4r_1|\xi|^2 + \frac{\sigma}{\nu^2}|\xi|^3 = \frac{\lambda_1}{\nu}\left(\frac{\lambda_1}{\nu} + 4|\xi|^2\left(1 - \frac{|\xi|}{r_1+|\xi|}\right) + \frac{\sigma|\xi|^3}{\nu\lambda_1}\right),$$

and U_{ij}, V_{ij} are the elements of the matrices

$$\begin{pmatrix} \xi_1^2((3r_1-|\xi|)\lambda_1 + \frac{\sigma}{\nu}|\xi|^2) & \xi_1\xi_2((3r_1-|\xi|)\lambda_1 + \frac{\sigma}{\nu}|\xi|^2) & i\xi_1 r_1\lambda_1(r_1-|\xi|) \\ \xi_1\xi_2((3r_1-|\xi|)\lambda_1 + \frac{\sigma}{\nu}|\xi|^2) & \xi_2^2((3r_1-|\xi|)\lambda_1 + \frac{\sigma}{\nu}|\xi|^2) & i\xi_2 r_1\lambda_1(r_1-|\xi|) \\ -i\xi_1 r_1\lambda_1(r_1-|\xi|) & -i\xi_2 r_1\lambda_1(r_1-|\xi|) & -|\xi| r_1\lambda_1(r_1-|\xi|) \end{pmatrix},$$

and

$$\begin{pmatrix} -\xi_1^2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -\xi_1\xi_2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_1\lambda_1(r_1^2+|\xi|^2) \\ -\xi_1\xi_2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -\xi_2^2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_2\lambda_1(r_1^2+|\xi|^2) \\ -i\xi_1|\xi|(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_2|\xi|(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & |\xi|\lambda_1(r_1^2+|\xi|^2) \end{pmatrix},$$

respectively. Notice that for the constant mode $\xi=(0,0)$ this reduces to

$$u_0^i(x_3)=-\frac{d_0^i}{\nu\sqrt{\lambda}}e^{-\sqrt{\lambda}x_3},\quad i=1,2,\qquad u_0^3\equiv 0,\qquad q_0\equiv -d_0^3,\qquad \rho_0=\frac{g_0}{\lambda},$$

and thus in the claim of the theorem $p\to -b_0^3$ for $x_3\to+\infty$. If $\gamma>|\boldsymbol{V}'|^2/\nu$ and $\operatorname{Re}\lambda\ge\gamma$, it holds

$$\frac{1}{c}|r_1|\le\frac{1}{2}(\sqrt{|\lambda|}+|\xi|)\le\sqrt{|\lambda|+|\xi|^2}\le\sqrt{|\lambda|}+|\xi|\le c|r_1|, \tag{4.8}$$

and the same estimate with λ_1. Moreover

$$|r_1+|\xi||\ge|r_1|,\quad |r_1+|\xi||\ge|\xi|, \tag{4.9}$$

and

$$|P|\ge c(\gamma)\left(\frac{\gamma^2}{\nu^2}+|\lambda_1||\xi|^2+|\lambda_1|^2+\sigma|\xi|^3\right),$$

from which

$$|P|^2\ge(|\xi|^6+|\xi|^4|\lambda_1|^2+|\xi|^2|\lambda_1|^3+|\lambda_1|^4). \tag{4.10}$$

The principal parts of the norms of e_i (see (2.2)) on $[0,+\infty)$ are estimated as

$$\begin{aligned} \|e_0\|^2_{\mathring{W}_2^\eta([0,+\infty))}&\le c|r_1|^{2\eta-1},\\ \|e_1\|^2_{\mathring{W}_2^\eta([0,+\infty))}&\le c\frac{|r_1|^{2\eta-1}+|\xi|^{2\eta-1}}{|r_1|^2}, \end{aligned} \tag{4.11}$$

for any $\eta\ge0$. Finally it is easy to show that for $\xi\in\mathbb{Z}^2$ and $\operatorname{Re}\lambda\ge\gamma$, it holds

$$\begin{aligned} |U_{ij}|^2+|V_{ij}|^2&\le c(|\xi|^2|\lambda_1|^4+|\xi|^4|\lambda_1|^3+|\xi|^6|\lambda_1|^2+|\xi|^8),\\ |U_{i3}|^2+|U_{3i}|^2+|V_{i3}|^2&\le c(|\xi|^2|\lambda_1|^4+|\xi|^6|\lambda_1|^2). \end{aligned} \tag{4.12}$$

Let us estimate $\|\boldsymbol{u}\|^2_{L^2(\mathbb{R}_+)}$. From (4.9)-(4.12) it is easy to see that

$$\frac{|U_{ij}|^2}{|r_1|^2|r_1+|\xi||^2|P|^2}\|e_0\|^2_{L^2(\mathbb{R}_+)}\le c\frac{|U_{ij}|^2}{|r_1|^3|\xi|^2|P|^2}\le\frac{c}{|r_1|^3},$$

$$\frac{|V_{ij}|^2}{|r_1+|\xi||^2|P|^2}\|e_1\|^2_{L^2(\mathbb{R}_+)} \le c\frac{|V_{ij}|^2}{|r_1|^2}\left(\frac{1}{|r_1||\xi|^2|P|^2}+\frac{1}{|\xi||r_1||\xi||P|^2}\right) \le \frac{c}{|r_1|^3},$$

and similarly

$$\frac{|\xi|^4|U_{ij}|^2}{|\lambda_1|^2|r_1|^2|r_1+|\xi||^2|P|^2}\|e_0\|^2_{L^2(\mathbb{R}_+)} \le c\frac{|\xi|^2}{|r_1|^3},$$

$$\frac{|\xi|^4|V_{ij}|^2}{|\lambda_1|^2|r_1+|\xi||^2|P|^2}\|e_1\|^2_{L^2(\mathbb{R}_+)} \le c\frac{|\xi|^2}{|r_1|^3},$$

which gives

$$|r_1|^{2(l+2)}\|\boldsymbol{u}_\xi\|^2_{L^2(\mathbb{R}_+)} \le c|r_1|^{2l+1}(|\boldsymbol{b}_\xi|^2+|\xi|^2|g_\xi|^2). \tag{4.13}$$

One proceed in the same way estimating the $\mathring{W}_2^{l+2}(\mathbb{R}_+)$ norm, using the fact that $|\xi|^{2l+3} \le |r_1|^{2l+3}$ in (4.11), to obtain

$$\|\boldsymbol{u}_\xi\|^2_{\mathring{W}_2^{l+2}(\mathbb{R}_+)} \le c|r_1|^{2l+1}(|\boldsymbol{b}_\xi|^2+|\xi|^2|g_\xi|^2). \tag{4.14}$$

To estimate the pressure, notice first that, for any $\xi \neq 0$, $\|e^{-|\xi|x_3}\|^2_{L^2(\mathbb{R}_+)} = 1/2|\xi|$, and thus

$$\begin{aligned}|r_1|^{2l}|\xi|^2\|q_\xi\|^2_{L^2(\mathbb{R}_+)} &\le c|r_1|^{2l+1}\Bigg[\left(\frac{|\xi|^5|r_1|^3}{|P|^2}+\frac{|\xi|^9}{|r_1||P|^2}\right)|g_\xi|^2+\\ &\left(\frac{|r_1||\lambda_1|^2|\xi|^3}{|P|^2}+\frac{|\xi|^7}{|r_1||P|^2}+\frac{|r_1|^3|\lambda_1|^2|\xi|}{|P|^2}+\frac{|\xi|^5|\lambda_1|^2}{|r_1||P|^2}\right)|\boldsymbol{d}_\xi|^2\Bigg]\\ &\le c|r_1|^{2l+1}\Bigg[\left(\frac{|\xi|^5|r_1|^3}{|P|^2}+|\xi|^2\right)|g_\xi|^2+\\ &\left(1+\frac{|r_1||\lambda_1|^2|\xi|^3}{|P|^2}+\frac{|r_1|^3|\lambda_1|^2|\xi|}{|P|^2}\right)|\boldsymbol{d}_\xi|^2\Bigg],\end{aligned}$$

where we used the fact that $|r_1| \ge |\xi|$ and (4.10) on the terms containing $|r_1|$ at the denominator. On the remaining terms we distinguish the cases in which $|\lambda_1| \le |\xi|^2$, which implies $|r_1| \le c|\xi|$, and $|\lambda_1| > |\xi|^2$, which implies $|r_1| \le c\sqrt{|\lambda_1|}$. In the first case, by (4.10),

$$\frac{|r_1||\lambda_1|^2|\xi|^3}{|P|^2}+\frac{|r_1|^3|\lambda_1|^2|\xi|}{|P|^2} \le c\frac{|\lambda_1|^2|\xi|^4}{P^2} \le c, \qquad \frac{|\xi|^5|r_1|^3}{|P|^2} \le c\frac{|\xi|^8}{|P|^2} \le c|\xi|^2;$$

in the second one, similarly

$$\frac{|r_1||\lambda_1|^2|\xi|^3}{|P|^2}+\frac{|r_1|^3|\lambda_1|^2|\xi|}{|P|^2} \le c\frac{|\lambda_1|^4}{P^2} \le c, \qquad \frac{|\xi|^5|r_1|^3}{|P|^2} \le c\frac{|\lambda_1|^4}{|P|^2} \le c.$$

All in all we have, (the case $\xi = 0$ is trivial)

$$|r_1|^{2l}|\xi|^2\|q_\xi\|^2_{L^2(\mathbb{R}_+)} \le c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + |\xi|^2|g_\xi|^2). \tag{4.15}$$

Similarly one can estimate the principal part of the $W_2^l(\mathbb{R}_+)$-norm of $\frac{dq_\xi}{dx_3}$, obtaining, through

$$\|e^{-|\xi|x_3}\|_{\mathring{W}_2^l(\mathbb{R}_+)} \le c|\xi|^{2l-1},$$

the inequality

$$\|\frac{dq_\xi}{dx_3}\|^2_{\mathring{W}_2^l(\mathbb{R}_+)} \le c|\xi|^2|\xi|^{2l}\|q_\xi\|^2_{L^2(\mathbb{R}_+)} \le c|r_1|^{2l}|\xi|^2\|q_\xi\|^2_{L^2(\mathbb{R}_+)}, \tag{4.16}$$

and the last term is bounded as before. Summing in $\xi \in \mathbb{Z}^2$ the inequalities (4.13), (4.14), (4.15), (4.16), and using (4.8), we get through Parceval identity

$$\begin{aligned}&\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Sigma^\infty)} + \|\nabla q\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Sigma^\infty)}\\ &\le c(\gamma)(\|\boldsymbol{d}\|^2_{W_2^{l+\frac12}(\Sigma)} + |\lambda|^{l+\frac12}\|\boldsymbol{d}\|^2_{L^2(\Sigma)} + \|g\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\Sigma)})\end{aligned}$$

To estimate q at $x_3 = 0$, one has, with the same method

$$|r_1|^{2l+1}|q_\xi(0)|^2 \le c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + (1+|\xi|^2)|g_\xi|^2),$$

which gives

$$\begin{aligned}&\|q(0)\|^2_{W_2^{l+\frac12}(\Sigma)} + |\lambda|^{l+\frac12}\|q(0)\|^2_{L^2(\Sigma)}\\ &\le c(\gamma)(\|\boldsymbol{d}\|^2_{W_2^{l+\frac12}(\Sigma)} + |\lambda|^{l+\frac12}\|\boldsymbol{d}\|^2_{L^2(\Sigma)} + \|g\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\Sigma)})\end{aligned}$$

To estimate $q - d_0^3$ on Σ^∞, for any $\xi \neq 0$ we have

$$\|q_\xi\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{2|\xi|}|q_\xi|^2 \le \frac{c}{|\xi|}(|\boldsymbol{d}_\xi|^2 + (1+|\xi|^2)|g_\xi|^2),$$

which gives (4.7), summing over $\xi \in \mathbb{Z} \setminus \{0\}$ and recalling that $q_0 = -d_0^3$.

We finally estimate ρ_ξ, using the relations

$$\begin{aligned}\lambda_1\rho_\xi &= g_\xi - u_\xi^3(0),\\ \sigma|\xi|^2\rho_\xi &= d_\xi^3 + p(0) - 2\nu\frac{du_\xi^3(0)}{dx_3} = d_\xi^3 + q_\xi(0) + 2\nu(i\xi_1 u_\xi^1 + i\xi_2 u_\xi^2).\end{aligned}$$

From the explicit formula for u_ξ and the bounds (4.9), (4.10), (4.12), we obtain

$$\begin{aligned}|\lambda_1|^2|\rho_\xi|^2 &\le c(\frac{|\boldsymbol{d}_\xi|^2}{|r_1|^2} + |g_\xi|^2)\\ |\xi|^4|\rho_\xi|^2 &\le c(|\boldsymbol{d}_\xi|^2 + (1+|\xi|^2)|g_\xi|^2).\end{aligned} \tag{4.17}$$

Since $|\lambda_1| \geq c|\lambda| \geq c\gamma$, and $|r_1| \geq c|\xi|$, we get from the first one

$$|\lambda|^{l+\frac{1}{2}}|\lambda\rho_\xi|^2 \leq |\lambda|^{l+\frac{1}{2}}(|\boldsymbol{d}_\xi|^2 + |g_\xi|^2),$$
$$|\lambda|^{l+\frac{1}{2}}|\xi|^2|\lambda\rho_\xi|^2 \leq |\lambda|^{l+\frac{1}{2}}(|\boldsymbol{d}_\xi|^2 + |\xi|^2|g_\xi|^2)$$

which, summed on $\xi \in \mathbb{Z}^2$, gives

$$|\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W^1_2(\Sigma)} \leq c|\lambda|^{l+\frac{1}{2}}(\|\boldsymbol{d}\|^2_{L^2(\Sigma)} + \|g\|^2_{W^1_2(\Sigma)}).$$

Moreover, also by the first inequality in (4.17) and $|r_1| \geq c|\xi|$, we get

$$|\xi|^{2l+3}|\lambda\rho_\xi|^2 \leq c(|\xi|^{2l+1}|\boldsymbol{d}_\xi|^2 + |\xi|^{2l+3}|g_\xi|^2),$$

which gives, together with $|\rho_\xi|^2 \leq c(|\boldsymbol{d}_\xi|^2 + |g_\xi|^2)$,

$$\|\lambda\rho\|^2_{W^{l+\frac{3}{2}}_2(\Sigma)} \leq c(\|\boldsymbol{d}\|^2_{W^{l+\frac{1}{2}}_2(\Sigma)} + \|g\|^2_{W^{l+\frac{3}{2}}_2(\Sigma)}).$$

The second inequality in (4.17), on the other hand gives

$$|\xi|^{2l+5}|\rho_\xi|^2 \leq c(|\xi|^{2l+1}|\boldsymbol{d}_\xi|^2 + |\xi|^{2l+3}|g_\xi|^2),$$

which gives, as before

$$\|\rho\|^2_{W^{l+\frac{5}{2}}_2(\Sigma)} \leq c(\|\boldsymbol{d}\|^2_{W^{l+\frac{1}{2}}_2(\Sigma)} + \|g\|^2_{W^{l+\frac{3}{2}}_2(\Sigma)}).$$

□

We now consider the full model problem (4.4).

Theorem 4.1.2 *Let $l \geq 0$ and $\boldsymbol{V}' = (V_1, V_2)$ a constant vector. Suppose h decays for $x_3 \to +\infty$ sufficiently rapidly, and h' is compactly supported in x_3. Moreover let all the right hand terms in (4.4) be Σ-periodic. For sufficiently large $\operatorname{Re}\lambda$, there is a unique periodic solution $\boldsymbol{u}$, q, ρ, with $\boldsymbol{u} \to 0$ and $q \to c$ for $x_3 \to +\infty$ to (4.4), which satisfies the estimate*

$$\begin{aligned}
&\|\boldsymbol{u}\|^2_{W^{l+2}_2(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Sigma^\infty)} + \|\nabla q\|^2_{W^l_2(\Sigma^\infty)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Sigma^\infty)} + \|q(0)\|^2_{W^{l+\frac{1}{2}}_2(\Sigma)} \\
&+ |\lambda|^{l+\frac{1}{2}}\|q(0)\|^2_{L^2(\Sigma)} + \|\rho\|^2_{W^{l+\frac{5}{2}}_2(\Sigma)} + \|\lambda\rho\|^2_{W^{l+\frac{3}{2}}_2(\Sigma)} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W^1_2(\Sigma)} \\
&\leq c\big(\|\boldsymbol{f}\|^2_{W^l_2(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + |\lambda|^{l+2}\|h'\|^2_{L^2(\Sigma^\infty)} \\
&+ \|h\|^2_{W^{l+1}_2(\Sigma^\infty)} + \|\boldsymbol{d}\|^2_{W^{l+\frac{1}{2}}_2(\Sigma)} + |\lambda|^{l+\frac{1}{2}}\|\boldsymbol{d}\|^2_{L^2(\Sigma)} + \|g\|^2_{W^{l+\frac{3}{2}}_2(\Sigma)} + |\lambda|^{l+\frac{1}{2}}\|g\|^2_{W^1_2(\Sigma)}\big).
\end{aligned} \tag{4.18}$$

Proof. Let us call $X(\boldsymbol{u}, q, \rho)$ the left hand side of (4.19) and $Y(\boldsymbol{f}, \boldsymbol{d}, g, h, \boldsymbol{F}, h')$ the right hand side.

First of all we solve the corresponding problem with solenoidal velocity, i.e. we consider the case $h = 0$. To this end, consider the problem

$$\begin{cases} \lambda \boldsymbol{u} + (\boldsymbol{V}' \cdot \nabla')\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \mathbb{R}^3_+, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \mathbb{R}^3_+, \\ \nu\big(u^3_{,x_j} + u^j_{,x_3}\big) = d^j, \quad j = 1, 2 & \text{in } \mathbb{R}^2, \\ -q + 2\nu u^3_{,x_3} - \sigma\Delta'\rho = d^3, & \text{in } \mathbb{R}^2, \\ \lambda\rho + \boldsymbol{V}' \cdot \nabla'\rho + u^3 = g, & \text{in } \mathbb{R}^2. \end{cases} \tag{4.19}$$

We claim that it has a unique solution satisfying the estimate of the theorem. To prove this, we can suppose that $\nabla \cdot \boldsymbol{f} = 0$ since otherwise we can subtract a pressure component p satisfying

$$\begin{cases} \Delta p = \nabla \cdot \boldsymbol{f} & \text{in } \Sigma^\infty, \\ p = 0 & \text{if } x_3 = 0. \end{cases}$$

The weak formulation

$$\int_{\Sigma^\infty} \nabla p \nabla \eta dx = \int_{\Sigma^\infty} \boldsymbol{f} \nabla \eta dx \leq c\|\boldsymbol{f}\|_{W^\eta(\Sigma^\infty)} \|\nabla\eta\|_{W_2^{-\eta}(\Sigma^\infty)},$$

gives by duality $\|\nabla p\|_{W_2^\eta(\Sigma^\infty)} \leq c\|\boldsymbol{f}\|_{W_2^\eta(\Sigma^\infty)}$ for any $\eta \geq 0$; thus it holds

$$\|\nabla p\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l \|\nabla p\|^2_{L^2(\Sigma^\infty)} \leq c(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l \|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)}).$$

Since $p = 0$ for $x_3 = 0$ by definition, the boundary conditions are unaltered for the triple $(\boldsymbol{u}, (q - p), \rho)$, and the previous estimate shows that a bound on $q - p$ implies one for q. If $\boldsymbol{f}$ is solenoidal, one can consider a solution of

$$\lambda \boldsymbol{v} - \nu\Delta\boldsymbol{v} + (\boldsymbol{V}' \cdot \nabla')\boldsymbol{v} = \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{v} = 0$$

obtained as following. We can extend $\boldsymbol{f}$ with preservation of class and solenoidality, and thus suppose $\boldsymbol{f}$ is defined and solenoidal in the whole $\mathbb{R}^3$. If $\widetilde{\boldsymbol{f}}(x_1, x_2, s)$ is the Fourier transform with respect to the x_3 variable , we define $\boldsymbol{u}$ by

$$\widetilde{\boldsymbol{v}}_\xi = \frac{\widetilde{\boldsymbol{f}}_\xi}{\lambda_1 + \nu|\xi|^2},$$

where as usual the index ξ indicates the ξ-th Fourier coefficient with respect to (x_1, x_2), and $\lambda_1 = \lambda + i\boldsymbol{V}' \cdot \xi$. It is easy to see that for $\operatorname{Re}\lambda > |\boldsymbol{V}'|^2/\nu$ it holds

$$|\lambda_1 + |\xi|^2|^2 \geq c((1 + |\xi|^2)^2 + |\lambda|^2),$$

and thus through Parceval identity

$$\|\boldsymbol{v}\|^2_{W_2^{l+2}(\Sigma\times\mathbb{R})} + |\lambda|^{l+2}\|\boldsymbol{v}\|^2_{L^2(\Sigma\times\mathbb{R})} \leq c(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)}).$$

Now the solution of (4.19) is obtained as $(\boldsymbol{v}+\boldsymbol{w},p,\rho)$ where $(\boldsymbol{w},p,\rho)$ solves (4.5) with right hand sides respectively $\boldsymbol{d}-\nu S(\boldsymbol{v})\boldsymbol{e}_3$ and $g-v^3$. Indeed one readily has

$$\|S(\boldsymbol{v})\boldsymbol{e}_3\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)} + \|v^3\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} \leq c\|\boldsymbol{v}\|^2_{W_2^{l+2}(\Sigma^\infty)} \leq c\|\boldsymbol{f}\|^2_{W_2^{l+2}(\Sigma^\infty)}, \tag{4.20}$$

by the properties of the restriction operator and

$$\begin{aligned}|\lambda|^{l+\frac{1}{2}}(\|v^3\|^2_{W_2^1(\Sigma)} + \|S(\boldsymbol{v})\|_{L^2(\Sigma)}) &\leq c|\lambda|^{l+\frac{1}{2}}\|\boldsymbol{v}\|^2_{W_2^1(\Sigma)} \leq c|\lambda|^{l+\frac{1}{2}}\|\boldsymbol{v}\|^2_{W_2^{\frac{3}{2}}(\Sigma^\infty)}\\ &\leq c(\|\boldsymbol{v}\|_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{v}\|^2_{L^2(\Sigma^\infty)})\\ &\leq c(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)}),\end{aligned} \tag{4.21}$$

by interpolation inequality (2.14), and these two estimates give (4.18) for the solution of (4.19).

We now can get rid of the term $(\boldsymbol{V}'\cdot\nabla')\boldsymbol{u}$ in the equation for the velocity by a standard iteration argument, defining $(\boldsymbol{u}_1,q_1,\rho_1)$ as the solution to (4.19), and $(\boldsymbol{u}_{n+1},q_{n+1},\rho_{n+1})$ as the solution of (4.19) with right hand side on the velocity equation $\boldsymbol{f}+(\boldsymbol{V}'\cdot\nabla')\boldsymbol{u}_n$. If $(\boldsymbol{w}_n,p_n,\mu_n) := (\boldsymbol{u}_n-\boldsymbol{u}_{n-1},q_n-q_{n-1},\rho_n-\rho_{n-1})$, notice that $(\boldsymbol{w}_{n+1},p_{n+1},\mu_{n+1})$ satisfies (4.19) with right hand side $(\boldsymbol{V}'\cdot\nabla')\boldsymbol{w}_n$ on the velocity equation and zero elsewhere. From the interpolation inequality

$$\begin{aligned}&\|(\boldsymbol{V}'\cdot\nabla')\boldsymbol{w}_n\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|(\boldsymbol{V}'\cdot\nabla')\boldsymbol{w}_n\|^2_{L^2(\Sigma^\infty)}\\ &\qquad\leq \frac{c}{|\lambda|}(\|\boldsymbol{w}_n\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{w}_n\|^2_{L^2(\Sigma^\infty)}),\end{aligned}$$

and the estimate (4.18) for problem (4.19) we get that

$$X(\boldsymbol{w}_{n+1},p_{n+1},\mu_{n+1}) \leq \frac{c}{|\lambda|}X(\boldsymbol{w}_n,p_n,\mu_n),$$

which in turn gives, for $c/|\lambda| < 1$, strong convergence of the sequence $(\boldsymbol{u}_n,q_n,\rho_n)$ to a solution of

$$\begin{cases}\lambda\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\nabla q=\boldsymbol{f} & \text{in } \mathbb{R}^3_+,\\ \nabla\cdot\boldsymbol{u}=0 & \text{in } \mathbb{R}^3_+,\\ \nu\big(u^3_{,x_j}+u^j_{,x_3}\big)=d^j, \quad j=1,2 & \text{in } \mathbb{R}^2,\\ -q+2\nu u^3_{,x_3}-\sigma\Delta'\rho=d^3, & \text{in } \mathbb{R}^2,\\ \lambda\rho+\boldsymbol{V}'\cdot\nabla'\rho+u^3=g, & \text{in } \mathbb{R}^2.\end{cases} \tag{4.22}$$

and the estimate $X(\boldsymbol{u},q,\rho) \le cY(\boldsymbol{f},\boldsymbol{d},g,0,0,0)$.

We finally take care of the divergence term, defining $\boldsymbol{w} = \nabla\psi$, where ψ is the stable periodic solution of

$$\begin{cases} \Delta\psi = h = \nabla\cdot\boldsymbol{F} + h' & \text{in } \mathbb{R}^3_+, \\ \psi = 0 & \text{on } \mathbb{R}^2. \end{cases}$$

From the energy inequality for this problem and standard coercive estimate, one has

$$\begin{aligned} &\|\boldsymbol{w}\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{w}\|^2_{L^2(\Sigma^\infty)} \\ &\le c\big(\|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|_{L^2(\Sigma^\infty)})\big). \end{aligned} \tag{4.23}$$

The solution will then be defined as $(\boldsymbol{w}+\boldsymbol{u},q,\rho)$, where $(\boldsymbol{u},q,\rho)$ solves (4.22) with right hand sides

$$\boldsymbol{f}_1 = \boldsymbol{f} - \lambda\boldsymbol{w} + \nu\Delta\boldsymbol{w}, \qquad \boldsymbol{d}_1 = \boldsymbol{d} - \nu S(\boldsymbol{w})\boldsymbol{e}_3, \qquad g_1 = g - w^3.$$

Proceeding as in (4.20), (4.21) and using (4.23) we obtain (4.18). □

The second model problem arises from the need of a correction in solenoidality, together with Dirichlet boundary conditions on the bottom surface.

Theorem 4.1.3 *Let $l \ge 0$. Suppose $\boldsymbol{f} \in W_2^l(\Sigma^\infty)$, $\boldsymbol{a} = (a^1,a^2,0) \in W_2^{l+\frac{3}{2}}(\Sigma)$, $h \in W^{l+1}(\Sigma^\infty)$ and $\boldsymbol{F}, h' \in L^2(\Sigma^\infty)$ are Σ-periodic, $h'(x) = 0$ whenever $x_3 \ge L$, $F^3 = 0$ for $x_3 = 0$ and $\int_{\Sigma^\infty} h'dx = 0$. For any $\operatorname{Re}\lambda \ge \gamma > 0$ there is a unique Σ-periodic solution to*

$$\begin{cases} \lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f}, & \text{in } \mathbb{R}^3_+ \\ \nabla\cdot\boldsymbol{u} = h = \nabla\cdot\boldsymbol{F} + h', & \text{in } \mathbb{R}^3_+ \\ \boldsymbol{u} = \boldsymbol{a}, & \text{in } \mathbb{R}^2; \end{cases} \tag{4.24}$$

it satisfies the estimate

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Sigma^\infty)} + \|\nabla p\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\nabla p\|^2_{L^2(\Sigma^\infty)} \\ &\le c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)} + \|\boldsymbol{a}\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{3}{2}}\|\boldsymbol{a}\|^2_{L^2(\Sigma)} \\ &+ \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|^2_{L^2(\Sigma^\infty)})\big). \end{aligned} \tag{4.25}$$

Proof. We start by looking for solutions of

$$\begin{cases} \lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = 0, & \text{in } \mathbb{R}^3_+ \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \mathbb{R}^3_+ \\ \boldsymbol{u} = \boldsymbol{a}, & \text{in } \mathbb{R}^2,\ a^3 = 0. \end{cases} \tag{4.26}$$

with $\boldsymbol{a} \in W_2^{l+\frac{3}{2}}(\Sigma)$. By considering for each x_3 the Fourier series expansion we get, for each $\xi \in \mathbb{Z}^2$, the system of ODE in $x_3 \geq 0$

$$\begin{cases} \nu\big(r^2 - \dfrac{d^2}{dx_3^2}\big)u_\xi^j(x_3) + i\xi_j' p_\xi = 0, \qquad j = 1, 2, \\ \nu\big(r^2 - \dfrac{d^2}{dx_3^2}\big)u_\xi^3(x_3) + \dfrac{dp_\xi(x_3)}{dx_3} = 0, \\ i\xi_1 u_\xi^1(x_{3)} + i\xi_2' u_\xi^2 + \dfrac{du_\xi^3(x_3)}{dx_3} = 0, \\ \boldsymbol{u}_\xi(0) = \boldsymbol{a}_\xi. \end{cases}$$

where $a_\xi^3 = 0$, $r = r(\lambda, \xi) = \sqrt{\frac{\lambda}{\nu} + |\xi|^2}$, $-\pi < \operatorname{Arg} r < \pi$. This can be solved explicitly, and the only stable solution for $\xi \neq 0$ is given by

$$\begin{aligned} \boldsymbol{u}_\xi(x_3) &= \boldsymbol{a}_\xi e_0(x_3) + P(\xi, \boldsymbol{a}_\xi)\big(i\frac{\xi}{|\xi|}, -1\big)e_1(x_3), \\ p_\xi(x_3) &= \nu P(\xi, \boldsymbol{a}_\xi)\big(1 + \frac{r}{|\xi|}\big)e^{-|\xi|x_3}, \end{aligned} \tag{4.27}$$

where

$$P(\xi, \boldsymbol{a}_\xi) = -i\xi \cdot \boldsymbol{a}_\xi', \qquad \boldsymbol{a}_\xi' = (a_\xi^1, a_\xi^2),$$

$$e_0(x_3) = e^{-rx_3}, \qquad e_1(x_3) = \frac{e^{-rx_3} - e^{-|\xi|x_3}}{r - |\xi|}.$$

For the constant mode $\xi = 0$ the solution is

$$u_0^j(x_3) = a_0^j e^{-\sqrt{\frac{\lambda}{\nu}}x_3}, \qquad j = 1, 2; \qquad u_0^3 \equiv p_0 \equiv 0.$$

Using (4.11) and proceeding as in the proof of theorem 4.1.1 one can see that for any mode ξ it holds

$$\begin{aligned} &\|\boldsymbol{u}_\xi\|^2_{\mathring{W}_2^{l+2}(\mathbb{R}_+)} + |r|^{2(l+2)}\|\boldsymbol{u}_\xi\|^2_{L^2(\mathbb{R}_+)} + \|\frac{dp_\xi}{dx_3}\|^2_{\mathring{W}_2^l(\mathbb{R}_+)} + |r|^{2l}|\xi|^2\|p_\xi\|^2_{L^2(\mathbb{R}_+)} \\ &\leq c\big(|r|^{2l+3} + |\xi||r|^{2l+2} + |\xi|^2|r|^{2l+1} + |\xi|^3|r|^{2l} + |\xi|^{2l+1}|r|^2 + |\xi|^{2l+3}\big)|\boldsymbol{a}_\xi|^2. \end{aligned}$$

Since $|r| \leq \sqrt{|\lambda|} + |\xi|$, repeated applications of Young inequality gives

$$|r|^{2l+3} + |\xi||r|^{2l+2} + |\xi|^2|r|^{2l+1} + |\xi|^3|r|^{2l} + |\xi|^{2l+1}|r|^2 + |\xi|^{2l+3} \leq |\xi|^{2l+3} + |\lambda|^{l+\frac{3}{2}},$$

and thus, summing in $\xi \in \mathbb{Z}^2$ the previous inequality and using Parceval identity, we obtain

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Sigma^\infty)} + \|\nabla p\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\nabla p\|^2_{L^2(\Sigma^\infty)} \\ &\leq c\big(\|\boldsymbol{a}'\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{3}{2}}\|\boldsymbol{a}'\|^2_{L^2(\Sigma)}\big). \end{aligned} \tag{4.28}$$

We now solve the inhomogeneous problem

$$\begin{cases} \lambda \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla q = \boldsymbol{f}, & \text{in } \mathbb{R}^3_+ \\ \nabla \cdot \boldsymbol{u} = h := \nabla \cdot \boldsymbol{F} + h', & \text{in } \mathbb{R}^3_+ \\ \boldsymbol{u} = 0, & \text{in } \mathbb{R}^2. \end{cases} \tag{4.29}$$

with $\boldsymbol{f}$, $\boldsymbol{F}$ and h' Σ-periodic, h' with compact support w.r.t x_3, and the compatibility conditions

$$F^3 = 0 \quad \text{on } \Sigma, \qquad \int_{\Sigma^\infty} h' dx = 0.$$

To correct the divergence, we consider the equation for $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{w}$, where $\boldsymbol{w} = \nabla \psi$, with ψ is a stable periodic solution of

$$\begin{cases} \Delta \psi = h = \nabla \cdot \boldsymbol{F} + h' & \text{in } \mathbb{R}^3_+, \\ \psi_{,x_3} = 0 & \text{on } \mathbb{R}^2. \end{cases}$$

From the energy inequality for this problem and the standard coercive estimate, one has

$$\begin{aligned} &\|\nabla \psi\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2} \|\nabla \psi\|^2_{L^2(\Sigma^\infty)} \\ &\le c\big(\|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|_{L^2(\Sigma^\infty)})\big). \end{aligned} \tag{4.30}$$

The new forcing term for the complex parameter Stokes equation for $\boldsymbol{v}$ is

$$\boldsymbol{f}_1 = \boldsymbol{f} - \lambda \boldsymbol{w} + \nu \Delta \boldsymbol{w},$$

which we reduce it in its solenoidal part by considering a stable periodic solution ϕ in $\mathbb{R}^3_+$ of

$$\begin{cases} \Delta \phi = \nabla \cdot \boldsymbol{f}_1 & \text{in } \mathbb{R}^3_+, \\ \phi_{x_3} = f_1^3 & \text{on } \mathbb{R}^2. \end{cases}$$

The compatibility conditions for this problem are clearly satisfied and we have the inequality

$$\|\nabla \phi\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l \|\nabla \phi\|^2_{L^2(\Sigma^\infty)} \le c(\|\boldsymbol{f}_1\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l \|\boldsymbol{f}_1\|^2_{L^2(\Sigma^\infty)}). \tag{4.31}$$

The couple $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}$, $p = q - \phi$ satisfies

$$\begin{cases} \lambda \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{f}_2 & \text{in } \mathbb{R}^3_+, \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \mathbb{R}^3_+, \\ \boldsymbol{v} = -\boldsymbol{w} & \text{in } \mathbb{R}^2. \end{cases}$$

where $\boldsymbol{f}_2 = \boldsymbol{f}_1 - \nabla\phi$ is solenoidal and $f_2^3 = 0$ for $x_3 = 0$; moreover from (4.30), (4.31) and the interpolation inequality, $\boldsymbol{f}_2$ satisfies

$$\|\boldsymbol{f}_2\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l \|\boldsymbol{f}_2\|^2_{L^2(\Sigma^\infty)} \le c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l \|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)} + \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|^2_{L^2(\Sigma^\infty)})\big).$$

By the condition $f_2^3 = 0$ on $x_3 = 0$, we can construct an extension $\widehat{\boldsymbol{f}}_2$ to the whole $\mathbb{R}^3$, Σ-periodic, with preservation of solenoidality and regularity and with $\widehat{f}_2^3$ odd. Letting $\widehat{\boldsymbol{v}}$ be the solution in $\mathbb{R}^3$

$$\begin{cases} \lambda\widehat{\boldsymbol{v}} - \nu\Delta\widehat{\boldsymbol{v}} + \nabla\widehat{p} = \widehat{\boldsymbol{f}}_2, \\ \nabla\cdot\widehat{\boldsymbol{v}} = 0, \end{cases}$$

uniqueness (for $\operatorname{Re}\lambda \gg 1$) of the solution and regularity estimates ensure that $\widehat{v}^3 = 0$ for $x_3 = 0$ and the inequality

$$\|\widehat{\boldsymbol{v}}\|^2_{W_2^{l+2}(\Sigma\times\mathbb{R})} + |\lambda|^{l+2}\|\widehat{\boldsymbol{v}}\|^2_{L^2(\Sigma\times\mathbb{R})} \le c\big(\|\widehat{\boldsymbol{f}}_2\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\widehat{\boldsymbol{f}}_2\|^2_{L^2(\Sigma^\infty)}\big). \quad (4.32)$$

Therefore the couple $\widehat{\boldsymbol{u}} = \boldsymbol{v} - \widehat{\boldsymbol{v}}$, p (the pressure $\widehat{p}$ vanishes) is the unique solution to problem (4.26), with $\boldsymbol{a} = -\boldsymbol{w} - \widehat{\boldsymbol{v}}$ and thus $a^3 = 0$. Estimates (4.30) and (4.32), together with the interpolation inequality (2.15), give

$$\|\boldsymbol{a}'\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{3}{2}}\|\boldsymbol{a}'\|^2_{L^2(\Sigma)} \le c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2()} + \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|^2_{L^2(\Sigma^\infty)})\big).$$

Thus applying estimate (4.28), (4.30), (4.31) and (4.32), we finally get

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Sigma^\infty)} + \|\nabla p\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\nabla p\|^2_{L^2(\Sigma^\infty)} \\ &\quad\le c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)} + \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} \\ &\qquad + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|^2_{L^2(\Sigma^\infty)})\big). \end{aligned} \quad (4.33)$$

Summing the solutions of (4.26) and (4.29), together with the estimates (4.28), (4.33) gives the claim. □

As an application we give and existence theorem for the periodic Stokes problem on the half-space. For $T \le +\infty$ we set $\Sigma_T^\infty = \Sigma^\infty \times [0, T)$.

Theorem 4.1.4 *Let* $l \in [0,1)$. *Given* Σ*-periodic* $\boldsymbol{f} \in W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)$, $\boldsymbol{a} \in W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)$ *with* $a^3 \equiv 0$ *for all* $t \ge 0$ *and* $\boldsymbol{v}_0 \in W_2^{l+1}(\Sigma^\infty)$ *such that*

$$\nabla\cdot\boldsymbol{v}_0 = 0, \qquad v_0^3\lfloor_\Sigma = 0,$$

there exists a unique Σ-periodic solution $\boldsymbol{v} \in W_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)$, $\nabla p \in W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)$ *of*

$$\begin{cases} \boldsymbol{v}_{,t} - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} & \text{in } \Sigma_T^\infty, \\ \nabla\cdot\boldsymbol{v} = 0 & \text{in } \Sigma_T^\infty, \\ \boldsymbol{v} = \boldsymbol{a} & \text{on } \Sigma,\ \forall t \geq 0, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0, & \boldsymbol{v} \to 0 \text{ for } x_3 \to +\infty, \end{cases} \tag{4.34}$$

and the following estimates holds:

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)} + \|\nabla p\|^2_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} \leq \\ &\leq c(T)(\|\boldsymbol{f}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} + \|\boldsymbol{v}_0\|^2_{W_2^{l+1}(\Sigma^\infty)} + \|\boldsymbol{a}\|^2_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)}); \end{aligned} \tag{4.35}$$

if $T \geq 1$,

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)} + \|\nabla p\|^2_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} \leq \\ &\leq c(\|\boldsymbol{f}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} + \|\boldsymbol{v}_0\|^2_{W_2^{l+1}(\Sigma^\infty)} + \|\boldsymbol{a}\|^2_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)} + \|\boldsymbol{v}\|_{L^2(\Sigma_T^\infty)}), \end{aligned} \tag{4.36}$$

with constant independent of T*; if* $T \leq 1$

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)} + \|\nabla p\|^2_{H_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} \leq \\ &\leq c(\|\boldsymbol{f}\|^2_{H_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} + \|\boldsymbol{v}_0\|^2_{W_2^{l+1}(\Sigma^\infty)} + \|\boldsymbol{a}\|^2_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)}), \end{aligned} \tag{4.37}$$

with constant c *independent of* T.

Proof. We follow the plan of section 1.3 and reduce (4.34) to a similar problem with homogeneous initial condition. We fix a large $T_0 \gg T+1$, and extend $\boldsymbol{f}$ and $\boldsymbol{a}$ with controlled norm for all $t \geq 0$ in such a way that both vanish for $t \geq T_0$, keeping the notation unchanged. This can be done by standard continuation theorems, and it holds

$$\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(Q_\infty)} \leq c(T)(\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(Q_T)}), \tag{4.38}$$

with constant $c(T)$ bounded for $T \geq 1$. For small T we use theorem 2.3.3, point 3, to obtain

$$\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(Q_\infty)} \leq c(\|\boldsymbol{f}\|_{H_2^{l,\frac{l}{2}}(Q_T)} + \|\boldsymbol{a}\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(Q_T)}), \tag{4.39}$$

with a constant independent of T if $T \leq 1$. In the following we will then suppose $T = +\infty$ and thus $\Sigma_\infty^\infty = \Sigma^\infty \times [0, +\infty)$.

By standard extension theorems for anisotropic Sobolev–Slobodetskii spaces, we find an extension $\boldsymbol{v}_1 \in W_2^{l+2,\frac{l}{2}+1}(\Sigma_\infty^\infty)$ of $\boldsymbol{v}_0$ such that, $\boldsymbol{v}_1 = 0$ for $t \geq T_0$,

$$\|\boldsymbol{v}_1\|_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Sigma^\infty)}.$$

and $v_1^3 \equiv 0$ on Σ. We then pick $\boldsymbol{v}_2(x,t) = \nabla\psi(x,t)$, where, for each t, ψ is the periodic solution of

$$\begin{cases} \Delta\psi = \nabla\cdot\boldsymbol{v}_1 & \text{in } \Sigma^\infty, \\ \psi_{,x_3} = 0 & \text{in } \Sigma, \\ \psi \to 0 & \text{for } x_3 \to +\infty, \end{cases} \tag{4.40}$$

noticing that from $\nabla\cdot\boldsymbol{v}_0 = 0$ we get $\boldsymbol{v}_2(0) = 0$, and $\boldsymbol{v}_2 = 0$ for $t \geq T_0$. By standard elliptic estimates, the vector $\boldsymbol{v}_2$ satisfies

$$\|\boldsymbol{v}_2\|_{W_2^{l+2,0}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_1\|_{W_2^{l+1,0}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Sigma^\infty)}.$$

Moreover taking the derivative w.r.t. t in the weak formulation of (4.40) we get

$$\int_{\Sigma^\infty} \boldsymbol{v}_{2,t}\cdot\nabla\eta dx = \int_{\Sigma^\infty} \boldsymbol{v}_{1,t}\cdot\nabla\eta dx$$

for all $t \geq 0$ and $\eta \in W_2^1(\Sigma^\infty)$. Reasoning in this way also for the discrete time differences, we get

$$\|\boldsymbol{v}_2\|_{W_2^{0,\frac{l}{2}+1}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_1\|_{W_2^{0,\frac{l}{2}+1}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Sigma^\infty)},$$

giving with the previous estimate

$$\|\boldsymbol{v}_2\|_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_1\|_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_\infty^\infty)} \leq c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Sigma^\infty)}. \tag{4.41}$$

Setting

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_1 - \boldsymbol{v}_2, \quad \widehat{\boldsymbol{f}} = \boldsymbol{f} - (\boldsymbol{v}_1 + \boldsymbol{v}_2)_{,t} + \nu\Delta(\boldsymbol{v}_1 + \boldsymbol{v}_2), \quad \widehat{\boldsymbol{a}} = \boldsymbol{a} - \boldsymbol{v}_1 - \boldsymbol{v}_2,$$

problem (4.34) is reduced to the homogeneous problem

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q = \widehat{\boldsymbol{f}} & \text{in } \Sigma_\infty^\infty, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Sigma_\infty^\infty, \\ \boldsymbol{u} = \widehat{\boldsymbol{a}} & \text{on } \Sigma,\ \forall t \geq 0, \\ \boldsymbol{u}(0) = 0, & \boldsymbol{u} \to 0 \text{ for } x_3 \to +\infty. \end{cases} \tag{4.42}$$

Here $\widehat{a}^3 = 0$ by construction, and from $\boldsymbol{v}_2(0) = 0$ and $\boldsymbol{a}(0) = \boldsymbol{v}_0 = \boldsymbol{v}_1(0)$ we get $\widehat{\boldsymbol{a}}(0) = 0$. By theorem 2.2.4, both $\widehat{\boldsymbol{f}}$ and $\widehat{\boldsymbol{a}}$ can be extended to zero in $(-\infty, 0]$ preserving regularity, and we can use the Laplace transform in time L to reduce the above problem to

$$\begin{cases} \lambda \boldsymbol{u}' - \nu\Delta\boldsymbol{u}' + \nabla q' = L\widehat{\boldsymbol{f}} & \text{in } \Sigma^\infty, \\ \nabla\cdot\boldsymbol{u}' = 0 & \text{in } \Sigma^\infty, \\ \boldsymbol{u}' = L\widehat{\boldsymbol{a}} & \text{on } \Sigma. \end{cases}$$

This problem is uniquely solvable for $\operatorname{Re}\lambda > \gamma' > 0$ using problems (4.26) and (4.29). Thus taking inverse Laplace transform and setting $\boldsymbol{u}' = L\boldsymbol{u}$, $p' = Lp$, we get a solution to (4.42) and thus to (4.34).

We now prove the estimates, letting

$$\Sigma^\infty_{\mathbb{R}} = \{(x,t) \in \Sigma^\infty \times \mathbb{R}\} \quad \text{and} \quad \Sigma_{\mathbb{R}} = \{(x,t) \in \Sigma \times \mathbb{R}\}.$$

From (4.28), (4.33) for the transformed problem and using the properties of the Laplace transform stated in section 2.4, for $\gamma > \gamma'$ we obtain the weighted estimate

$$\|\boldsymbol{u}\|^2_{W^{l+2,\frac{l}{2}+1}_{2,\gamma}(\Sigma^\infty_{\mathbb{R}})} + \|\nabla q\|^2_{W^{l,\frac{l}{2}}_{2,\gamma}(\Sigma^\infty_{\mathbb{R}})} \le c(\gamma)(\|\widehat{\boldsymbol{f}}\|^2_{W^{l,\frac{l}{2}}_{2,\gamma}(\Sigma^\infty_{\mathbb{R}})} + \|\widehat{\boldsymbol{a}}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2,\gamma}(\Sigma_{\mathbb{R}})}).$$

Notice that by theorem 2.3.1,

$$\begin{aligned} \|\widehat{\boldsymbol{f}}\|^2_{W^{l,\frac{l}{2}}_{2,\gamma}(\Sigma^\infty_{\mathbb{R}})} + \|\widehat{\boldsymbol{a}}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2,\gamma}(\Sigma_{\mathbb{R}})} &\le \|\widehat{\boldsymbol{f}}\|^2_{W^{l,\frac{l}{2}}_{2,\gamma}(\Sigma^\infty_\infty)} + \|\widehat{\boldsymbol{a}}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2,\gamma}(\Sigma_\infty)} \\ &\le c(\gamma, 2T_0)(\|\widehat{\boldsymbol{f}}\|^2_{W^{l,\frac{l}{2}}_{2}(\Sigma^\infty_{2T_0})} + \|\widehat{\boldsymbol{a}}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2}(\Sigma_{2T_0})}), \end{aligned}$$

since $\widehat{\boldsymbol{f}}$ and $\widehat{\boldsymbol{a}}$ vanish for $t \ge T_0 \ge T+1$. Moreover, from (4.41) we have

$$\begin{aligned} &\|\widehat{\boldsymbol{f}}\|^2_{W^{l,\frac{l}{2}}_{2}(\Sigma^\infty_{2T_0})} + \|\widehat{\boldsymbol{a}}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2}(\Sigma_{2T_0})} \\ &\qquad \le c(\|\boldsymbol{f}\|^2_{W^{l,\frac{l}{2}}_{2}(\Sigma^\infty_\infty)} + \|\boldsymbol{a}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2}(\Sigma_\infty)} + \|\boldsymbol{v}_0\|^2_{W^{l+1}_2(\Sigma^\infty)}). \end{aligned}$$

Now (4.38) gives

$$\begin{aligned} \|\boldsymbol{u}\|^2_{W^{l+2,\frac{l}{2}+1}_{2,\gamma}(\Sigma^\infty_T)} + \|\nabla q\|^2_{W^{l,\frac{l}{2}}_{2,\gamma}(\Sigma^\infty_T)} &\le c(\gamma,T)(\|\boldsymbol{u}\|^2_{W^{l+2,\frac{l}{2}+1}_{2,\gamma}(\Sigma^\infty_{\mathbb{R}})} + \|\nabla q\|^2_{W^{l,\frac{l}{2}}_{2,\gamma}(\Sigma^\infty_{\mathbb{R}})}) \\ &\le c(\gamma,T)(\|\boldsymbol{f}\|^2_{W^{l,\frac{l}{2}}_{2}(\Sigma^\infty_T)} + \|\boldsymbol{a}\|^2_{W^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}_{2}(\Sigma^\infty_T)} + \|\boldsymbol{v}_0\|^2_{W^{l+1}_2(\Sigma^\infty)}), \end{aligned}$$

and summing back $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and using again (4.41) proves (4.35). For small T, notice that $\boldsymbol{u}$ and ∇p vanish for $t<0$; applying (2.29) and (4.39) gives

$$\begin{aligned}\|\boldsymbol{u}\|^2_{H_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)}+\|\nabla q\|^2_{H_2^{l,\frac{l}{2}}(\Sigma_T^\infty)}&\le ce^{2\gamma T}(\|\boldsymbol{u}\|^2_{W_{2,\gamma}^{l+2,\frac{l}{2}+1}(\Sigma_{\mathbb{R}}^\infty)}+\|\nabla q\|^2_{W_{2,\gamma}^{l,\frac{l}{2}}(\Sigma_{\mathbb{R}}^\infty)})\\ &\le c(\gamma,T_0)e^{2\gamma T}(\|\boldsymbol{f}\|^2_{H_2^{l,\frac{l}{2}}(\Sigma_T^\infty)}+\|\boldsymbol{a}\|^2_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T^\infty)}+\|\boldsymbol{v}_0\|^2_{W_2^{l+1}(\Sigma^\infty)}).\end{aligned}$$

Since by (2.29) and (4.41), it holds

$$\|\boldsymbol{v}_1+\boldsymbol{v}_2\|_{H_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)}\le c\|\boldsymbol{v}_1+\boldsymbol{v}_2\|_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_\infty^\infty)}\le c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Sigma^\infty)},$$

we can add back $\boldsymbol{v}_1+\boldsymbol{v}_2$ to $\boldsymbol{u}$ and obtain (4.37).

Let us prove uniqueness in $[0,T)$. To this end it suffice to show that the only solution $(\boldsymbol{v},\nabla p)\in W_2^{l+2,\frac{l}{2}+1}(Q_T)\times W_2^{l,\frac{l}{2}}(Q_T)$ to (4.34) in $[0,T)$ with $\boldsymbol{f}=\boldsymbol{a}=\boldsymbol{v}_0=0$ is the vanishing one. Let $(\boldsymbol{v},\nabla p)$ be such a solution: we extend it for $t\ge T$ with $\nabla\cdot\boldsymbol{v}=0$ and $\boldsymbol{v}\lfloor_\Sigma=0$, supposing $(\boldsymbol{v},p)=0$ for $t\ge T+1$, and since $\frac{l}{2}<\frac{1}{2}$, we can extend it to $t<0$ as zero. The extension (still denoted by $(\boldsymbol{v},\nabla p)$) belongs to $W_{2,\gamma}^{l+2,\frac{l}{2}+1}(\Omega\times\mathbb{R})\times W_{2,\gamma}^{l,\frac{l}{2}}(\Omega\times\mathbb{R})$, and if $\boldsymbol{f}=\boldsymbol{v}_{,t}-\nu\Delta\boldsymbol{v}+\nabla p$, clearly $\boldsymbol{f}\in W_{2,\gamma}^{l,\frac{l}{2}}(\Omega\times\mathbb{R})$. We define $\boldsymbol{f}_T(x,t):=\boldsymbol{f}(x,t+T)$, which also belongs to $W_{2,\gamma}^{l,\frac{l}{2}}(\Omega\times\mathbb{R})$ since $\boldsymbol{f}(x,t)=0$ for $t\le T$, and let $(\boldsymbol{v}_{,t},\nabla p_T)$ be the solution constructed above in the infinite time interval for right hand side $\boldsymbol{f}_T$, and vanishing initial data and Dirichlet boundary condition. Taking Laplace transform and using property 2 of theorem 2.4.1 together with the uniqueness of the solution to the resulting problem of the type (4.42), we get that

$$L(\boldsymbol{v}_{,t},\nabla p_T)(x,z)=e^{Tz}L(\boldsymbol{v},\nabla p),$$

which implies

$$(\boldsymbol{v}_{,t}(x,t),\nabla p_T(x,t))=(\boldsymbol{v}(x,t+T),\nabla p(x,t+T)).$$

Since by construction $\boldsymbol{v}_{,t}=\nabla p_T=0$ for $t<0$, this gives $\boldsymbol{v}=\nabla p=0$ for $t<T$ and the claimed uniqueness in $[0,T)$.

To obtain (4.36), assume (4.35) for any solution, with a constant $c(T)$ bounded for $1\le T\le 2$. For $T\ge 1$ arbitrary, we consider a partition of unity $\varphi_k=\varphi(t-\frac{3}{4}k)$ with $\varphi=1$ for $|t|\le\frac{1}{4}$, $\varphi=0$ iff $|t|\ge\frac{1}{2}$. We define

$$T_k=\max\{0,\frac{3}{4}k-\frac{1}{2}\},\quad k=0,\dots M:=\left[\frac{4}{3}(T+\frac{1}{2})\right]+1.$$

Finally we modify φ_M in such a way that $\varphi_M=1$ for $t\ge T_M+\frac{1}{4}$. The supports of the φ_k in $[0,T]$ have diameter in $[1,2]$, and at most two supports intersect at a time. Moreover, T_k is the starting point of the support of φ_k. The couple $(\boldsymbol{u}_k, q_k) := \varphi_k(\boldsymbol{v}, p)$ solves (4.34) in $I_k := [T_k, \min\{T_k+1, T\}]$ with right hand side $\varphi_k \boldsymbol{f} + \varphi_{k,t}\boldsymbol{v}$ and starting value at $t = T_k$ equal zero for $k \geq 1$, $\boldsymbol{v}_0$ for $k = 0$. Applying (4.35) to these solutions, and (2.20), we have

$$\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)} + \|\nabla p\|_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} = \|\sum_{k=0}^{M} \boldsymbol{u}_k\|^2_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)} + \|\sum_{k=0}^{M} \nabla q_k\|_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} \\
&\quad \leq \sum_{k=0}^{M} \|\boldsymbol{u}_k\|^2_{W_2^{l+2,\frac{l}{2}+1}(\Sigma^\infty \times I_k)} + \|\nabla q_k\|_{W_2^{l,\frac{l}{2}}(\Sigma^\infty \times I_k)} \\
&\quad \leq c \sum_{k=0}^{M} \|\varphi_k \boldsymbol{f}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma^\infty \times I_k)} + \|\varphi_k \boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma^\infty \times I_k)} + \|\varphi_{k,t}\boldsymbol{v}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma^\infty \times I_k)} \\
&\quad \leq c \sum_{k=0}^{M} \|\boldsymbol{f}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma^\infty \times I_k)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma^\infty \times I_k)} + \|\boldsymbol{v}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma^\infty \times I_k)} \\
&\quad \leq 2c(\|\boldsymbol{f}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T^\infty)} + \|\boldsymbol{v}\|^2_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)}),
\end{aligned}$$

where c is a constant which depends only on φ_k and $\sup_{1\leq s\leq 2} c(s)$ from (4.35), and thus is independent of T. Now the interpolation inequality

$$\|\boldsymbol{v}\|_{W_2^{l,\frac{l}{2}}(\Sigma_T^\infty)} \leq \varepsilon \|\boldsymbol{v}\|_{W_2^{l+2,\frac{l}{2}+1}(\Sigma_T^\infty)} + c(\varepsilon)\|\boldsymbol{v}\|_{L^2(\Sigma_T^\infty)},$$

gives (4.36) (notice again that $c(\varepsilon)$ is independent of T for T bounded away from 0). □

The estimate (4.36) is performed only to illustrate the method, and it can be improved through a spectral analysis for problem (4.24). Indeed the fact that theorem 4.1.3 holds for any $\lambda \geq 0$ implies that the spectrum of the linear problem lies in the complex left semiplane. Once one can show that the linear operator is compact (and thus the real part of its spectrum is bounded by $\gamma < 0$), standard methods allows to get rid of the $\|\boldsymbol{v}\|_{L^2(\Sigma_T^\infty}$ term in (4.36). However, the compactness properties of the linear problem is not *a-priori* clear, due to the unboundedness of the domain involved, and a detailed discussion is omitted.

4.2 Parameter dependent linear problem

In this section we prove the solvability and the coercive estimates, for sufficiently large $\operatorname{Re}\lambda$, of the problem

$$\begin{cases} \lambda \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = h = \nabla\cdot\boldsymbol{F} & \text{in } \Omega, \\ \mathbb{T}(\boldsymbol{u}, q) + \sigma L\rho\boldsymbol{N} = \boldsymbol{d}, & \text{on } \mathcal{G}, \\ \lambda\rho + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = g, & \text{on } \mathcal{G}, \\ \boldsymbol{u} = \boldsymbol{a}, & \text{on } \Sigma, \end{cases} \tag{4.43}$$

where $a^3 = F^3 = 0$ on Σ and, recalling (3.5),

$$\begin{aligned} L\rho &= -\frac{1}{\sqrt{g_b}}\Delta_{\mathcal{G}}\rho + \boldsymbol{b}\cdot\nabla\rho \\ &= -\frac{1}{\sqrt{g_b}}\Delta_{\mathcal{G}}\rho - \frac{1}{g_b^{\frac{5}{2}}}\nabla'\phi_b\cdot\nabla'|\nabla'\phi_b|^2\nabla\phi_b\cdot\nabla'\rho + \frac{\nabla'|\nabla'\phi_b|^2\cdot\nabla'\rho}{g_b^{\frac{3}{2}}}. \end{aligned}$$

We start with a lemma which allows to extend the equation $h = \nabla\cdot\boldsymbol{F}$ from Ω to $\mathbb{R}^3_+$ controlling the norms.

Lemma 4.2.1 *Let $h, h' \in W_2^{l+1}(\Omega)$, $\boldsymbol{F} \in W_2^{l+2}(\Omega)$ be Σ-periodic and such that*

$$h = \nabla\cdot\boldsymbol{F} + h',$$

holds in Ω. There exist a Σ-periodic extensions $\overline{h}$ of h to $\mathbb{R}^3_+$ and $\overline{\boldsymbol{F}} \in W_2^{l+2}(\Sigma^\infty)$ such that

$$\overline{h} = \nabla\cdot\overline{\boldsymbol{F}},$$

in $\mathbb{R}^3_+$, $\overline{h} = \overline{\boldsymbol{F}} = 0$ for sufficiently large x_3, $\overline{\boldsymbol{F}}\cdot\boldsymbol{n} = \boldsymbol{F}\cdot\boldsymbol{n}$ on Σ and

$$\begin{aligned} &\|\overline{h}\|_{W_2^{l+1}(\Sigma^\infty)} \le c\|h\|_{W_2^{l+1}(\Omega)}, \\ &\|\overline{\boldsymbol{F}}\|_{L^2(\Sigma^\infty)} \le c(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|h'\|_{L^2(\Omega)}). \end{aligned} \tag{4.44}$$

Proof. Let ψ be the periodic solution of

$$\begin{cases} \Delta\psi = h = \nabla\cdot\boldsymbol{F} + h' & \text{in } \Omega, \\ \psi = 0 & \text{on } \mathcal{G}, \\ \frac{\partial\psi}{\partial\boldsymbol{n}} = \boldsymbol{F}\cdot\boldsymbol{n} & \text{on } \Sigma. \end{cases} \tag{4.45}$$

Standard elliptic estimates guarantee that

$$\|\psi\|_{\mathring{W}_2^{l+3}(\Omega)} \le c\|h\|_{W_2^{l+1}(\Omega)}$$

and the weak formulation of (4.45) reads

$$\int_\Omega \nabla\psi \cdot \nabla\eta dx = \int_\Omega \boldsymbol{F} \cdot \nabla\eta - h'\eta dx$$

for all $\eta \in C^\infty(\overline{\Omega})$ such that $\eta\lfloor_{\mathcal{G}}= 0$, which gives

$$\|\nabla\psi\|^2_{L^2(\Omega)} \leq \|\boldsymbol{F}\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + \|h'\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}. \tag{4.46}$$

Since $\psi = 0$ on $\mathcal{G}$, a form of Poincaré inequality gives

$$\|\psi\|_{L^2(\Omega)} \leq c\|\nabla\psi\|_{L^2(\Omega)},$$

and thus (4.46) becomes

$$\|\nabla\psi\|_{L^2(\Omega)} \leq c(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)}). \tag{4.47}$$

We now extend $\nabla\psi$ to as a vector field $\overline{\boldsymbol{F}}$ defined in the whole $\mathbb{R}^3_+$, with controlled norms and supposing that it vanishes for sufficiently large x_3. Setting then $\overline{h} := \nabla \cdot \overline{\boldsymbol{F}}$ gives the claim, since

$$\|\overline{h}\|_{W_2^{l+1}(\Sigma^\infty)} \leq \|\overline{\boldsymbol{F}}\|_{W_2^{l+2}(\Sigma^\infty)} \leq c\|\nabla\psi\|_{W_2^{l+2}(\Omega)} \leq c\|h\|_{W_2^{l+1}(\Omega)},$$

while the inequality for $\overline{\boldsymbol{F}}$ follows from (4.47). □

We will use the following proposition

Proposition 4.2.2 *Let $l \geq 0$. For any sufficiently large $\operatorname{Re}\lambda$, there is a unique periodic solution to the problem*

$$\begin{cases} \lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} = 0 & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma, \end{cases} \tag{4.48}$$

for any $\boldsymbol{f} \in W_2^l(\Omega)$, and it satisfies the inequality

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Omega)} + \\ &+ \|q\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|q\|^2_{L^2(\mathcal{G})} \leq c(\|\boldsymbol{f}\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Omega)}). \end{aligned} \tag{4.49}$$

Proof. The existence of a weak solution $\boldsymbol{u} \in \mathcal{J} := \{\boldsymbol{v} \in W_2^1(\Omega) : \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{v}\lfloor_\Sigma = 0\}$ can be proved through Lax-Milgram theorem, since the weak formulation of (4.48) is

$$\lambda \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\phi} + \frac{\nu}{2} \int_\Omega \mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{\phi}) dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\phi}, \qquad \forall \boldsymbol{\phi} \in W_2^1(\Omega), \quad \boldsymbol{\phi}\lfloor_\Sigma = 0.$$

Indeed, Korn inequality gives coerciveness of the bilinear form defined from the left hand side, and Sobolev inequality the continuity of the right hand side, for $\boldsymbol{f} \in L^2(\Omega)$. The pressure can be recovered through standard methods, see e.g. [32]. The estimate (4.49) follows, for example, from Shauder localisation method and the analogous estimates for the related problems in the half-space. □

The following lemma will be useful to estimate perturbed linear differential operator. It has easy generalisation for arbitrary dimensions, but we will prove it only for dimension 2 and 3.

Lemma 4.2.3 *Let η, ψ be smooth functions such that* supp $\eta \in B(0,\delta) \subset \Omega$, *for some smooth bounded $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. If*

$$\sup_{B(0,\delta)} |\eta| + \delta||\nabla\eta| + \delta^2|\nabla^2\eta| + \frac{|\psi|}{\delta} + |\nabla\psi| + |\nabla^2\psi| \leq k, \tag{4.50}$$

for k independent of δ, then for any $f \in W_2^l(\Omega)$,

$$\|\eta\psi f\|^2_{W_2^l(\Omega)} \leq c_1\delta\|f\|^2_{W_2^l(\Omega)} + c_2(\delta)\|f\|^2_{L^2(\Omega)},$$

where $c_2(\delta)$ depends on δ, η, ψ, l and k and c_1 only on k.

Proof. We consider an extension f^* of f to the whole $\mathbb{R}^N$, with controlled norm. Let us consider the case $N = 3$ first. From (4.50) we get

$$\|\eta\psi\|^2_{W_2^2(\mathbb{R}^3)} \leq ck^4\delta.$$

If $l \leq \frac{3}{2}$ we use (2.7) to obtain

$$\|\eta\psi f\|^2_{W_2^l(\Omega)} \leq c\|\eta\psi f^*\|^2_{W_2^l(\mathbb{R}^3)} \leq \|\eta\psi\|^2_{W_2^2(\mathbb{R}^3)}\|f^*\|_{W_2^l(\mathbb{R}^3)} \leq ck^4\delta\|f\|_{W_2^l(\Omega)},$$

otherwise we use (2.8) with $\min\{2, l\} > s > \frac{3}{2}$, to obtain

$$\begin{aligned}\|\eta\psi f\|^2_{W_2^l(\Omega)} &\leq \|\eta\psi\|^2_{W_2^s(\mathbb{R}^3)}\|f^*\|^2_{W_2^l(\mathbb{R}^3)} + \|\eta\psi\|^2_{W_2^l(\mathbb{R}^3)}\|f^*\|^2_{W_2^s(\mathbb{R}^3)} \\ &\leq \delta\|f^*\|^2_{W_2^l(\mathbb{R}^3)} + c(\delta)\|f^*\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

where we used the interpolation inequality

$$\|f^*\|^2_{W_2^s(\mathbb{R}^3)} \leq \delta \|f^*\|^2_{W_2^l(\mathbb{R}^3)} + c(\delta)\|f^*\|^2_{L^2(\mathbb{R}^3)}.$$

In the case $N = 2$ we proceed in a similar way, this time using the norm $\|\eta\psi\|_{W_2^{\frac{3}{2}}(\mathbb{R}^2)}$ instead of the W_2^2 norm in the estimates. It holds

$$\|\eta\psi\|^2_{W_2^{\frac{3}{2}}(\mathbb{R}^2)} \leq ck\delta,$$

which follows from

$$\|\eta\psi\|^2_{W_2^2(\mathbb{R}^2)} \leq ck^4, \qquad \|\eta\psi\|^2_{L^2(\mathbb{R}^2)} \leq ck^4\delta^4,$$

and the interpolation inequality

$$\|\eta\psi\|^2_{W_2^{\frac{3}{2}}(\mathbb{R}^2)} \leq c(\delta\|\eta\psi\|^2_{W_2^2(\mathbb{R}^2)} + \frac{1}{\delta^3}\|\eta\psi\|^2_{L^2(\mathbb{R}^2)}) \leq ck^4\delta.$$

The rest of the proof is entirely analogous. □

We can now prove the existence theorem.

Theorem 4.2.4 *Let $l \geq 0$. For any sufficiently large* $\operatorname{Re}\lambda$, *there exists a unique periodic solution of* (4.43), *for any choice of periodic $\boldsymbol{f} \in W_2^l(\Omega)$, $\boldsymbol{d} \in W_2^{l+\frac{1}{2}}(\mathcal{G})$, $g \in W_2^{l+\frac{3}{2}}(\Omega)$, $h \in W_2^{l+1}(\Omega)$ and $\boldsymbol{F} \in W_2^1(\Omega)$ with $F^3\lfloor_\Sigma = 0$ and this solution satisfies the estimate:*

$$\begin{gathered}\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Omega)} + \|q\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} \\ + |\lambda|^{l+\frac{1}{2}}\|q\|^2_{L^2(\mathcal{G})} + \|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})} \leq \\ c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Omega)} + \|h\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + \\ |\lambda|^{l+\frac{1}{2}}\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|g\|^2_{W_2^1(\mathcal{G})} + \|\boldsymbol{a}\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{3}{2}}\|\boldsymbol{a}\|_{L^2(\Sigma)}\big),\end{gathered} \tag{4.51}$$

and the inequality

$$\begin{gathered}\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Omega)} + \|q\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} \\ + |\lambda|^{l+\frac{1}{2}}\|q\|^2_{L^2(\mathcal{G})} + \|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{3}{2}}\|\lambda\rho\|^2_{L^2(\mathcal{G})} \leq \\ c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Omega)} + \|h\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} \\ + |\lambda|^{l+\frac{1}{2}}\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{3}{2}}\|g\|^2_{L^2(\mathcal{G})} + \|\boldsymbol{a}\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{3}{2}}\|\boldsymbol{a}\|_{L^2(\Sigma)}\big),\end{gathered} \tag{4.52}$$

Proof. We first show that it suffice to prove the existence of a solution of

$$\begin{cases} \lambda \boldsymbol{u} - \nu\Delta \boldsymbol{u} + \nabla q = 0 & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} + \sigma L\rho \boldsymbol{N} = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \lambda\rho + \nabla'\phi \cdot \boldsymbol{u} - u^3 + \boldsymbol{v}_b \cdot \nabla'\rho = g & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma. \end{cases} \tag{4.53}$$

Indeed one can extend $\boldsymbol{f}$ with preservation of class and controlled norms, as well as apply lemma 4.2.1 to $\boldsymbol{F}$ and h. We consider a solution $\boldsymbol{v}_1$, p_1 of (4.24) with those right hand sides, then a solution $\boldsymbol{v}_2$, p_2 of (4.48) with right hand side ∇p_1. Given a solution $\boldsymbol{v}_3$, q_3, ρ of (4.53) with right hand sides

$$\widetilde{\boldsymbol{d}} := \boldsymbol{d} - \nu S(\boldsymbol{v}_1), \qquad \widetilde{g} := g - \nabla'\phi_b \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2) + v_1^3 + v_2^3,$$

the triple

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3, \quad p_2 + p_3, \quad \rho,$$

satisfies (4.43). From the estimates (4.25) and (4.49) for problems (4.24) and (4.48) respectively, we readily get

$$\begin{aligned} &\|\widetilde{\boldsymbol{d}}\|^2_{W_2^{l+\frac12}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\widetilde{\boldsymbol{d}}\|^2_{L^2(\mathcal{G})} + \|\widetilde{g}\|^2_{W_2^{l+\frac32}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\widetilde{g}\|^2_{W_2^1(\mathcal{G})} \\ &\quad \le c\big(\|\boldsymbol{d}\|^2_{W_2^{l+\frac12}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac32}(\mathcal{G})} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\mathcal{G})} \\ &\quad + \|\boldsymbol{v}_1\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{v}_1\|^2_{L^2(\Sigma^\infty)} + \|\boldsymbol{v}_2\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{v}_2\|^2_{L^2(\Sigma^\infty)}\big) \\ &\quad \le c\big(\|\boldsymbol{f}\|^2_{W_2^l(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Sigma^\infty)} + \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} \\ &\quad + \|\boldsymbol{d}\|^2_{W_2^{l+\frac12}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac32}(\mathcal{G})} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\mathcal{G})} \\ &\quad + \|\boldsymbol{a}\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac32}\|\boldsymbol{a}\|_{L^2(\Sigma)}\big), \end{aligned}$$

Also by the same estimates, it is clear that a bound of the form

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Omega)} + \|q\|^2_{W_2^{l+\frac12}(\mathcal{G})} \\ &\quad + |\lambda|^{l+\frac12}\|q\|^2_{L^2(\mathcal{G})} + \|\rho\|^2_{W_2^{l+\frac52}(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac32}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})} \\ &\quad \le c\big(\|\boldsymbol{d}\|^2_{W_2^{l+\frac12}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac32}(\mathcal{G})} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\mathcal{G})}\big), \end{aligned}$$

for the solution of (4.53), implies (4.51) for the solution just constructed.

We now proof the existence of a solution to (4.53). For any fixed $\delta > 0$, we consider a finite covering of $\mathcal{G}$ with balls $B(x_i, \delta)$, $x_i \in \mathcal{G}$, and this can be done in such a way that the number of balls containing any point of Ω_b is bounded independently of δ. We now choose a periodic partition of unity φ_i, with each φ_i having support in $B(x_i, 2\delta)$, $\sum_i \varphi_i = 1$ on $V := \Omega_b \cap \cup_i B(x_i, \delta) \subset \Omega_b \cap \cup_i B(x_i, 2\delta) =: U$. For each φ_i we choose η_i, with support in $B(x_i, 3\delta)$ such that $\varphi_i = \eta_i \varphi_i$. Any norm of the φ_i, η_i is bounded by a suitable constant depending only on δ, and in particular we can suppose

$$|\nabla \varphi_i| + |\nabla \eta_i| \leq \frac{c}{\delta}, \qquad |\nabla^2 \varphi_i| + |\nabla^2 \eta_i| \leq \frac{c}{\delta}^2. \tag{4.54}$$

Moreover, $\boldsymbol{N}_i$ will be the normal to $\mathcal{G}$ at x_i, $\boldsymbol{V}_i = \boldsymbol{v}_b(x_i)$, Π_i the projection on the tangent space to $\mathcal{G}$ at x_i, C_i an isometry bringing $\boldsymbol{N}_i$ to $-\boldsymbol{e}_3$ and we will write $\boldsymbol{N}'_i = C_i \boldsymbol{N}$, $\boldsymbol{V}'_i = C_i \boldsymbol{V}_i$, $\mathcal{G}' = \mathcal{C}\mathcal{G}$. For each i we will set, as in (3.1)

$$y_i = C_i x, \qquad y_i = e_{\phi_i}(z_i)$$

where ϕ_i is defined through $C_i(x', \phi_b(x')) = (z'_i, -\phi_i(z'_i))$ and e_{ϕ_i} is the transformation defined in (3.1). Here we suppose that $\phi_i^* = \theta_i(z_{i3})\phi_i(z'_i)$ with $\theta_i = 1$ on the support of $\varphi_i(x(z_i))$. Recall that for any isometry C, it holds

$$\nabla_x = C^T \nabla_y = C^{-1} \nabla_y, \qquad \Delta_x = C_{ij} C_{kj} \frac{\partial^2}{\partial y_i \partial y_k} = \Delta_y, \tag{4.55}$$

and thus these formulas hold for each of the C_i with respect to the coordinates y_i. Moreover, as has been calculated in section 4.1, $\nabla_{y_i} = \mathcal{L}_i^{-T} \nabla_{z_i}$, where $\mathcal{L}_i$ is the Jacoby matrix of the transformation e_{ϕ_i}.

Since we supposed ϕ_b smooth, it holds, for $z' \in \Sigma \cap B(0, 2\delta)$

$$|\phi_i(z')| \leq c|z'|^2, \qquad |\nabla' \phi_i(z')| \leq c|z'|. \tag{4.56}$$

Notice that by this estimate and (4.54), (4.50) holds for $\eta = \eta_i$ and $\psi = \frac{\partial \phi_i}{\partial z_j}$. Therefore, by the previous lemma and proposition 2.1.3, for any function h and $m = m(z, \phi_i, \nabla \phi_i)$ it holds

$$\|\eta_i \frac{\partial \phi_i}{\partial z_j} m h\|^2_{W_2^\mu(\Sigma)} \leq c \|\eta_i \frac{\partial \phi_i}{\partial z_j} h\|^2_{W_2^\mu(\Sigma)} \leq c\delta \|h\|^2_{W_2^\mu(\Sigma)} + c(\delta)\|h\|^2_{L^2(\Sigma)}, \tag{4.57}$$

and the same inequality for the norms on Σ^∞, with constants depending also on m. In the following we will shorten somewhat the notation. We may for example simply write $z(x) = e_{\phi_i}^{-1}(C_i(x))$ (and similar expressions) whenever the dependance on i will be clear.

We define a linear operator $R(\boldsymbol{d}, g) = (\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho})$, where we will construct $\widehat{\boldsymbol{u}}$, $\widehat{q}$ and $\widehat{\rho}$ linearly in $\boldsymbol{d}$ and g in the following. We let

$$\boldsymbol{v} = \sum_i \eta_i C_i^{-1} \boldsymbol{v}_i(z_i(x)), \qquad p = \sum_i \eta_i p_i(z_i(x)) \qquad \rho = \sum_i \eta_i \rho_i(z_i(x))$$

where $\boldsymbol{v}_i = \boldsymbol{v}_i(z)$, $p_i = p_i(z)$, ρ_i are periodic and solve a problem of the type (4.5), namely

$$\begin{cases} \lambda \boldsymbol{v}_i - \nu\Delta_z \boldsymbol{v}_i + (\boldsymbol{V}'_i \cdot \nabla_z)\boldsymbol{v}_i + \nabla_z p_i = 0, & \text{in } \mathbb{R}^3_+ \\ \nabla_z \cdot \boldsymbol{v}_i = 0, & \text{in } \mathbb{R}^3_+ \\ \nu\big(\frac{\partial v_i^3}{\partial z_j} + \frac{\partial v_i^j}{\partial z_3}\big) = \eta_i (C_i \boldsymbol{d})^j(x(z)), \quad j = 1,2 & \text{in } \mathbb{R}^2 \\ -p_i + 2\nu \frac{\partial v_i^3}{\partial z_3} - \sigma\Delta'\rho_i = \varphi_i (C_i \boldsymbol{d})^3(x(z)), & \text{in } \mathbb{R}^2 \\ \lambda\rho_i + \boldsymbol{V}'_i \cdot \nabla'\rho_i + v_i^3 = \varphi_i g(x(z)), & \text{in } \mathbb{R}^2, \end{cases} \tag{4.58}$$

We then set

$$\begin{cases} \lambda \boldsymbol{v} - \nu\Delta_x \boldsymbol{v} + \nabla_x p =: \widehat{\boldsymbol{f}}, \\ \nabla_x \cdot \boldsymbol{v} =: \widehat{h}, \\ \mathbb{T}(p, \boldsymbol{v})\boldsymbol{N} + \sigma L\rho \boldsymbol{N} =: \boldsymbol{d} + \boldsymbol{A}(\boldsymbol{d}, g), \\ \lambda\rho + \nabla'\phi_b \cdot \boldsymbol{v} - v^3 + \boldsymbol{v}_b \cdot \nabla'\rho =: g + A(\boldsymbol{d}, g). \end{cases}$$

noting that both $\boldsymbol{v}$ and p vanish in a neighbourhood of Σ. We can write the divergence h as

$$\widehat{h} = \nabla \cdot \widehat{\boldsymbol{F}} + \widehat{h}',$$

for sufficiently regular $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$ that will be specified later. We can apply lemma 4.2.1 on $\widehat{h}$, $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$ and suppose that $\widehat{h}' = 0$ and $\widehat{h}$, $\widehat{\boldsymbol{F}}$ are defined in the whole Σ^∞ with controlled norms. We also extend $\widehat{\boldsymbol{f}}$ to Σ^∞ with preservation of class and controlled norm. We keep the notation unchanged for the extensions of $\widehat{h}$ and $\widehat{\boldsymbol{f}}$, and call $\overline{\boldsymbol{F}}$ the vector arising from lemma 4.2.1. We define $\overline{\boldsymbol{v}}_1$, $\overline{p}_1$ as the periodic solution to

$$\begin{cases} \lambda \overline{\boldsymbol{v}}_1 - \nu\Delta \overline{\boldsymbol{v}}_1 + \nabla \overline{p}_1 = \widehat{\boldsymbol{f}} & \text{in } \mathbb{R}^3_+, \\ \nabla \cdot \overline{\boldsymbol{v}}_1 = -\widehat{h} = -\nabla \cdot \overline{\boldsymbol{F}} & \text{in } \mathbb{R}^3_+, \\ \overline{\boldsymbol{v}}_1 = 0 & \text{on } \Sigma, \end{cases}$$

(notice that $\overline{F}^3\lfloor_\Sigma = 0$, since all the η_i vanish in a neighbourhood of Σ), for which it holds the estimate (4.33), which, together with (4.44), implies

$$\begin{aligned} &\|\overline{\boldsymbol{v}}_1\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\overline{\boldsymbol{v}}_1\|^2_{L^2(\Omega)} + \|\nabla\overline{p}_1\|^2_{W_2^l(\Omega)} + |\lambda|^l \|\nabla\overline{p}_1\|^2_{L^2(\Omega)} \\ &\le c\big(\|\widehat{\boldsymbol{f}}\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\widehat{\boldsymbol{f}}\|^2_{L^2(\Omega)} + \|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)})\big). \end{aligned} \tag{4.59}$$

Finally we let $\overline{\boldsymbol{v}}_2$, $\overline{p}_2$ be the solution of (4.48) with right hand side $\nabla\overline{p}_1$. We let

$$R(\boldsymbol{d}, g) = (\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho}) := (\boldsymbol{v} + \overline{\boldsymbol{v}}_1 + \overline{\boldsymbol{v}}_2, p + \overline{p}_2, \rho). \tag{4.60}$$

This triple solves

$$\begin{cases} \lambda\widehat{\boldsymbol{u}} - \nu\Delta\widehat{\boldsymbol{u}} + \nabla\widehat{q} = 0 & \text{in } \Omega, \\ \nabla\cdot\widehat{\boldsymbol{u}} = 0 & \text{in } \Omega, \\ \mathbb{T}(\widehat{\boldsymbol{u}},\widehat{q})\boldsymbol{N} + \sigma L\widehat{\rho}\boldsymbol{N} = \boldsymbol{d} + \widehat{\boldsymbol{A}}(\boldsymbol{d},g) & \text{on } \mathcal{G}, \\ \lambda\widehat{\rho} + \nabla'\phi_b\cdot\widehat{\boldsymbol{u}}(e_\rho) - \widehat{u}^3(e_\rho) + \boldsymbol{v}_b\cdot\nabla'\widehat{\rho} = g + \widehat{A}(\boldsymbol{d},g) & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma. \end{cases}$$

where

$$\widehat{\boldsymbol{A}}(\boldsymbol{d},g) = \boldsymbol{A}(\boldsymbol{d},g) + \nu\mathbb{D}(\overline{\boldsymbol{v}}_1)\boldsymbol{N}, \qquad \widehat{A}(\boldsymbol{d},g) = A(\boldsymbol{d},g) + \nabla'\phi_b\cdot(\overline{\boldsymbol{v}}_1+\overline{\boldsymbol{v}}_2) - \overline{v}_1^3 - \overline{v}_2^3.$$

We will prove that $(\widehat{\boldsymbol{A}},\widehat{A})$ is a contraction operator from $W_2^{l+\frac{1}{2}}(\mathcal{G})\times W_2^{l+\frac{3}{2}}(\mathcal{G})$ to itself, therefore establishing the invertibility of $I + (\widehat{\boldsymbol{A}},\widehat{A})$, and obtaining the solution $R(I + (\widehat{\boldsymbol{A}},\widehat{A}))^{-1}(\boldsymbol{d},g)$. Instead of using the usual norm, we will perform the estimates w.r.t. the equivalent norm

$$[\boldsymbol{d}]_\lambda^2 = \sum_i \|\varphi_i\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\varphi_i\boldsymbol{d}\|^2_{L^2(\mathcal{G})},$$

$$[g]_\lambda^2 = \sum_i \|\varphi_i g\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\varphi_i g\|^2_{W_2^1(\mathcal{G})}.$$

These norms are equivalent to the usual weighted norm, with constant independent of δ and λ: this follows from well known properties of partitions of unity, and more precisely

$$D^k\sum_i \varphi_i g = \sum_i \varphi_i D^k g,$$

for any derivative of order k $D^k g$, which certainly holds in a neighbourhood of $\mathcal{G}$. Thus in particular, it holds

$$([\boldsymbol{d}]_\lambda^2 + [g]_\lambda^2)^{\frac{1}{2}} \le c(\|\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}(\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^1(\mathcal{G})}))^{\frac{1}{2}}, \tag{4.61}$$

with constant independent of δ and λ, and we will denote by $\|(\boldsymbol{d},g)\|_\lambda$ the right hand side in the previous inequality.

We will split the estimates in several steps, always supposing $\delta < 1 < |\lambda|$ in the following.

1. Estimate of $\widehat{\boldsymbol{f}}$.
Transforming coordinates near each point x_i and using (4.55) and(4.58) we get the explicit expression

$$\begin{aligned}\widehat{\boldsymbol{f}} &= \sum_i \eta_i C_i^{-1}\left[-\nu(\Delta_y - \Delta_z)\boldsymbol{v}_i + (\nabla_y - \nabla_x)p_i\right]\\ &\quad - \sum_i C_i^{-1}\nu(2\nabla_x\eta_i\nabla_x\boldsymbol{v}_i + \boldsymbol{v}_i\Delta_x\eta_i + \frac{\eta_i}{\nu}(\boldsymbol{V}'_i\cdot\nabla_z)\boldsymbol{v}_i)\\ &\quad + \sum_i p_i\nabla_x\eta_i.\end{aligned}$$

The terms in $\widehat{\boldsymbol{f}}$ are of three kind: those of higher order (in the first line), those of lower order in $\boldsymbol{v}_i$ (on the second line) and the term $\sum_i p_i\nabla_x\eta_i$. We start estimating the lower order terms, writing them in the z coordinates. Recall that $\nabla_x = C_i^{-1}\mathcal{L}_i^{-T}\nabla_z$ near x_i, and clearly for any $r \geq 0$ and any i,

$$\|\mathcal{L}_i^{-T}\|_{W_2^r(\Sigma^\infty)} \leq c\|\phi_b\|_{W_2^{r+1}(\Sigma)},$$

by the explicit form (3.3). This allows to perform all the estimates equivalently in the z coordinates, and we have, by interpolation inequality

$$\begin{aligned}&\|\sum_i C_i^{-1}\nu(2\nabla_x\eta_i\nabla_x\boldsymbol{v}_i + \boldsymbol{v}_i\Delta_x\eta_i + \frac{\eta_i}{\nu}(\boldsymbol{V}'_i\cdot\nabla_z)\boldsymbol{v}_i)\|^2_{W_2^l(\Omega)}\\ &\quad +|\lambda|^l\|\sum_i C_i^{-1}\nu(2\nabla_x\eta_i\nabla_x\boldsymbol{v}_i + \boldsymbol{v}_i\Delta_x\eta_i + \frac{\eta_i}{\nu}(\boldsymbol{V}'_i\cdot\nabla_z)\boldsymbol{v}_i)\|^2_{L^2(\Omega)}\\ &\quad \leq c(\delta)\sum_i\|\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^l\|\boldsymbol{v}_i\|^2_{W_2^1(\Sigma^\infty)}\\ &\quad \leq \frac{c(\delta)}{|\lambda|}\sum_i\|\boldsymbol{v}_i\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{v}_i\|^2_{L^2(\Sigma^\infty)}\\ &\quad \leq \frac{c(\delta)}{|\lambda|}([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda),\end{aligned}$$

where we used the estimates of theorem 4.1.1 for the solution of (4.58). For the higher order terms, i.e.

$$\sum_i \eta_i C_i^{-1}(\nu(\Delta_y - \Delta_z)\boldsymbol{v}_i + (\nabla_y - \nabla_z)p_i),$$

we recall that the operators $\Delta_y - \Delta_z$ and $\nabla_y - \nabla_z$ have already been explicitly calculated in section 4.1. Recalling that $\nabla_y = \mathcal{L}_i^{-T}\nabla_z$ we have indeed

$$\nabla_y - \nabla_z = (\mathcal{L}_i^{-T} - I)\nabla_z,$$

$$\Delta_y - \Delta_z = (\mathcal{L}_i^{-T} - I)(\mathcal{L}_i^{-T} + I) : D_z^2 + \mathcal{L}_i^{-T} D_z \mathcal{L}_i^{-T} \nabla_z.$$

The last term in the previous formula is still a lower order term which can be estimated as before, while by (3.3), the terms involving $\mathcal{L}_i^{-T} - I$ have coefficients of the form $\frac{\partial \phi_i^*}{\partial z_k} m_k$ with smooth m_k's depending on z, ϕ_i^* and $\nabla \phi_i^*$. Therefore we have to estimate terms of the form

$$\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j}, \qquad \text{and} \qquad \eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l},$$

in the W_2^l and L^2 norm. The L^2 norm is readily estimated using (4.56), giving

$$\begin{aligned}
|\lambda|^l \sum_{i,k,j,l} &\|\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j}\|_{L^2(\Sigma^\infty)}^2 + \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l}\|_{L^2(\Sigma^\infty)}^2 \\
&\le c\delta^2 \sum_i |\lambda|^l (\|\nabla p_i\|_{L^2(\Sigma^\infty)}^2 + \|\boldsymbol{v}_i\|_{W_2^2(\Sigma^\infty)}^2) \\
&\le c\delta^2 \sum_i \|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)}^2 + |\lambda|^{l+2} \|\boldsymbol{v}_i\|_{L^2(\Sigma^\infty)}^2 + |\lambda|^l \|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2 \\
&\le c\delta^2 ([\boldsymbol{d}]_\lambda^2 + [g]_\lambda^2).
\end{aligned}$$

by the a-priori estimate for problem (4.58). For the W_2^l norm, suppose first that $l > 0$. We use (4.57), obtaining:

$$\begin{aligned}
&\sum_{i,k,j,l} \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j}\|_{W_2^l(\Sigma^\infty)}^2 + \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l}\|_{W_2^l(\Sigma^\infty)}^2 \\
&\le c \sum_i \delta \|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)}^2 + c(\delta) \|\boldsymbol{v}_i\|_{W_2^2(\Sigma^\infty)}^2 + \delta \|\nabla_z p_i\|_{W_2^l(\Sigma^\infty)}^2 + c(\delta) \|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2 \\
&\le c \sum_i \delta (\|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)}^2 + \|\nabla_z p_i\|_{W_2^l(\Sigma^\infty)}^2) + c(\delta)(\|\boldsymbol{v}_i\|_{L^2(\Sigma^\infty)}^2 + \|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2) \\
&\le c(\delta + \frac{c(\delta)}{|\lambda|^l})([\boldsymbol{d}]_\lambda^2 + [g]_\lambda^2),
\end{aligned}$$

by the interpolation inequality

$$c(\delta) \|\boldsymbol{v}_i\|_{W_2^2(\Sigma^\infty)}^2 \le \delta \|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)}^2 + c'(\delta) \|\boldsymbol{v}_i\|_{L^2(\Sigma)}^2,$$

and the coercive estimates for problem (4.58). For $l = 0$, (4.56) directly gives

$$\begin{aligned}
\sum_{i,k,j,l} &\|\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j}\|_{L^2(\Sigma^\infty)}^2 + \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l}\|_{L^2(\Sigma^\infty)}^2 \\
&\le c\delta^2 \sum_i (\|\boldsymbol{v}_i\|_{W_2^2(\Sigma^\infty)}^2 + \|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2) \le c\delta^2 ([\boldsymbol{d}]_\lambda^2 + [g]_\lambda^2)
\end{aligned}$$

To estimate the term $\sum_i p_i \nabla \eta_i$, we let

$$\overline{d}_i = -\int_\Sigma \varphi_i (C_i \boldsymbol{d})^3 (x(z)) dS$$

and use (4.6) and (4.7) to obtain

$$\begin{aligned}\|\sum_i p_i \nabla_x \eta_i\|^2_{L^2(\Omega)} &\le c(\delta) \sum_i \|p_i\|^2_{L^2(\Sigma^\infty)} \\ &\le c(\delta) \sum_i \|p_i - \overline{d}_i\|^2_{L^2(\Sigma^\infty)} + \|\varphi_i \boldsymbol{d}\|^2_{L^2(\mathcal{G})} \\ &\le c(\delta) \sum_i \|\varphi_i \boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|\varphi_i g\|^2_{W^1_2(\mathcal{G})},\end{aligned} \tag{4.62}$$

and thus

$$|\lambda|^l \|\sum_i p_i \nabla_x \eta_i\|^2_{L^2(\Omega)} \le \frac{c(\delta)}{\sqrt{|\lambda|}} ([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).$$

Moreover, by interpolation and the coercive estimates for problem (4.58)

$$\begin{aligned}\|\sum_i p_i \nabla_x \eta_i\|^2_{W^l_2(\Omega)} &\le c(\delta) \sum_i \|p_i\|^2_{W^l_2(\Sigma^\infty)} \\ &\le \delta \sum_i \|\nabla_z p_i\|^2_{W^l_2(\Sigma^\infty)} + c(\delta) \sum_i \|p_i\|^2_{L^2(\Sigma^\infty)} \\ &\le (\delta + \frac{c(\delta)}{\sqrt{|\lambda|}}) ([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda),\end{aligned}$$

where we used (4.62). All in all we have obtained

$$\|\widehat{\boldsymbol{f}}\|^2_{W^l_2(\Omega)} + |\lambda|^l \|\widehat{\boldsymbol{f}}\|^2_{L^2(\Omega)} \le \big(\delta + \frac{c(\delta)}{|\lambda|^\theta}\big) ([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda), \tag{4.63}$$

where $\theta > 0$ (equal to l if $0 < l < \frac{1}{2}$, and $\frac{1}{2}$ otherwise).

2. Construction of $\widehat{\boldsymbol{F}}$ *and* h'.

We now prove that $\widehat{h}$ can be written as the sum $\widehat{h} = \nabla_x \cdot \widehat{\boldsymbol{F}} + \widehat{h}'$ in a satisfactory way. More precisely we claim that for some tensors M^i_r, $r = 0, \ldots, 3$ and functions m_i, smooth and depending only on ϕ_b and $\{\eta_i\}$

$$\widehat{\boldsymbol{F}} = \sum_i M^i_0 \boldsymbol{v}_i + \frac{1}{\lambda} \sum_i (M^i_1 \nabla_y \boldsymbol{v}_i - M^i_2 p_i) \tag{4.64}$$

$$\widehat{h}' = \sum_i \frac{1}{\lambda} (M^i_3 \nabla_y \boldsymbol{v}_i - m_i p_i) \tag{4.65}$$

where, in the z coordinates,

$$|M_0^i| \leq c|\nabla \phi_i|, \tag{4.66}$$

for some constant depending only on ϕ_b. To prove this representation, first notice that from (4.55) we get

$$\nabla_x \cdot C_i^T \boldsymbol{w} = C_i^T \nabla_y \cdot C_i^T \boldsymbol{w} = \nabla_y \boldsymbol{w}, \tag{4.67}$$

for any vector $\boldsymbol{w}$, being C an isometry. Therefore

$$\nabla_x \cdot (\eta_i C_i^{-1} \boldsymbol{v}_i) = \nabla_y \cdot (\eta_i \boldsymbol{v}_i),$$

Thus, by the solenoidality (in the z coordinates) of $\boldsymbol{v}_i$, it holds

$$\widehat{h} = \nabla_x \cdot (\eta_i C^T \boldsymbol{v}_i) = \nabla_y \eta_i \cdot \boldsymbol{v}_i + \eta_i (\nabla_y - \nabla_z) \cdot \boldsymbol{v}_i. \tag{4.68}$$

Recall that $\mathcal{L}_i^T \nabla_y = \nabla_z$, and call l_{jk}^i the entries of $\mathcal{L}_i$. Using convention (3.4), and summation convention on repeated indexes except i, we can write

$$\Delta_z v_i^m = l_{kj}^i l_{sj,k}^i v_{i,s}^m + l_{kj}^i l_{sj}^i v_{i,sk}^m, \quad \nabla_z p_i = (l_{mk}^i p_{i,k}),$$

and since $\boldsymbol{v}_i = (\Delta_z \boldsymbol{v}_i - \nabla_z p_i)/\lambda$, we have

$$\begin{aligned}
\nabla_y \eta_i \cdot \boldsymbol{v}_i &= \eta_{i,m} (l_{kj}^i l_{sj,k}^i v_{i,s}^m + l_{kj}^i l_{sj} v_{i,sk}^m - l_{mk}^i p_{i,k}) \\
&= \frac{1}{\lambda} \big(\eta_{i,m} (l_{kj}^i l_{sj}^i v_{i,s}^m - l_{mk}^i p_i) \big)_{,k} \\
&\quad + \frac{1}{\lambda} \left[-(\eta_{i,m} l_{kj}^i l_{sj})_{,k} v_{i,s}^m + (\eta_{i,m} l_{mk}^i)_{,k} p_i + \eta_{i,m} l_{kj}^i l_{sj,k} v_{i,s}^m \right]
\end{aligned}$$

One can proceed in a similar way for the term $\eta_i (\nabla_y - \nabla_z) \cdot \boldsymbol{v}_i$. From $\nabla_z = \mathcal{L}_i^T \nabla_y$, we define the matrices a^i whose entries are $a_{hk}^i = (I - \mathcal{L}_i^T)_{hk}$. Notice that, using (3.2)

$$|a_{hk}^i| \leq c(|\nabla' \phi_i| + |\phi_i|). \tag{4.69}$$

Proceeding as before, in the y coordinates we have

$$\begin{aligned}
\eta_i (\nabla_y - \nabla_z) \cdot \boldsymbol{v}_i &= \eta_i a_{mt}^i v_{i,t}^m = (\eta_i a_{mt}^i v_i^m)_{,t} - \eta_i a_{mt,t}^i v^m \\
&= (\eta_i a_{mk}^i v_i^m)_{,k} - \frac{\eta_i a_{mt,t}^i}{\lambda} (l_{kj}^i l_{sj,k}^i v_{i,s}^m + l_{kj}^i l_{sj} v_{i,sk}^m - l_{mk}^i p_{i,k}) \\
&= \Big(\eta_i a_{mk}^i v_i^m - \frac{1}{\lambda} \eta_i a_{mt,t}^i (l_{kj}^i l_{sj}^i v_{i,s}^m - l_{mk}^i p_i) \Big)_{,k} \\
&+ \frac{1}{\lambda} \left[(\eta_i a_{mt,t}^i l_{kj}^i l_{sj})_{,k} v_{i,s}^m - (\eta_i a_{mt,t}^i l_{mk}^i)_{,k} p_i - \eta_i a_{mt,t}^i l_{kj}^i l_{sj,k} v_{i,s}^m \right].
\end{aligned}$$

If the hk-entry of C_i is denoted by C_i^{hk}, we now define the tensors M_r^i, $r = 0, \dots, 3$ as follows

$$\begin{aligned}(M_0^i)^h_m &= C_i^{kh}\eta_i a^i_{mk};\\ (M_1^i)^h_{sm} &= C_i^{kh}(\eta_{i,m} - \eta_i a^i_{mt,t})l^i_{kj}l^i_{sj};\\ (M_2^i)^h &= C_i^{kh}(\eta_{i,m} - \eta_i a^i_{mt,t})l^i_{mk};\\ (M_3^i)_{sm} &= (\eta_{i,m} - \eta_i a^i_{mt,t})l^i_{kj}l^i_{sj,k} - ((\eta_{i,m} - \eta_i a^i_{mt,t})l^i_{kj}l^i_{sj})_{,k};\end{aligned}$$

and the functions

$$m_i = \big((\eta_{i,m} - \eta_i a^i_{mt,t})l^i_{mk}\big)_{,k}.$$

Here the contraction symbols in the tensors M_1^i and M_3^i are the bottom one, i.e., for example, $M_1^i \cdot \nabla \boldsymbol{v}_i$ is the vector whose h-th component is

$$(M_1^i)^h_{sm} v^m_{i,s} = C_i^{kh}(\eta_{i,m} - \eta_i a^i_{mt,t})l^i_{kj}l^i_{sj}v^m_{i,s}.$$

Applying (4.67) for the terms in the divergence and gathering the previous equalities, we obtain

$$\nabla_x \cdot \boldsymbol{v} = \nabla_x \cdot \left[\sum_i M_0^i \boldsymbol{v}_i + \frac{M_1^i \nabla \boldsymbol{v}_i - M_2^i p_i}{\lambda}\right] + \sum_i \frac{M_3^i \nabla \boldsymbol{v}_i - m_i p_i}{\lambda},$$

which gives (4.64) and (4.65), while (4.69) and (4.56) give (4.66) for small δ.

3. Estimate for $\widehat{h}$, $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$.

For $\widehat{h}$, using formula (4.68), we can split the estimate in local coordinates:

$$\|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} \le c(\delta)\sum_i \|\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} + c\sum_i \|\eta_i(\nabla_y - \nabla_z)\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)}.$$

The first sum has only lower order terms which can be estimated through interpolation inequality, while by (3.3) the second one has addends of the form

$$\eta_i \frac{\partial \phi_i^*}{\partial z_k} m^i_{kj} \frac{\partial \boldsymbol{v}_i}{\partial z_j}, \tag{4.70}$$

for some smooth functions m^i_{jk} depending only on ϕ_b. As before, its W_2^{l+1} square norm is estimated through (4.57) and interpolation, giving

$$\begin{aligned}\sum_i \|\eta_i(\nabla_y - \nabla_z)\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} &\le c\sum_i \delta\|\nabla_z \boldsymbol{v}_i\|_{W_2^{l+1}(\Sigma^\infty)} + c(\delta)\|\boldsymbol{v}_i\|_{W_2^1(\Sigma^\infty)},\\ &\le c\sum_i \delta\|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)} + c(\delta)\|\boldsymbol{v}_i\|_{L^2(\Sigma^\infty)} \le c(\delta + \frac{c(\delta)}{|\lambda|^{l+2}})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).\end{aligned}$$

All in all, we get

$$\|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} \leq c(\delta + \frac{c(\delta)}{|\lambda|})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).$$

To estimate $\widehat{\boldsymbol{F}}$, using the expression (4.64), (4.66) and (4.56), we have

$$\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} \leq c \sum_i \frac{1}{|\lambda|^2} \Big(\|\boldsymbol{v}_i\|^2_{W_2^1(\Sigma^\infty)} + \|p_i\|^2_{L^2(\Sigma^\infty)} \Big) + \delta^2 \|\boldsymbol{v}_i\|_{L^2(\Sigma^\infty)}$$

and proceeding as in (4.62) for the pressure term, one obtains

$$|\lambda|^{l+2} \|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} \leq c(\delta^2 + \frac{c(\delta)}{\sqrt{|\lambda|}})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).$$

The estimate for $\widehat{h}'$, due to the form (4.65), is even simpler, and is omitted. The full estimate then reads

$$\|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)}) \leq c(\delta + \frac{c(\delta)}{\sqrt{|\lambda|}})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda). \quad (4.71)$$

4. Estimate of $\widehat{\boldsymbol{A}}$

The term $\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}$ is readily estimated through (4.59), the continuity of the restriction operator and interpolation inequality, giving

$$\begin{aligned} \|\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}} \|\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}\|^2_{L^2(\mathcal{G})} &\leq c(\|\overline{\boldsymbol{v}}_1\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\overline{\boldsymbol{v}}_1\|^2_{L^2(\Omega)}) \\ \leq c\big(\|\widehat{\boldsymbol{f}}\|^2_{W_2^l(\Omega)} + |\lambda|^l \|\widehat{\boldsymbol{f}}\|^2_{L^2(\Omega)} + \|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} &+ |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)})\big). \end{aligned}$$

From the previous estimates for $\widehat{\boldsymbol{f}}$, $\widehat{h}$, $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$ one thus obtains

$$\|\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}} \|\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}\|^2_{L^2(\mathcal{G})} \leq c(\delta + \frac{c(\delta)}{|\lambda|^\theta})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).$$

Regarding $\boldsymbol{A}$ we have, using $\boldsymbol{d} = \sum_i \varphi_i \eta_i \boldsymbol{d}$,

$$\begin{aligned} \mathbb{T}_x(p, \boldsymbol{v})\boldsymbol{N} + \sigma L\rho \boldsymbol{N} - \boldsymbol{d} = &\sum_i \eta_i \big(- p_i \boldsymbol{N} + C_i^{-1} \mathbb{D}_x(\boldsymbol{v}_i)\boldsymbol{N} + \sigma L\rho_i \boldsymbol{N} - \varphi_i \boldsymbol{d} \big) \\ &+ \sum_i C_i^{-1} \nabla_x \varphi_i \otimes \boldsymbol{v}_i \cdot \boldsymbol{N} + \sigma(L(\eta_i\rho_i) - \eta_i L\rho_i)\boldsymbol{N}, \end{aligned}$$

in the x coordinates. The second sum is a lower order term, and thus it can be estimated via interpolation with $\frac{c(\delta)}{|\lambda|}([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda)$. We transform the first sum in the z coordinates and use the boundary conditions for (4.58), obtaining a sum whose addends are

$$\eta_i\big(-p_i(\boldsymbol{N}'-\boldsymbol{e}_3)+(\mathbb{D}_y-\mathbb{D}_z)(\boldsymbol{v}_i)\boldsymbol{N}'+\mathbb{D}_z(\boldsymbol{v}_i)(\boldsymbol{N}'-\boldsymbol{e}_3)\\ +\sigma L\rho_i(\boldsymbol{N}'-\boldsymbol{e}_3)+\sigma(L\rho_i+\Delta'\rho_i)\boldsymbol{e}_3\big).$$

Notice that $\eta_i(\mathbb{D}_y-\mathbb{D}_z)(\boldsymbol{v}_i)$ and $\eta_i\mathbb{D}_z\boldsymbol{v}_i(\boldsymbol{N}'-\boldsymbol{e}_3)$ can be computed explicitly using (3.3) and both are linear combination of terms of the form (4.70). Similarly

$$p_i(\boldsymbol{N}'-\boldsymbol{e}_3)=p_i\frac{\partial\phi_i}{\partial z_j}\boldsymbol{m}^i_j,\quad L\rho_i(\boldsymbol{N}'-\boldsymbol{e}_3)=\frac{\partial\phi_i}{\partial z_j}\boldsymbol{m}^i_{jkl}\frac{\partial^2\rho_i}{\partial z_k\partial z_l}+\frac{\partial\phi_i}{\partial z_j}\boldsymbol{m}^i_{jk}\frac{\partial\rho_i}{\partial z_k},$$

for some smooth vectors $\boldsymbol{m}^j$, $\boldsymbol{m}^i_{jk}$ and $\boldsymbol{m}^i_{jkl}$ depending only on ϕ_b. Moreover, letting $g_i=1+|\nabla'\phi_i|^2$, we have

$$L\rho+\Delta'\rho=\frac{|\nabla'\phi_i|^2}{(1+\sqrt{g})\sqrt{g}}\Delta'\rho_i+\frac{\phi_{i\alpha}\phi_{i\beta}}{g^{\frac{5}{2}}}\rho_{\alpha\beta}+m^i_{\alpha\beta}\phi_{i\alpha}\rho_{i\beta},$$

which has the same structure of $L\rho_i(\boldsymbol{N}'-\boldsymbol{e}_3)$. These terms are thus estimated using (4.57) and interpolation inequalities as follows: for the pressure term

$$\begin{aligned}\sum_i\|\eta_i p_i(\boldsymbol{N}'-\boldsymbol{e}_3)\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)}&+|\lambda|^{l+\frac{1}{2}}\|\eta_i p_i(\boldsymbol{N}'-\boldsymbol{e}_3)\|^2_{L^2(\Sigma)}\\ &\le c\sum_i\delta(\|p_i\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)}+|\lambda|^{l+\frac{1}{2}}\|p_i\|^2_{L^2(\Sigma)})+c(\delta)\|p_i\|^2_{L^2(\Sigma)}\\ &\le c(\delta+\frac{c(\delta)}{|\lambda|^{l+\frac{1}{2}}})([\boldsymbol{d}]^2_\lambda+[g]^2_\lambda),\end{aligned}$$

while

$$\begin{aligned}\sum_i\|\eta_i(\mathbb{D}_y-\mathbb{D}_z)(\boldsymbol{v}_i)\boldsymbol{N}'\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)}&+|\lambda|^{l+\frac{1}{2}}\|\eta_i(\mathbb{D}_y-\mathbb{D}_z)(\boldsymbol{v}_i)\boldsymbol{N}'\|^2_{L^2(\Sigma)}+\\ &+\|\eta_i\mathbb{D}_z(\boldsymbol{v}_i)(\boldsymbol{N}'-\boldsymbol{e}_3)\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)}+|\lambda|^{l+\frac{1}{2}}\|\eta_i\mathbb{D}_z(\boldsymbol{v}_i)(\boldsymbol{N}'-\boldsymbol{e}_3)\|^2_{L^2(\Sigma)}\\ &\le c\sum_i\delta(\|\nabla_z\boldsymbol{v}_i\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)}+|\lambda|^{l+\frac{1}{2}}\|\nabla_z\boldsymbol{v}_i\|^2_{L^2(\Sigma)})+c(\delta)\|\nabla_z\boldsymbol{v}_i\|^2_{L^2(\Sigma)}\\ &\le c\sum_i\delta(\|\boldsymbol{v}_i\|^2_{W_2^{l+2}(\Sigma^\infty)}+|\lambda|^{l+2}\|\boldsymbol{v}_i\|^2_{L^2(\Sigma^\infty)})+c(\delta)\|\boldsymbol{v}_i\|^2_{L^2(\Sigma^\infty)}\\ &\le c(\delta+\frac{c(\delta)}{|\lambda|^{l+2}})([\boldsymbol{d}]^2_\lambda+[g]^2_\lambda),\end{aligned}$$

and finally

$$\begin{aligned}
\sum_i &\|\eta_i L\rho_i(\boldsymbol{N}'-\boldsymbol{e}_3)\|^2_{W_2^{l+\frac12}(\Sigma)} + |\lambda|^{l+\frac12}\|\eta_i L\rho_i(\boldsymbol{N}'-\boldsymbol{e}_3)\|^2_{L^2(\Sigma)} + \\
&\sum_i \|\eta_i(L\rho_i+\Delta'\rho_i)\boldsymbol{e}_3\|^2_{W_2^{l+\frac12}(\Sigma)} + |\lambda|^{l+\frac12}\|\eta_i(L\rho_i+\Delta'\rho_i)\boldsymbol{e}_3\|^2_{L^2(\Sigma)} \\
&\le c\sum_i \delta(\|\rho_i\|^2_{W_2^{l+\frac52}(\Sigma)} + |\lambda|^{l+\frac12}\|\rho_i\|^2_{W_2^2(\Sigma)}) + c(\delta)(\|\rho_i\|^2_{W_2^2(\Sigma)} + \|\rho_i\|^2_{W_2^{l+\frac32}(\Sigma)}) \\
&\le c\sum_i \delta(\|\rho_i\|^2_{W_2^{l+\frac52}(\Sigma)} + |\lambda|^{l+\frac52}\|\rho_i\|^2_{L^2(\Sigma)}) + c(\delta)\|\rho_i\|^2_{L^2(\Sigma)} \\
&\le c(\delta + \frac{c(\delta)}{|\lambda|^{l+\frac52}})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).
\end{aligned}$$

All in all we have

$$\|\widehat{\boldsymbol{A}}(\boldsymbol{d},g)\|^2_{W_2^{l+\frac12}(\mathcal{G})} + |\lambda|^{l+\frac12}\|\widehat{\boldsymbol{A}}(\boldsymbol{d},g)\|^2_{L^2(\mathcal{G})} \le c(\delta + \frac{c(\delta)}{|\lambda|^{l+\frac12}})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).$$

5. Estimate of $\widehat{A}$

The estimate for $\nabla'_x\phi_b\cdot(\overline{\boldsymbol{v}}_1+\overline{\boldsymbol{v}}_2)-\overline{v}_1^3-\overline{v}_2^3$ follows from (4.59) for $\overline{\boldsymbol{v}}_1$ and (4.49) for $\overline{v}_2$. The argument is very similar to those given above, and is omitted. For the estimate of A we localize in the z coordinates, obtaining, via

$$(\nabla'_x\phi_b,-1)C_i^{-1}\boldsymbol{v}_i = (-\nabla'_z\phi_i,1)\boldsymbol{v}_i, \qquad \boldsymbol{v}_b\cdot\nabla'_x\rho_i = C_i\boldsymbol{v}_b\cdot\nabla'_z\rho_i,$$

the explicit representation

$$\begin{aligned}
\lambda\rho+&\nabla'_x\phi_b\cdot\boldsymbol{v} - v^3 + \boldsymbol{v}_b\cdot\nabla_x\rho - g \\
&= \sum_i \eta_i(\lambda\rho_i + (\nabla'_x\phi_b,-1)\cdot C_i^{-1}\boldsymbol{v}_i + \boldsymbol{v}_b\cdot\nabla'_x\rho_i - \varphi_i g) + \sum_i \rho_i\boldsymbol{v}_b\cdot\nabla_x\eta_i \\
&= \sum_i \eta_i\left[C_i(\boldsymbol{v}_b - \boldsymbol{v}_b(x_i))\nabla'_z\rho_i - \nabla'_z\phi_i\cdot\boldsymbol{v}_i\right] + \sum_i C_i\rho_i\boldsymbol{v}_b\cdot\nabla_z\eta_i.
\end{aligned}$$

The second summand is a lower order term, which can be estimated through interpolation inequality as

$$\begin{aligned}
\|\sum_i &\rho_i C_i\boldsymbol{v}_b\nabla_z\eta_i\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|\sum_i \rho_i C_i\boldsymbol{v}_b\nabla_z\eta_i\|^2_{W_2^1(\Sigma)} \\
&\le c(\delta)\sum_i \|\rho_i\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|\rho_i\|^2_{W_2^1(\Sigma)} \\
&\le \frac{c(\delta)}{|\lambda|}\sum_i \|\rho_i\|^2_{W_2^{l+\frac52}(\Sigma)} + |\lambda|^{l+\frac32}\|\rho_i\|^2_{W_2^1(\Sigma)} \le \frac{c(\delta)}{|\lambda|}([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda)
\end{aligned}$$

For the higher order terms let us look at $\eta_i \nabla' \phi_i \cdot \boldsymbol{v}_i$: it is readily estimated through (4.57) as

$$\begin{aligned}
&\|\sum_i \eta_i \nabla' \phi_i \cdot \boldsymbol{v}_i\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}} \|\sum_i \eta_i \nabla' \phi_i \cdot \boldsymbol{v}_i\|^2_{W_2^1(\Sigma)} \\
&\quad \le \sum_i \delta(\|\boldsymbol{v}_i\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}} \|\boldsymbol{v}_i\|^2_{W_2^1(\Sigma)}) + c(\delta)|\lambda|^{l+\frac{1}{2}} \|\boldsymbol{v}_i\|^2_{L^2(\Sigma)} \\
&\quad \le \sum_i \delta(\|\boldsymbol{v}_i\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+\frac{3}{2}} \|\boldsymbol{v}_i\|^2_{L^2(\Sigma)}) + c(\delta)|\lambda|^{l+\frac{1}{2}} \|\boldsymbol{v}_i\|^2_{L^2(\Sigma)}
\end{aligned}$$

by standard restriction theorem and interpolation inequality (2.14). We now apply interpolation inequality (2.15) to the L^2 terms, to obtain

$$\begin{aligned}
&\|\sum_i \eta_i \nabla' \phi_i \cdot \boldsymbol{v}_i\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}} \|\sum_i \eta_i \nabla' \phi_i \cdot \boldsymbol{v}_i\|^2_{W_2^1(\Sigma)} \le \\
&(\delta + \frac{c(\delta)}{|\lambda|}) \sum_i (\|\boldsymbol{v}_i\|^2_{W_2^{l+2}(\Sigma^\infty)} + |\lambda|^{l+2} \|\boldsymbol{v}_i\|^2_{L^2(\Sigma^\infty)}) \le c(\delta + \frac{c(\delta)}{|\lambda|})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda).
\end{aligned}$$

For the remaining terms of the form $\eta_i(\boldsymbol{v}_b - \boldsymbol{v}_b(x_i))\nabla'_z \rho_i$, by the smoothness of $\boldsymbol{v}_b$ we can assume that $|v_b^k - v_b^k(x_i)| \le \delta$ on the support of η_i and thus apply lemma 4.2.3 with $\psi = \boldsymbol{v}_b - \boldsymbol{v}_b(x_i)$ to obtain

$$\begin{aligned}
&\|\sum_i \eta_i C_i(\boldsymbol{v}_b - \boldsymbol{v}_b(x_i))\nabla'_z \rho_i\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}} \|\sum_i \eta_i C_i(\boldsymbol{v}_b - \boldsymbol{v}_b(x_i))\nabla'_z \rho_i\|^2_{W_2^1(\Sigma)} \\
&\quad \le \sum_i \delta\big(\|\nabla\rho\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}} \|\nabla\rho\|^2_{W_2^1(\Sigma)}\big) + c(\delta)|\lambda|^{l+\frac{1}{2}} \|\nabla\rho\|^2_{L^2(\Sigma)} \\
&\quad \le \sum_i \delta\big(\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\Sigma)} + |\lambda|^{l+\frac{5}{2}} \|\rho\|^2_{L^2(\Sigma)}\big) + \frac{c(\delta)}{|\lambda|^2} |\lambda|^{l+\frac{1}{2}} \|\lambda\rho\|^2_{W_2^1(\Sigma)} \\
&\quad \le c(\delta + \frac{c(\delta)}{|\lambda|^2})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda),
\end{aligned}$$

which completes the proof of the inequality

$$\|\widehat{A}(\boldsymbol{d}, g)\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}} \|\widehat{A}(\boldsymbol{d}, g)\|^2_{W_2^1(\mathcal{G})} \le c(\delta + \frac{c(\delta)}{|\lambda|})([\boldsymbol{d}]^2_\lambda + [g]^2_\lambda). \quad (4.72)$$

6. Existence and estimates for the solution

The estimates (4.2) and (4.72), together with (4.61) give that

$$\|(\widehat{\boldsymbol{A}}(\boldsymbol{d}, g), \widehat{A}(\boldsymbol{d}, g))\|_\lambda \le c(\delta + \frac{c(\delta)}{|\lambda|^\theta})\|(\boldsymbol{d}, g)\|_\lambda.$$

Therefore, choosing δ sufficiently small, and then $|\lambda|$ sufficiently large, we get that $(\widehat{\boldsymbol{A}}, \widehat{A})$ is a contraction on $W_2^{l+\frac{1}{2}}(\mathcal{G}) \times W_2^{l+\frac{3}{2}}(\mathcal{G})$ normed with $\| \quad \|_\lambda$, as in (4.61). The existence thus follows, and we will suppose henceforth that δ is fixed. a

We now prove the estimate (4.51). To this end, it suffice to prove the continuity of the operator R defined in (4.60) with respect to the norm $\|(\boldsymbol{u}, p, \rho)\|_\lambda$ defined as

$$\|(\boldsymbol{u}, p, \rho)\|_\lambda^2 := \|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Omega)} + \|q\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|q\|^2_{L^2(\mathcal{G})} + \|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})}.$$

Now from (4.59) and (4.49) we have that

$$\|\overline{\boldsymbol{v}}_1 + \overline{\boldsymbol{v}}_2\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\overline{\boldsymbol{v}}_1 + \overline{\boldsymbol{v}}_2\|^2_{L^2(\Omega)} + \|\nabla p_2\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla p_2\|^2_{L^2(\Omega)} + \|p_2\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|p_2\|^2_{L^2(\mathcal{G})} \leq c\big(\|\widehat{\boldsymbol{f}}\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\widehat{\boldsymbol{f}}\|^2_{L^2(\Omega)} + \|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)})\big),$$

and the right hand side is bounded by $\|(\boldsymbol{d}, g)\|_\lambda$ by (4.63), (4.71). Looking at the definition of $(\boldsymbol{v}, p, \rho)$ we have, applying proposition 2.1.3 and the estimates for problem (4.58)

$$\|(\boldsymbol{v}, p, \rho)\|_\lambda \leq c(\delta)\|(\boldsymbol{d}, g)\|_\lambda,$$

which completes the proof of

$$\|R(\boldsymbol{d}, g)\|_\lambda \leq c\|(\boldsymbol{d}, g)\|,$$

since δ is fixed. To prove (4.52) we notice that, by the equation for ρ:

$$\lambda\rho = g - \nabla'\phi \cdot \boldsymbol{u} + u^3 - \boldsymbol{v}_b \cdot \nabla'\rho,$$

and, by interpolation inequality (2.15),

$$\begin{aligned}
&|\lambda|^{l+\frac{3}{2}}\|\lambda\rho\|^2_{L^2(\mathcal{G})} \\
&\quad\leq |\lambda|^{l+\frac{3}{2}}\|g\|^2_{L^2(\mathcal{G})} + c(\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + |\lambda|^{l+\frac{3}{2}}\|\nabla'\rho\|^2_{L^2(\mathcal{G})}) \\
&\quad\leq |\lambda|^{l+\frac{3}{2}}\|g\|^2_{L^2(\mathcal{G})} + c(\|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \frac{c}{|\lambda|}|\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})}).
\end{aligned}$$

Noting that

$$|\lambda|^{l+\frac{1}{2}}\|g\|^2_{W_2^1(\mathcal{G})} \leq c(\|g\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{3}{2}}\|g\|^2_{L^2(\mathcal{G})}),$$

we obtain the claim from (4.51).

7. Uniqueness

By taking the difference of two solutions, it suffice to show that problem (4.43) with vanishing right hand sides has only the trivial solution. Suppose then that $(\boldsymbol{u}, p, \rho)$ is such a solution. Taking the scalar product with $\boldsymbol{u}$ in the first equation and integrating by parts gives, by the second and third equation

$$\lambda \int_\Omega |\boldsymbol{u}|^2 dx + \frac{\nu}{2} \int_\Omega |\mathbb{D}(\boldsymbol{u})|^2 dx = -\sigma \int_{\mathcal{G}} L\rho \boldsymbol{N} \cdot \boldsymbol{u} dS, \tag{4.73}$$

(the boundary terms on Σ vanish due to $\boldsymbol{u}\lfloor_\Sigma = 0$). The right hand side can be rewritten using the equation for ρ and

$$\boldsymbol{N} = (-\nabla'\phi_b, 1)/\sqrt{g_b}, \qquad g_b = 1 + |\nabla\phi_b|^2, \qquad dS = \sqrt{g_b} dx',$$

giving

$$\int_{\mathcal{G}} L\rho \boldsymbol{N} \cdot \boldsymbol{u} dS = \int_\Sigma L\rho(-\nabla'\rho \cdot \boldsymbol{u} + u^3) dx' = \int_\Sigma L\rho(\lambda\rho + \boldsymbol{v}_b \cdot \nabla'\rho) dx'.$$

If $\lambda = s + it$, taking the real part in (4.73) thus gives

$$s \int_\Omega |\boldsymbol{u}|^2 dx + \frac{\nu}{2} \int_\Omega |\mathbb{D}(\boldsymbol{u})|^2 dx = -\sigma B_s(\rho) < 0,$$

by lemma 3.1.1, for s sufficiently large. Therefore, for $s = \operatorname{Re}\lambda$ sufficiently large we get $\boldsymbol{u} = 0$, and $\nabla\rho = 0$ from $B_s(\rho) = 0$ and (3.9). From the equation for ρ we thus get $\rho = 0$ and from the boundary condition on the stress tensor, $p = 0$ on $\mathcal{G}$. Since the velocity equation now reads $\nabla p = 0$, we conclude that p vanishes in the whole Ω, and thus $(\boldsymbol{u}, p, \rho) = 0$. □

4.3 Time dependent linear problem

In this section we prove the solvability of the time-dependent linear problem (4.1).

We first need to consider the perturbed version of problem (4.43), i.e.

$$\begin{cases} \lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q - \widehat{\boldsymbol{\Phi}}_1(\boldsymbol{u}, \rho) = \boldsymbol{f} & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{u} - \Phi_2(\rho) = h = \nabla \cdot \boldsymbol{F} & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \widetilde{\boldsymbol{\Phi}}(\rho) = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \lambda\rho + \nabla'\phi_b \cdot \boldsymbol{u} - u^3 + \nabla'\rho \cdot \boldsymbol{v}_b = g & \text{on } \mathcal{G}, \\ \boldsymbol{u} = \boldsymbol{a} & \text{on } \Sigma, \end{cases} \tag{4.74}$$

where $\widehat{\mathbf{\Phi}}_1$ is given as in (3.18) substituting the term $\rho^*_{,t}$ with $\lambda\rho^*$, and, as usual, $a^3 = F^3 = 0$ on Σ.

Theorem 4.3.1 *Let $l \geq 0$. For any sufficiently large $\mathrm{Re}\,\lambda$, there exists a unique periodic solution of (4.74), for any choice of periodic $\boldsymbol{f} \in W_2^l(\Omega_b)$, $\boldsymbol{d} \in W_2^{l+\frac{1}{2}}(\mathcal{G})$, $g \in W_2^{l+\frac{3}{2}}(\mathcal{G})$, $h \in W_2^{l+1}(\Omega_b)$ and $\boldsymbol{F} \in W_2^1(\Omega_b)$ with $F^3\lfloor_\Sigma = 0$ and this solution satisfies (4.51) and(4.52).*

Proof. We start estimating the λ-weighted norm of the various Φ_i. For $\widehat{\mathbf{\Phi}}_1$, we see from the definition (3.18) that all its terms are of the form

$$\nabla\rho^* \cdot \boldsymbol{M}_k^1, \quad D^2\rho^* \cdot \boldsymbol{m}_k^1, \quad \lambda\rho^*\boldsymbol{m}_k^2, \quad \boldsymbol{u} \cdot \boldsymbol{M}_k^2, \quad \text{or} \quad \boldsymbol{m}_k^3 \cdot \nabla\boldsymbol{u},$$

for some smooth vectors and matrices $\boldsymbol{m}_k^h$ and $\boldsymbol{M}_k^h$ depending on $\boldsymbol{v}_b$ and p_b. Each of these terms can be estimated in the $W_2^l(\Omega_b)$ norm through proposition 2.1.3. One considers separately the terms containing the spatial derivatives of ρ and those containing the derivatives of θ to obtain, for $\mathrm{Re}\,\lambda \gg 1$,

$$\|\widehat{\mathbf{\Phi}}_1\|_{W_2^l(\Omega_b)} \leq c(\|\nabla\rho\|_{W_2^{l+1}(\mathcal{G})} + |\lambda|\|\rho\|_{W_2^l(\mathcal{G})} + \|\boldsymbol{u}\|_{W_2^{l+1}(\Omega_b)})$$

where c is a constant depending on the higher order norms of $\boldsymbol{v}_b$, p_b and θ. Applying interpolation inequality one then obtains, again for $\mathrm{Re}\,\lambda >> 1$,

$$\begin{aligned}\|\widehat{\mathbf{\Phi}}_1\|^2_{W_2^l(\Omega_b)} \leq \frac{c}{\sqrt{|\lambda|}}(\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} \\ + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{L^2(\mathcal{G})} + \|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega_b)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega_b)}).\end{aligned}$$

For the L^2 norm one has

$$\|\widehat{\mathbf{\Phi}}_1\|_{L^2(\Omega_b)} \leq c(\|\nabla\rho\|_{W_2^1(\mathcal{G})} + \|\lambda\rho\|_{L^2(\mathcal{G})} + \|\boldsymbol{u}\|_{W_2^1(\Omega_b)}),$$

and thus, bounding $\|\rho\|_{L^2(\mathcal{G})}$ with $\|\rho\|_{W_2^1(\mathcal{G})}$ and using interpolation inequalities

$$\begin{aligned}|\lambda|^l\|\widehat{\mathbf{\Phi}}_1\|^2_{L^2(\Omega_b)} \leq \frac{c}{\sqrt{|\lambda|}}(\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{3}{2}}\|\rho\|^2_{W_2^1(\mathcal{G})} \\ + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})} + \|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega_b)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega_b)}).\end{aligned}$$

Therefore, for $\mathrm{Re}\,\lambda > 1$, we have

$$\begin{aligned}\|\widehat{\mathbf{\Phi}}_1\|^2_{W_2^l(\Omega_b)} + |\lambda|^l\|\widehat{\mathbf{\Phi}}_1\|^2_{L^2(\Omega_b)} \leq \frac{c}{\sqrt{\lambda}}(\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})} \\ + \|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega_b)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega_b)}).\end{aligned} \tag{4.75}$$

For Φ_2, recall by (3.21), that it is a linear combination of terms of the form $\nabla\rho^* \cdot \boldsymbol{m}^1$ and $\rho \cdot \boldsymbol{m}^2$, and therefore

$$\|\Phi_2(\rho)\|_{W_2^{l+1}(\Omega_b)} \le c(\|\nabla\rho\|_{W_2^{l+1}(\mathcal{G})} + \|\rho\|_{W_2^{l+1}(\mathcal{G})}),$$

giving

$$\|\Phi_2(\rho)\|^2_{W_2^{l+1}(\Omega_b)} \le \frac{c}{\sqrt{|\lambda|}}(\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{3}{2}}\|\rho\|^2_{W_2^1(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})}). \quad (4.76)$$

Moreover, recalling that $\Phi_2(\rho) = \nabla \cdot (I - \widehat{\mathcal{L}})\boldsymbol{v}_b$, we have for $(I - \widehat{\mathcal{L}})\boldsymbol{v}_b$,

$$|\lambda|^{l+2}\|(I - \widehat{\mathcal{L}})\boldsymbol{v}_b\|^2_{L^2(\Omega_b)} \le c|\lambda|^{l+2}\|\rho\|^2_{W_2^1(\mathcal{G})} \le \frac{c}{\sqrt{|\lambda|}}|\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})}. \quad (4.77)$$

Finally both $\boldsymbol{\Phi}_3$ and Φ_4 are linear combinations of terms of the form $\nabla'\rho \cdot \boldsymbol{M}$ with $\boldsymbol{M}$ depending on $\boldsymbol{v}_b$ and ϕ_b, therefore, for $\operatorname{Re}\lambda > 1$,

$$\begin{aligned}\|\widetilde{\boldsymbol{\Phi}}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\widetilde{\boldsymbol{\Phi}}\|^2_{L^2(\mathcal{G})} &\le c(\|\nabla'\rho\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\nabla'\rho\|^2_{L^2(\mathcal{G})}) \\ &\le \frac{c}{\sqrt{|\lambda|}}(\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})}).\end{aligned} \quad (4.78)$$

This concludes the preliminary estimates for the linear perturbations.

To construct a solution, we recursively define the sequence $(\boldsymbol{v}_n, p_n, \rho_n) \in X := W_2^{l+2}(\Omega) \times W_2^{l+1}(\Omega) \times W_2^{l+\frac{5}{2}}(\mathcal{G})$, equipped with the weighted norm whose square is

$$\begin{aligned}[(\boldsymbol{v}, p, \rho)]^2_{l,\lambda} &:= \|\boldsymbol{u}\|^2_{W_2^{l+2}(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{W_2^l(\Omega)} + |\lambda|^l\|\nabla q\|^2_{L^2(\Omega)} \\ &+ \|q\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|q\|^2_{L^2(\mathcal{G})} + \|\rho\|^2_{W_2^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\lambda\rho\|^2_{W_2^1(\mathcal{G})}.\end{aligned}$$

We set $(\boldsymbol{v}_0, p_0, \rho_0) = (0, 0, 0)$, while $(\boldsymbol{v}_{n+1}, p_{n+1}, \rho_{n+1})$ solves (4.43) with right hand sides, respectively

$$\boldsymbol{f}_n = \boldsymbol{f} + \widehat{\boldsymbol{\Phi}}_1(\boldsymbol{v}_n, \rho_n), \qquad h_n = h + \Phi_2(\rho_n), \qquad \boldsymbol{d}_n = \boldsymbol{d} + \widetilde{\Phi}(\rho_n), \qquad g_n = g.$$

Notice that $h_n = \nabla \cdot \boldsymbol{F}_n$ with

$$\boldsymbol{F}_n = \boldsymbol{F} + (I - \widehat{\mathcal{L}}_n)\boldsymbol{v}_b,$$

and thus, by the condition $F^3 = 0$ on Σ and (3.23), it holds $F_n^3 = 0$ on Σ. We claim that if $\operatorname{Re}\lambda$ is sufficiently large, $\{(\boldsymbol{v}_n, p_n, \rho_n)\}$ is a Cauchy sequence in X.

Indeed the difference $(\boldsymbol{w}_{n+1}, q_{n+1}, \sigma_{n+1}) := (\boldsymbol{v}_{n+1} - \boldsymbol{v}_n, p_{n+1} - p_n, \rho_{n+1} - \rho_n)$ satisfies (4.43) with right hand side, respectively

$$\boldsymbol{\Phi}_1(\boldsymbol{w}_n, \sigma_n), \qquad \Phi_2(\sigma_n), \qquad \widetilde{\Phi}(\sigma_n), \qquad 0,$$

and thus (4.51), together with the previously proved estimates (4.75)–(4.78), shows that

$$[(\boldsymbol{w}_{n+1}, q_{n+1}, \sigma_{n+1})]^2_{l,\lambda} \leq \frac{c}{\sqrt{|\lambda|}}[(\boldsymbol{w}_n, q_n, \sigma_n)]^2_{l,\lambda},$$

and thus the claim for $\frac{c}{\sqrt{|\lambda|}} \leq \frac{1}{2}$. Therefore $(\boldsymbol{v}_n, p_n, \rho_n) \to (\boldsymbol{v}, p, \rho) \in X$ and the latter clearly solves problem (4.74). The a priori estimate (4.51) follows from

$$[(\boldsymbol{v}, p, \rho)]^2_{l,\lambda} \leq \frac{c}{\sqrt{|\lambda|}}[(\boldsymbol{v}, p, \rho)]^2_{l,\lambda} + c\big(\|\boldsymbol{f}\|^2_{W^l_2(\Omega)} + |\lambda|^l\|\boldsymbol{f}\|^2_{L^2(\Omega)} + \|h\|^2_{W^{l+1}_2(\Omega)} +$$
$$|\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|\boldsymbol{d}\|^2_{L^2(\mathcal{G})} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + |\lambda|^{l+\frac{1}{2}}\|g\|^2_{W^1_2(\mathcal{G})}\big),$$

again for $\frac{c}{\sqrt{|\lambda|}} \leq \frac{1}{2}$, and this also gives uniqueness. One can proceed as in the proof of theorem 4.2.4 to obtain also estimate (4.52) for the solution.
□

To shorten somehow the notation we define

$$\begin{aligned}\|(\boldsymbol{u}, p, \rho)\|_{W,l,T} =& \|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} + \|\nabla p\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|p\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} \\ &+ \|\rho\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)} + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)}.\end{aligned} \tag{4.79}$$

$$\begin{aligned}\|(\boldsymbol{u}, p, \rho)\|_{H,l,T} =& \|\boldsymbol{u}\|_{H_2^{l+2,\frac{l}{2}+1}(Q_T)} + \|\nabla p\|_{H_2^{l,\frac{l}{2}}(Q_T)} + \|p\|_{H_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} \\ &+ \|\rho\|_{H_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)} + \|\rho_{,t}\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)}.\end{aligned} \tag{4.80}$$

Theorem 4.3.2 *Let $l \in (\frac{1}{2}, 1)$ and $T < +\infty$. For any Σ-periodic choice of $\boldsymbol{f} \in W_2^{l,\frac{l}{2}}(Q_T)$, $h \in W_2^{l+1,0}(Q_T)$, $\boldsymbol{F} \in W_2^{0,\frac{l}{2}+1}(Q_T)$ with $F^3\lfloor_\Sigma = 0$, $\boldsymbol{d} \in W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)$, $g \in W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)$, $\boldsymbol{a} \in W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma)$ with $a^3 \equiv 0$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$ and $\boldsymbol{u}_0 \in W_2^{l+1}(\Omega)$ such that*

$$\begin{cases} \nabla \cdot \boldsymbol{u}_0(x) = \Phi_2(\rho_0)(x) + \nabla \cdot \boldsymbol{F}(x, 0) & \text{for } x \in \Omega_b, \\ \nu\Pi_b\mathbb{D}(\boldsymbol{u}_0)(x)\boldsymbol{N}(x) = \boldsymbol{\Phi}_3(\rho_0)(x) + \Pi_b\boldsymbol{d}(x, 0) & \text{for } x \in \mathcal{G}, \\ \boldsymbol{u}_0(x) = \boldsymbol{a}(x, 0) & \text{for } x \in \Sigma, \end{cases} \tag{4.81}$$

there exists a unique solution to (4.1), *and it holds the estimate*

$$\begin{aligned}\|(\boldsymbol{u}, q,\rho)\|_{W,l,T} \le c(T)(&\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|h\|_{W_2^{l+1,0}(Q_T)} \\ &+ \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} + \|\boldsymbol{d}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \\ &+ \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}).\end{aligned} \tag{4.82}$$

Moreover, if $T \ge 1$*, it holds*

$$\begin{aligned}\|(\boldsymbol{u}, q,\rho)\|_{W,l,T} \le c(&\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \\ &+ \|\boldsymbol{d}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)} \\ &+ \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}\|_{L^2(Q_T)} + \|\rho\|_{L^2(G_T)}).\end{aligned} \tag{4.83}$$

with constant independent of T *and, if* $T \le 1$*,*

$$\begin{aligned}\|(\boldsymbol{u}, q,\rho)\|_{H,l,T} \le c\big(&\|\boldsymbol{f}\|_{H_2^{l,\frac{l}{2}}(Q_T)} + \|h\|_{H_2^{l+1,0}(Q_T)} \\ &+ \|\boldsymbol{F}\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)} + \|\boldsymbol{d}\|_{H_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|\boldsymbol{a}\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)} \\ &+ \|g\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\big)\end{aligned} \tag{4.84}$$

also with constant independent of T*.*

Proof. We follow the plan described in section 2.4, and reduce problem (4.1) to a similar one with homogeneous initial data in order to apply Laplace transform and use theorem 4.3.1 to get the solution. First of all we fix $T_0 \ge T + 1$ and extend all the right hand terms except h and $\boldsymbol{F}$ (keeping the notation unchanged) to Q_∞ and G_∞ with controlled norm, supposing furthermore that all the terms vanish for $t > T_0$. For $T \ge 1$, this can be done with constants independent of T, i.e.

$$\begin{aligned}&\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} + \|\boldsymbol{d}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_\infty)} + \|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_\infty)} \\ &\le c(\|\boldsymbol{f}\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|\boldsymbol{d}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} + \|\boldsymbol{a}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)}).\end{aligned}$$

For $T \le 1$ we can use theorem 2.3.3 to obtain the same estimate with, on the left side, the $H_2^{\eta,\frac{\eta}{2}}$ norms instead of the $W_2^{\eta,\frac{\eta}{2}}$ ones. To construct the extensions of h and $\boldsymbol{F}$, we define, for all $t \le T$, $\boldsymbol{w}_0 = \nabla\psi$, where ψ is the solution of

$$\begin{cases}\Delta\psi = h = \nabla\cdot\boldsymbol{F} & \text{in } \Omega_b,\\ \psi = 0 & \text{on } \mathcal{G},\\ \frac{\partial\psi}{\partial\boldsymbol{n}} = \boldsymbol{F}\cdot\boldsymbol{n} = 0 & \text{on } \Sigma.\end{cases} \tag{4.85}$$

From standard elliptic estimates we have, for each $t \le T$,

$$\|\boldsymbol{w}_0\|_{W_2^{l+2,0}(Q_T)} \le c\|h\|_{W_2^{l+1,0}(Q_T)},$$

and

$$\sup_{t\le T}\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \le c\sup_{t\le T}\|h\|_{W_2^{l}(\Omega)}.$$

Differentiating in t the weak formulation of (4.85), one gets

$$\int_{\Omega_b} \boldsymbol{w}_{0,t}\cdot\nabla\eta dx = \int_{\Omega_b} \boldsymbol{F}_{,t}\cdot\nabla\eta dx, \qquad \forall\eta\in C^\infty, \quad \eta\lfloor_{\mathcal{G}}= 0,$$

and a similar identity for the finite differences in time of $\boldsymbol{w}_{0,t}$, which implies

$$\|\boldsymbol{w}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \le c\|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)},$$

$$\frac{1}{T^l}\int_0^T \|\boldsymbol{w}_{0,t}\|^2_{L^2(\Omega)}dt \le \frac{1}{T^l}\int_0^T \|\boldsymbol{F}_{,t}\|^2_{L^2(\Omega)}dt.$$

Therefore it holds

$$\|\boldsymbol{w}_0\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} \le c(\|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}),$$

$$\|\boldsymbol{w}_0\|_{H_2^{l+2,\frac{l}{2}+1}(Q_T)} \le c(\|h\|_{H_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)}),$$

We extend $\boldsymbol{w}_0$ in such a way that $\boldsymbol{w}_0 = 0$ for $t \ge T_0$, $w_0^3 \equiv 0$, and with $W_2^{l+2,\frac{l}{2}+1}(Q_\infty)$ norm controlled by the $W_2^{l+2,\frac{l}{2}+1}(Q_T)$-norm of $\boldsymbol{w}_0$ if $T \ge 1$, and, using theorem 2.3.3, by its $H_2^{l+2,\frac{l}{2}+1}(Q_T)$-norm if $T \le 1$. Both controls are made with a constant independent of T. We then define, for all $t \ge 0$,

$$\boldsymbol{F}_0 = \boldsymbol{w}_0, \qquad h = \nabla\cdot\boldsymbol{w}_0,$$

and $\boldsymbol{f}_0 = \boldsymbol{w}_{0,t} - \nu\Delta\boldsymbol{w}_0$. It is clear that problem (4.1) is equivalent to the same problem with $\boldsymbol{F}_0$ instead of $\boldsymbol{F}$. It holds

$$\|h\|_{W_2^{l+1,0}(Q_\infty)}+\|\boldsymbol{F}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)} \le \begin{cases} c(\|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}) & \text{if } T \ge 1,\\ c(\|h\|_{H_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)}) & \text{if } T \le 1,\end{cases}$$

with constants independent of T, and

$$\begin{aligned}\|\boldsymbol{f}_0\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} &\le \|\boldsymbol{w}_0\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)}\\ &\le c(\|h\|_{W_2^{l+1,0}(Q_\infty)} + \|\boldsymbol{F}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)}).\end{aligned} \tag{4.86}$$

We now let

$$\sigma_0 = g(0) - \boldsymbol{v}_b \cdot \nabla' \rho_0 - \nabla' \phi_b \cdot \boldsymbol{u}_0 + u_0^3,$$

and we construct ρ_1 in such a way that

$$\rho_1 \lfloor_{t=0} = \rho_0, \qquad \rho_{1,t} \lfloor_{t=0} = \sigma_0,$$

$$\begin{aligned} &\|\rho_1\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} \leq c(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\sigma_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}) \\ &\leq c\big(\|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}\big) \end{aligned} \tag{4.87}$$

To construct such ρ_1 we first find, through theorem 2.1.6, $r_1 \in W_2^{l+\frac{5}{2}}(G_\infty)$ such that $r_1 \lfloor_{t=0} = \rho_0$, $r_{1,t} \lfloor_{t=0} = 0$ and

$$\|r_1\|_{W_2^{l+\frac{5}{2}}(G_\infty)} \leq c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}.$$

Then we construct, through theorem 2.2.1, point 4, $r_2 \in W_2^{l+\frac{7}{2},\frac{l}{2}+\frac{7}{4}}(G_\infty)$ such that $r_2 \lfloor_{t=0} = 0$, $r_{2,t} \lfloor_{t=0} = \sigma_0$ and

$$\begin{aligned} \|r_2\|_{W_2^{l+\frac{7}{2},\frac{l}{2}+\frac{7}{4}}(G_\infty)} &\leq \|\sigma_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})} \\ &\leq c\big(\|g(0)\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + \|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\big) \\ &\leq c\big(\|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}\big), \end{aligned}$$

for a constant c depending only on $\boldsymbol{v}_b$ and ϕ_b. The sum $\rho_1 = r_1 + r_2$ clearly satisfies the initial boundary conditions and

$$\|\rho_1\|_{W_2^{l+\frac{5}{2},0}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} \leq c(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\sigma_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}).$$

Finally, from the inequality

$$(1 + |\xi_0| + |\xi|^2)^{l+\frac{5}{2}} \leq c_l\big((1 + |\xi|^2)^{l+\frac{5}{2}} + (1 + |\xi_0| + |\xi|^2)^{l+\frac{1}{2}}|\xi_0|^2\big),$$

we get, through local coordinates, Fourier transform and Parceval identity

$$\|\rho_1\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_\infty)} \leq c(\|\rho_1\|_{W_2^{l+\frac{5}{2},0}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_\infty)}),$$

and thus (4.87). clearly we can modify ρ_1 so as to obtain $\rho_1 = 0$ for $t \geq T_0$, without affecting the latter inequality. Then we take out a part of the divergence, considering $\boldsymbol{w}_1 = \nabla\psi$ where ψ is the periodic solution of

$$\begin{cases} \Delta\psi = \Phi_2(\rho_1) = \nabla \cdot \boldsymbol{F}_1 & \text{in } \Omega_b, \\ \psi = 0 & \text{on } \mathcal{G}, \\ \frac{\partial\psi}{\partial\boldsymbol{n}} = \boldsymbol{F}_1 \cdot \boldsymbol{n} = 0 & \text{on } \Sigma, \end{cases}$$

where $\boldsymbol{F}_1 = (I - \widehat{\mathcal{L}}(\rho_1))\boldsymbol{v}_b$, which vanishes in a neighbourhood of Σ. Notice that since ρ_1 vanishes for $t \geq T_0$ this is also true for $\boldsymbol{w}_1$. We set $\boldsymbol{f}_1 = \boldsymbol{w}_{1,t} - \nu\Delta\boldsymbol{w}_1$. With the same argument as for problem (4.85), we get

$$\|\boldsymbol{f}_1\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \leq c\|\boldsymbol{w}_1\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} \leq c(\|\Phi_2(\rho_1)\|_{W_2^{l+1}(Q_\infty)} + \|\boldsymbol{F}_1\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)}),$$

and looking at the explicit form (3.21), (3.20) of Φ_2 and $\boldsymbol{F}_1$ we get

$$\begin{aligned}\|\boldsymbol{w}_1\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} &\leq c(\|\rho_1\|_{W_2^{l+2,0}(Q_\infty)} + \|\nabla\rho_1\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)} + \|\rho_1\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)}) \\ &\leq c(\|\rho_1\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(Q_\infty)} + \|\nabla\rho_{1,t}\|_{W_2^{0,\frac{l}{2}}(Q_\infty)}) \\ &\leq c(\|\rho_1\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(Q_\infty)} + \|\rho_{1,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(Q_\infty)})\end{aligned}$$

which gives, by (4.87),

$$\begin{aligned}\|\boldsymbol{f}_1\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} &\leq c\|\boldsymbol{w}_1\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} \\ &\leq c(\|g\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}).\end{aligned} \tag{4.88}$$

We then choose $\boldsymbol{w}_2 \in W_2^{l+2,\frac{l}{2}+1}(Q_\infty)$ in such a way that

$$\nabla\cdot\boldsymbol{w}_2 = 0, \quad \forall t \geq 0, \qquad \boldsymbol{w}_2(\cdot,0) = \boldsymbol{u}_0(\cdot) - \boldsymbol{w}_0(\cdot) - \boldsymbol{w}_1(\cdot,0),$$

with $\boldsymbol{w}_2 = 0$ for $t > T_0$, and optimal regularity estimates. To do this, notice that $w_1^3 = w_0^3 = 0$ on Σ, and $\nabla\cdot(\boldsymbol{u}_0(x) - \boldsymbol{w}_0(x,0) - \boldsymbol{w}_1(x,0)) = 0$ in Ω_b by the first condition in (4.81). By a result of Bogowskii [7], $\boldsymbol{u}_0(x) - \boldsymbol{w}_0(x,0) - \boldsymbol{w}_1(x,0)$ can be extended with preservation of class and solenoidality for all $x \in \mathbb{R}^3$, as a vector $\boldsymbol{w}_2^*$. We can set

$$\boldsymbol{w}_2(x,t) = \phi(t)\int_{\mathbb{R}^3}\Gamma(x-y,t)\boldsymbol{w}_2^*(y)dy,$$

where $\phi(t)$ is a smooth function equal to one for small t and vanishing for $t \geq T_0$, and

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{|x|^2}{4t}},$$

is the fundamental solution of the heat equation. Well known estimates of the heat potential give

$$\|\boldsymbol{w}_2\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} \leq c\|\boldsymbol{w}_2^*\|_{W_2^{l+1}(\mathbb{R}^3)} \leq c\|\boldsymbol{u}_0 - \boldsymbol{w}_1\|_{W_2^{l+1}(\Omega_b)}.$$

Letting $\boldsymbol{f}_2 = \boldsymbol{w}_{2,t} - \nu\Delta\boldsymbol{w}_2$, we get

$$\begin{aligned}\|\boldsymbol{f}_2\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} &\le c\|\boldsymbol{w}_2\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)}\\ &\le c(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\boldsymbol{w}_0\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} + \|\boldsymbol{w}_1\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)})\end{aligned} \tag{4.89}$$

Finally we set $\boldsymbol{f}_3 = -\boldsymbol{\Phi}_1(\boldsymbol{w}_0 + \boldsymbol{w}_1 + \boldsymbol{w}_2, \rho_1)$. From the explicit structure of $\boldsymbol{\Phi}_1$ given in (3.18) and applying (2.20), one sees that

$$\begin{aligned}\|\boldsymbol{f}_3\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \le & c\Big(\sum_{i=0}^{3}\|\boldsymbol{w}_i\|_{W_2^{l+1,\frac{l}{2}+\frac{1}{2}}(Q_\infty)}\\ &+ \|\rho_1\|_{W_2^{l+2,\frac{l}{2}+1}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l,\frac{l}{2}}(G_\infty)}\Big),\end{aligned}$$

which gives, *a fortiori*,

$$\begin{aligned}\|\boldsymbol{f}_3\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \le & c\Big(\sum_{i=0}^{3}\|\boldsymbol{w}_i\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)}\\ &+ \|\rho_1\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{2}}(G_\infty)}\Big)\end{aligned} \tag{4.90}$$

Letting

$$\sigma_2 = \nu\boldsymbol{N}\cdot\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} + \sigma L\rho_0 - \boldsymbol{d}(0)\cdot\boldsymbol{N} - \Phi_4(\rho_0),$$

we have $\sigma_2 \in W_2^{l-\frac{1}{2}}(\mathcal{G})$ (the four terms have regularity, respectively, l, l, $l-\frac{1}{2}$ and $l+1$) and thus we can extend it to the whole Ω_b as $\widehat{\sigma}_2 \in W_2^l(\Omega_b)$ with controlled norm, and subsequently define $p_1 \in W^{l+1,\frac{l}{2}+\frac{1}{2}}(Q_\infty)$ as an extension of $\widehat{\sigma}_2$ to Q_∞, also with controlled norm. Therefore p_1 satisfies

$$p_1(0)\lfloor_{\mathcal{G}} = \nu\boldsymbol{N}\cdot\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} + \sigma L\rho_0 - \boldsymbol{d}(0)\cdot\boldsymbol{N} - \Phi_4(\rho_0),$$

$$\|p_1\|_{W_2^{l+1,\frac{l}{2}+\frac{1}{2}}(Q_\infty)} \le c(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{d}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_\infty)}), \tag{4.91}$$

where we also used the fact that $\Phi_4(\rho_0)$, given in (3.28), is of the type $\nabla\rho_0\cdot\boldsymbol{M}$ for regular $\boldsymbol{M}$'s depending on $\boldsymbol{v}_b$ and ϕ_b.

We finally define $\boldsymbol{w} = \boldsymbol{w}_0 + \boldsymbol{w}_1 + \boldsymbol{w}_2$ and

$$\begin{aligned}&(\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho}) = (\boldsymbol{u} - \boldsymbol{w}, q - p_1, \rho - \rho_1)\\ &\widehat{\boldsymbol{f}} = \boldsymbol{f} - \nabla p_1 - \sum_{i=0}^{3}\boldsymbol{f}_i,\\ &\widehat{\boldsymbol{d}} = \boldsymbol{d} - \nu\mathbb{D}(\boldsymbol{w})\boldsymbol{N} - \sigma L\rho_1\boldsymbol{N} + p_1\boldsymbol{N} + \widetilde{\boldsymbol{\Phi}}(\rho_1),\\ &\widehat{g} = g - \rho_{1,t} - \nabla'\phi_b\cdot\boldsymbol{w} + w^3 - \boldsymbol{v}_b\cdot\nabla'\rho_1,\\ &\widehat{\boldsymbol{a}} = \boldsymbol{a} - \boldsymbol{w}.\end{aligned}$$

Problem (4.1) is then reduced to

$$\begin{cases} \widehat{\boldsymbol{u}}_{,t} - \nu\Delta\widehat{\boldsymbol{u}} + \nabla\widehat{q} - \boldsymbol{\Phi}_1(\widehat{\boldsymbol{u}},\widehat{\rho}) = \widehat{\boldsymbol{f}} & \text{in } \Omega_b, \\ \nabla\cdot\widehat{\boldsymbol{u}} - \Phi_2(\widehat{\rho}) = 0 & \text{in } \Omega_b, \\ \mathbb{T}(\widehat{\boldsymbol{u}},\widehat{q})\boldsymbol{N} + \sigma L\widehat{\rho}\boldsymbol{N} - \widetilde{\boldsymbol{\Phi}}(\widehat{\rho}) = \widehat{\boldsymbol{d}} & \text{on } \mathcal{G}, \\ \widehat{\rho}_t + \nabla'\phi_b\cdot\widehat{\boldsymbol{u}} - \widehat{u}^3 + \nabla'\widehat{\rho}\cdot\boldsymbol{v}_b = \widehat{g} & \text{on } \mathcal{G}, \\ \widehat{\boldsymbol{u}} = \widehat{\boldsymbol{a}} \text{ on } \Sigma, \text{ for all } t\geq 0, \\ \widehat{\boldsymbol{u}}(x,0) = 0, \quad x\in\Omega_b, \quad \widehat{\rho}(x',0) = 0, \quad x\in\Sigma, \end{cases} \tag{4.92}$$

where $\widehat{g}(0) = 0$ by the definition of ρ_1, $\widehat{\boldsymbol{a}}(0) = 0$ by the third condition in (4.81) and the definition of $\boldsymbol{w}_2$, and $\widehat{\boldsymbol{d}}(0) = 0$ by the second condition in (4.81) and the definition of $p_1(0)$. By theorem 2.2.4 and the condition $l < 1$, $\widehat{\boldsymbol{f}}$, $\widehat{\boldsymbol{d}}$, $\widehat{g}$ and $\widehat{\boldsymbol{a}}$ can be extended as 0 for $t < 0$ preserving regularity, therefore we can apply the Laplace transform to convert problem (4.92) to a problem of the form (4.74). The latter is solvable for $\operatorname{Re}\lambda \geq \gamma > 0$ for γ sufficiently large by theorem 4.3.1. Taking inverse Laplace transform gives a solution in weighted anisotropic Sobolev-Slobodetskii space $W^{\eta,\frac{\eta}{2}}_{2,\gamma'}$ (see (2.23)) for $\gamma' > \gamma$, defined for all t and vanishing for $t < 0$.

The rest of the proof is analogous as the one of theorem 4.1.4. We obtain a weighted estimated, which can be localised in $[0, T)$ on the left hand side with the suitable norms ($W^{\mu,\frac{\mu}{2}}_2$ if $T \geq 1$, $H^{\mu,\frac{\mu}{2}}_2$ otherwise). For the right hand side, we can control it through a non weighted estimate since all the terms vanish for $t \geq T_0$. If T is large, this procedure of eliminating the weights gives rise to a constant $C(T)$ in the bounds, which does not appear if $T \leq 1$. Moreover one can easily check, from (4.86)–(4.91), that it holds the inequality

$$\begin{aligned} &\|\widehat{\boldsymbol{f}}\|_{W^{l,\frac{l}{2}}_2(Q_\infty)} + \|\widehat{\boldsymbol{d}}\|_{W^{l+\frac12,\frac{l}{2}+\frac14}_2(G_\infty)} + \|\widehat{g}\|_{W^{l+\frac32,\frac{l}{2}+\frac34}_2(G_\infty)} + \|\widehat{\boldsymbol{a}}\|_{W^{l+\frac32,\frac{l}{2}+\frac34}_2(\Sigma_\infty)} \\ &\leq c\big(\|\boldsymbol{f}\|_{W^{l,\frac{l}{2}}_2(Q_\infty)} + \|h\|_{W^{l+1,0}_2(Q_\infty)} + \|\boldsymbol{F}_0\|_{W^{0,\frac{l}{2}+1}_2(Q_\infty)} + \|\boldsymbol{d}\|_{W^{l+\frac12,\frac{l}{2}+\frac14}_2(G_\infty)} \\ &+ \|g\|_{W^{l+\frac32,\frac{l}{2}+\frac34}_2(G_\infty)} + \|\boldsymbol{a}\|_{W^{l+\frac32,\frac{l}{2}+\frac34}_2(\Sigma_\infty)} + \|\boldsymbol{u}_0\|_{W^{l+1}_2(\Omega_b)} + \|\rho_0\|_{W^{l+2}_2(\mathcal{G})}\big), \end{aligned}$$

and the right hand side is bounded by

$$\begin{aligned} &c\big(\|\boldsymbol{f}\|_{W^{l,\frac{l}{2}}_2(Q_T)} + \|h\|_{W^{l+1,0}_2(Q_T)} + \|\boldsymbol{F}_0\|_{W^{0,\frac{l}{2}+1}_2(Q_T)} + \|\boldsymbol{d}\|_{W^{l+\frac12,\frac{l}{2}+\frac14}_2(G_T)} \\ &+ \|g\|_{W^{l+\frac32,\frac{l}{2}+\frac34}_2(G_T)} + \|\boldsymbol{a}\|_{W^{l+\frac32,\frac{l}{2}+\frac34}_2(\Sigma_T)} + \|\boldsymbol{u}_0\|_{W^{l+1}_2(\Omega_b)} + \|\rho_0\|_{W^{l+2}_2(\mathcal{G})}\big), \end{aligned}$$

if $T \geq 1$, or by

$$\begin{aligned} &c\big(\|\boldsymbol{f}\|_{H^{l,\frac{l}{2}}_2(Q_T)} + \|h\|_{H^{l+1,0}_2(Q_T)} + \|\boldsymbol{F}_0\|_{\widehat{W}^{0,\frac{l}{2}+1}_2(Q_T)} + \|\boldsymbol{d}\|_{H^{l+\frac12,\frac{l}{2}+\frac14}_2(G_T)} \\ &+ \|g\|_{H^{l+\frac32,\frac{l}{2}+\frac34}_2(G_T)} + \|\boldsymbol{a}\|_{H^{l+\frac32,\frac{l}{2}+\frac34}_2(\Sigma_T)} + \|\boldsymbol{u}_0\|_{W^{l+1}_2(\Omega_b)} + \|\rho_0\|_{W^{l+2}_2(\mathcal{G})}\big), \end{aligned}$$

if $T \leq 1$, both with constant independent of T. The same inequalities also hold for $\|(\boldsymbol{w}, p_1, \rho_1)\|_{W,l,T}$ and $\|(\boldsymbol{w}, p_1, \rho_1)\|_{H,l,T}$ in the two cases, and thus summing back those term to $(\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho})$ doesn't affect the estimates.

Finally, uniqueness and the proof of (4.83) are obtained in exactly the same way described in the proof of theorem 4.1.4. $\square$

Chapter 5

The nonlinear problem

In this chapter we study the original nonlinear problem (1.2), proving two types of result. The first one is an abstract linearization principle, which roughly speaking states that if the linearized problem is stable, then the nonlinear problem has a global smooth solution if the initial data are sufficiently small. The second one is a local in time existence and uniqueness result for the nonlinear problem, with arbitrary initial data.

In the first section we will prove (at least) quadratic estimates for the nonlinear terms appearing after the Hanzawa transformation. This is done with a different method than the one used, for example, in [30], where a modification of the time-related part of norm is performed.

In the second section the abstract linearization principle is proved, constructing the global solution in a sufficiently large interval and then repeating the construction step by step on multiples of the initial interval. It is worth noting that the presence of a nonlinearity in the equation for the divergence requires a splitting method for the construction of the solution, used for example in [30].

In the third section we apply the linearization principle to obtain exponential stability of the rest state. While the hypothesis required in the linearization principle seem rather abstract, in this case an explicit estimate on the spectrum is possible.

In the last section we give a rather sketchy description of the proof the local existence of the solution. The methods developed in [19] certainly apply to this case, once one has coercive estimates for the linear problem. However, we chose to safeguard consistency and prove that a sub-optimal (in the regularity sense) choice of Hanzawa transformation can still give the same result.

5.1 Estimate of the nonlinear terms

Our aim is to obtain estimates of the form

$$\|l_i(\boldsymbol{u},q,\rho)\| \leq C\big(\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} + \|\nabla p\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|p\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} \\ + \|\rho\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)} + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)}\big)^2,$$

where for each term $i = 0, \dots 5$ we will give such bounds for a suitable norm $\|l_i\|$, depending on l. Our main assumption will be that the hanzawa transformation (3.1) is well defined, and thus we require that $\sup_\rho \ll \mu(\theta)$. This ensures that all the nonlinear terms are polynomials in the derivatives of $\boldsymbol{u}$, p and ρ multiplied by a nonlinear term which is of the form $f(x,\rho,\nabla\rho)$, with a supposedly smooth f. Indeed the only singularity in the nonlinear terms appears in the Jacobian of the Hanzawa transformation, where

$$\det\mathcal{L}^{-1} = \frac{1}{1+\theta'\rho}.$$

Notice that, as long as T is bounded away from zero, say $T \geq 1$, it holds, for any $r > 1$

$$\sup_\Sigma |\rho| \leq c\|\rho\|_{W_2^r(\Sigma)} \leq c\|\rho\|_{W_2^r(\mathcal{G})},$$

with a constant independent of T. In the following, we will call μ any positive number such that

$$\mu \ll \frac{1}{\sup_\Sigma |\theta'|}. \tag{5.1}$$

so that for example the condition

$$\|\rho\|_{W_2^{l+2}(\mathcal{G})} \leq \mu$$

will ensure $|\rho\theta'| < \frac{1}{2}$ and the smoothness of the nonlinear terms.

More precisely we will prove the following theorem.

Theorem 5.1.1 *Let $l \in (\frac{1}{2},1)$. Suppose that $\|(\boldsymbol{u},p,\rho)\|_{W,l,T} \leq \mu$ such that (5.1) holds. There exists $c(\mu)$, bounded for bounded μ such that*

$$\|\widetilde{\boldsymbol{l}}_0(\boldsymbol{u},\rho)\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|\widetilde{\boldsymbol{l}}_1(\boldsymbol{u},p,\rho)\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|\widetilde{l}_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)} \\ + \|\boldsymbol{G}(\boldsymbol{u},\rho)\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} + \|\widetilde{\boldsymbol{l}}_3(\boldsymbol{u},\rho)\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|\widetilde{l}_4(\boldsymbol{u},\rho)\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} \\ + \|\widetilde{l}_5(\boldsymbol{u},\rho)\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \leq c(\mu)\|(\boldsymbol{u},p,\rho)\|^2_{W,l,T}.$$

The constant $c(\mu)$ also depends on $\boldsymbol{v}_b$, p_b, ϕ_b and T.

In the rest of this section we will thus always suppose $l \in (\frac{1}{2}, 1)$, $\|(\boldsymbol{u}, p, \rho)\|_l \leq \mu$. Moreover, for any given function $g : X \times Y \to \mathbb{R}$, and positive η, η' we will we use the following notation

$$\|g\|_{W_2^{\eta}(X)} = \|g(\cdot, y)\|_{W_2^{\eta}(X)}, \qquad \|g\|_{W_2^{\eta'}(Y)} = \|g(x, \cdot)\|_{W_2^{\eta'}(Y)},$$

the right hand sides being functions of y and x respectively.

Since $\rho^* = \theta\rho$ and θ is C^∞, any norm of ρ^* in Ω_b or Q_T is bounded by the same norm of ρ in $\mathcal{G}$ or G_T. Notice then that, letting from now on

$$\|\rho\|_l = \|\rho\|_{W_2^{l+\frac{5}{2}, \frac{l}{2}+\frac{5}{4}}(G_T)} + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2}, \frac{l}{2}+\frac{3}{4}}(G_T)},$$

theorem 2.2.3 gives

$$\sup_{t<T} \|\rho\|_{W_2^{l+2}(\mathcal{G})} \leq c\big(\|\rho\|_{W_2^{l+\frac{5}{2},0}(G_T)} + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2},0}(G_T)}\big) \leq c\|\rho\|_l. \tag{5.2}$$

From (2.5), it follows

$$\sup_{Q_T} |\rho^*| + |\nabla\rho^*| \leq c(\sup_{G_T} |\rho| + |\nabla'\rho|) \leq c\|\rho\|_l. \tag{5.3}$$

We will also frequently use the following bounds:

$$\begin{aligned}
&\sup_{\mathcal{G}} \|\rho\|_{W_2^{\frac{l}{2}+\frac{5}{4}}(0,T)} + \|\nabla\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} \\
&\quad \leq c\big(\sup_{G_T}(|\rho| + |\nabla\rho|) + \sup_{\mathcal{G}} \|\rho_{,t}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)} + \|\nabla\rho_{,t}\|_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)}\big) \\
&\quad \leq c\big(\|\rho\|_l + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2}, \frac{l}{2}+\frac{3}{4}}(G_T)} + \|\nabla\rho_{,t}\|_{W_2^{l+\frac{1}{2}, \frac{l}{2}+\frac{1}{4}}(G_T)}\big) \leq c\|\rho\|_l,
\end{aligned} \tag{5.4}$$

$$\sup_{\Omega_b} \|\boldsymbol{u}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)} \leq c\|\boldsymbol{u}\|_{W_2^{l+2, \frac{l}{2}+1}(Q_T)}, \tag{5.5}$$

Indeed (5.4) and (5.5) follow from repeated application (twice for ρ and thrice for $\boldsymbol{u}$) of the standard estimate for the restriction operator in the anisotropic Sobolev–Slobodetskii spaces. The constant in these inequalities depends on T, $\mathcal{G}$ and Ω_b, and remain bounded as long as T is bounded away from 0, which will always be the case in the following.

From now on we will suppose that $\|\rho\|_l \leq \mu$ for μ satisfying (5.1).

Lemma 5.1.2 *Let $l \in (\frac{1}{2}, 1)$, and suppose $\|\rho\|_l \leq \mu$. Given a smooth function $f : \mathcal{G} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, there exists a constant $c_f(\mu)$, bounded for bounded μ, such that for any function $g = g(x, t)$ and any $\eta \leq 1 + l$, $\eta' \leq \frac{l}{2} + \frac{3}{4}$, it holds*

$$\|f(x, \rho, \nabla\rho)g\|_{W_2^{\eta,0}(G_T)} \leq c_f(\mu)\|g\|_{W_2^{\eta,0}(G_T)}, \tag{5.6}$$

$$\|f(x, \rho, \nabla\rho)g\|_{W_2^{0,\eta'}(G_T)} \leq c_f(\mu)\|g\|_{W_2^{0,\eta'}(G_T)}. \tag{5.7}$$

The constant $c_f(\mu)$ also depends on $\mathcal{G}$ and T.

Proof. We claim that

$$\sup_{t<T} \|f(x,\rho,\nabla\rho)\|_{W_2^{1+l}(\mathcal{G})} \le c_f(\mu). \tag{5.8}$$

To prove this, first notice that by (5.3) and the smoothness of f,

$$f(x,\rho,\nabla\rho), \quad f_x(x,\rho,\nabla\rho), \quad \text{and} \quad f_s(x,\rho,\nabla\rho)\nabla\rho,$$

are bounded by a constant $c_f(\mu)$ independent of t, and thus the same holds true for their $L^2(\mathcal{G})$ norm. Denoting by $\mathring{W}_2^{1+l}(\mathcal{G})$ the principal part of the norm $W_2^{1+l}(\mathcal{G})$, its square is bounded by

$$\|f_x(x,\rho,\nabla\rho)\|^2_{W_2^l(\mathcal{G})} + \|f_s(x,\rho,\nabla\rho)\nabla\rho\|^2_{W_2^l(\mathcal{G})} + \|f_p(x,\rho,\nabla\rho)D^2\rho\|^2_{W_2^l(\mathcal{G})}, \tag{5.9}$$

and we have to estimate these three terms. The first one is readily bounded as

$$\begin{aligned}\|f_x(x,\rho,\nabla\rho)\|^2_{W_2^l(\mathcal{G})} \le & \|f_x(x,\rho,\nabla\rho)\|^2_{L^2(\mathcal{G})} + \|f_{xx}(x,\rho,\nabla\rho)\|^2_{L^2(\mathcal{G})} + \\ & + \|f_{xs}(x,\rho,\nabla\rho)\nabla\rho\|^2_{L^2(\mathcal{G})} + \|f_{xp}(x,\rho,\nabla\rho)D^2\rho\|^2_{L^2(\mathcal{G})},\end{aligned}$$

since the first three addends are bounded, and the fourth is estimated through (5.2). For the second term in (5.9) we apply proposition 2.1.3

$$\|f_s(x,\rho,\nabla\rho)\nabla\rho\|_{W_2^l(\mathcal{G})} \le c\|f_s(x,\rho,\nabla\rho)\|_{W_2^l(\mathcal{G})}\|\nabla\rho\|_{W_2^{1+l}(\mathcal{G})},$$

and the first factor can be estimated as $f_x(x,\rho,\nabla\rho)$ above, while the second is less than μ by (5.2). Let us now estimate the third term in (5.9): applying the mean value theorem we have

$$\begin{aligned}\|f_p(x,\rho,\nabla\rho)D^2\rho\|^2_{W_2^l(\mathcal{G})} \le c_f(\mu) + c\int_{|z|\le 1} \|f_p(x,\rho,\nabla\rho)\Delta_{-z}D^2\rho\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}} \\ + c\int_{|z|\le 1}\|D^2\rho Df_p(\boldsymbol{\xi}_z)(z,\Delta_{-z}\rho,\Delta_{-z}\nabla\rho)\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}},\end{aligned}$$

for some uniformly bounded function $\boldsymbol{\xi}_z$. Since $f_p(x,\rho,\nabla\rho)$ is bounded,

$$\int_{|z|\le 1}\|f_p(x,\rho,\nabla\rho)\Delta_{-z}D^2\rho\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}} \le c_f(\mu)\|\rho\|_{W_2^{l+2}(\mathcal{G})}.$$

Moreover, by proposition 2.1.3, point 1

$$\begin{aligned}&\int_{|z|\le 1}\|D^2\rho Df_p(\boldsymbol{\xi}_z)(z,\Delta_{-z}\rho,\Delta_{-z}\nabla\rho)\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}} \\ &\le c_f(\mu)\int_{|z|\le 1}\|D^2\rho(|z|+\Delta_{-z}\rho+\Delta_{-z}\nabla\rho)\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}} \le \\ &c_f(\mu)\big(\|D^2\rho\|^2_{L^2(\mathcal{G})} + \|D^2\rho\|^2_{W_2^l(\mathcal{G})}\int_{|z|\le 1}\|\Delta_{-z}\rho\|^2_{W_2^{1-l}(\mathcal{G})} + \|\Delta_{-z}\nabla\rho\|^2_{W_2^{1-l}(\mathcal{G})}\frac{dz}{|z|^{2+2l}}\big) \\ &\le c_f(\mu)\mu^2\big(1+\int_{|z|\le 1}\|\Delta_{-z}\rho\|^2_{L^2(\mathcal{G})} + \|\Delta_{-z}\nabla\rho\|^2_{L^2(\mathcal{G})} + \|\Delta_{-z}D^2\rho\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}}\big),\end{aligned}$$

where we bounded the W_2^{1-l} with the W_2^1 one. The last integral is bounded by $\|\rho\|^2_{W_2^{l+2}(\mathcal{G})}$, and using (5.2), we obtain a bound depending only on f and μ. Taking the supremum in $t < T$ in all these bounds gives (5.8).

For the time derivative, we claim

$$\sup_{\mathcal{G}} \|f(x,\rho,\nabla\rho)\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} \le c_f(\mu). \tag{5.10}$$

Indeed the $L^2(0,T)$ norm is bounded again by (5.3), while with the same argument as before we get, for any $x \in \mathcal{G}$

$$\begin{aligned}
&\|f_s(x,\rho,\nabla\rho)\rho_{,t}\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \le c_f\Big[\|\rho_{,t}(x,\cdot)\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \\
&+ \int_0^T \frac{dh}{h^{\frac{1+l}{2}}}\|\Delta_{-h} f(x,\rho,\nabla\rho)\rho_{,t}\|^2_{L^2(0,T)}\Big] \le c_f\Big[\|\rho_{,t}(x,\cdot)\|_{W^{l+1,\frac{l}{2}+1}(G_T)} \\
&+ \|\rho_{,t}(x,\cdot)\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \int_0^T \big(\|\Delta_{-h}\rho\|^2_{W_2^{\frac{3}{4}-\frac{l}{2}}(0,T)} + \|\Delta_{-h}\nabla\rho\|^2_{W_2^{\frac{3}{4}-\frac{l}{2}}(0,T)} + h^2\big)\frac{dh}{h^{\frac{1+l}{2}}}\Big] \\
&\le c_f\mu^2\Big[1 + \int_0^T \big(\|\Delta_{-h}\rho\|^2_{L^2(0,T)} + \|\Delta_{-h}\nabla\rho\|^2_{L^2(0,T)} + \|\Delta_{-h}\rho_{,t}\|^2_{L^2(0,T)} \\
&+ \|\Delta_{-h}\nabla\rho_{,t}\|^2_{L^2(0,T)}\big)\frac{dh}{h^{\frac{1+l}{2}}}\Big] \le c_f\mu^2\big(1 + \|\rho(x,\cdot)\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} + \|\nabla\rho(x,\cdot)\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \\
&+ \|\rho_{,t}(x,\cdot)\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} + \|\nabla\rho_{,t}(x,\cdot)\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)}\big) \\
&\le c_f\mu^2\big(1 + \|\rho\|^2_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} + \|\rho_{,t}\|^2_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)}\big) = c_f(\mu)
\end{aligned}$$

and with exactly the same procedure

$$\sup_{\mathcal{G}} \|f_p(x,\rho,\nabla\rho)\nabla\rho_{,t}\|_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \le c_f(\mu).$$

Now we can apply proposition 2.1.3, noting that

$$\|fg\|^2_{W_2^{\eta,0}(G_T)} = \int_0^T \|fg\|^2_{W_2^{\eta}(\mathcal{G})} dt \le \sup_{t<T}\|f\|^2_{W_2^{1+l}(\mathcal{G})} \int_0^T \|g\|^2_{W_2^{\eta}(\mathcal{G})} dt, \tag{5.11}$$

$$\|fg\|^2_{W_2^{0,\eta'}(G_T)} = \int_{\mathcal{G}} \|fg\|^2_{W_2^{\eta'}(0,T)} dx \le \sup_{\mathcal{G}}\|f\|^2_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} \int_{\mathcal{G}} \|g\|^2_{W_2^{\eta'}(0,T)} dt, \tag{5.12}$$

and thus (5.8) and (5.10) for $f = f(x,\rho,\nabla\rho)$ give the claim of the lemma. □

Remark 5.1.3 *In the following, we will have to estimate also functions of the form $f(x,\rho^*(x),\nabla\rho^*(x))$ for $x \in \Omega_b$. The proof of* (5.10) *and* (5.8) *carries over in this case, using the fact that any norm of ρ^* on Ω_b is bounded by the same norm of ρ on $\mathcal{G}$. For the final step, we recall that $1 + l > \frac{3}{2}$ and thus proposition 2.1.3 still applies.*

Remark 5.1.4 *In the proof of theorem 5.1.1 we will actually need values of* η' *in the range* $\eta' \in [0, \frac{l}{2} + \frac{1}{4}]$. *Since* $\frac{l}{2} + \frac{1}{4} < 1$, *one can prove instead of* (5.10) *the estimate*

$$\sup_{\mathcal{G}} \|f(x, \rho, \nabla\rho)\|_{W_2^1(0,T)} \leq c_f(\mu),$$

which actually allows to prove the lemma for $\eta' \leq 1$. *This is simpler to prove since the* L^2 *norm is bounded and*

$$\begin{aligned} &\|f_s(x, \rho, \nabla\rho)\rho_{,t}\|_{L^2(0,T)} + \|f_p(x, \rho, \nabla\rho)\nabla\rho_{,t}\|_{L^2(0,T)} \\ &\qquad \leq c_f(\mu)(\sup_{\mathcal{G}} \|\rho\|_{W_2^1(0,T)} + \|\nabla\rho\|_{W_2^1(0,T)}), \end{aligned}$$

which allows to conclude by (5.4) *since* $\frac{l}{2} + \frac{3}{4} > 1$.

From now on c will denote a constant depending on μ, the base state of the system $(\Omega_b, \boldsymbol{v}_b, p_b)$ and a finite set of functions $f = f(x, s, p)$, which can change from line to line but will be anyway denoted by c.

Estimate of $\|\boldsymbol{l}_0\|_{W_2^{l,\frac{l}{2}}(Q_T)}$.

For the norm $\|\boldsymbol{l}_0\|_{W_2^{l,0}(Q_T)}$, recalling the explicit formula (3.19) and (3.15), we notice that the various addends (except $\boldsymbol{u} \cdot \nabla\boldsymbol{u}$) of $\widetilde{\boldsymbol{l}}_0$ are linear combinations of terms of the type

$$\rho^*_{,x_i}\rho^* g(x, \rho^*, \nabla\rho^*) v^j w^k_{,x_l},$$

with $\boldsymbol{v}$ and $\boldsymbol{w}$ equal to $\boldsymbol{v}_b$ or $\boldsymbol{u}$ and all possible ways. These terms are estimated in the $W_2^{l,\frac{l}{2}}(\Omega_b)$ norm through Lemma 5.1.2 and a repeated application of (5.11), (5.12). We have, for any $\eta \leq 1 + l$ and $\eta' \leq \frac{l}{2} + \frac{1}{4}$:

$$\begin{aligned} &\|\rho^*_{,x_i}\rho^* g(x, \rho^*, \nabla\rho^*) v^j w^k_{,x_l}\|_{W_2^{\eta,0}(Q_T)} \\ &\leq c \sup_{t<T} \|\nabla\rho\|_{W_2^{1+l}(\mathcal{G})} \sup_{t<T} \|\rho\|_{W_2^{1+l}(\mathcal{G})} \sup_{t<T} \|\boldsymbol{v}\|_{W_2^{1+l}(\Omega_b)} \|\nabla\boldsymbol{w}\|_{W_2^{\eta,0}(Q_T)}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} &\|\rho^*_{x_i}\rho^* g(x, \rho^*, \nabla\rho^*) v^j w^k_{x_l}\|_{W_2^{0,\eta'}(Q_T)} \\ &\leq c \sup_{\mathcal{G}} \|\nabla\rho\|_{W_2^1(0,T)} \sup_{\mathcal{G}} \|\rho\|_{W_2^1(0,T)} \sup_{\Omega_b} \|\boldsymbol{v}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)} \|\nabla\boldsymbol{w}\|_{W_2^{0,\eta'}(Q_T)}. \end{aligned} \tag{5.14}$$

Now letting $\eta = l$ and $\eta' = \frac{l}{2}$, we apply (5.2)–(5.5) to obtain at least quadratic estimates in $\|\rho\|_l$ and $\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}$ for these terms. For the term $\boldsymbol{u} \cdot \nabla\boldsymbol{u}$ one proceeds in a similar way, obtaining the same estimate without the norms involving ρ.

For $\widetilde{\boldsymbol{l}}_0$ one thus have the estimate

$$\|\widetilde{\boldsymbol{l}}_0\|_{W_2^{l,\frac{l}{2}}(Q_T)} \leq c\big(\|\rho\|_l^2 + \|\rho\|_l \|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} + \|\boldsymbol{u}\|^2_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}\big).$$

Estimate of $\|\widetilde{\boldsymbol{l}}_1\|_{W_2^{l,\frac{l}{2}}(Q_T)}$.

We start from $\boldsymbol{l}_1(\boldsymbol{u}, q, \rho)$, as given in (3.8) : recall that $\widetilde{\nabla} = \mathcal{L}^T\nabla$ and

$$I - \mathcal{L}^{-T} = \frac{\nabla\rho^*}{1+\theta'\rho} \otimes \boldsymbol{e}_3, \tag{5.15}$$

therefore $(\nabla - \widetilde{\nabla})q$ is a linear combination of terms $\rho_{,x_i} f(x,\rho^*,\nabla\rho^*)q_{,x_j}$ and can be estimated as in (5.13), (5.14) with $\rho^* g(x,\rho^*,\nabla\rho^*)v^j = f(x,\rho^*,\nabla\rho^*)$ and $w^k = q$

$$\|\rho_{,x_i} f(x,\rho^*,\nabla\rho^*)q_{,x_j}\|_{W_2^{l,0}(Q_T)} \le c\sup_{t<T}\|\nabla\rho\|_{W_2^{1+l}(\mathcal{G})}\|\nabla q\|_{W_2^{l,0}(Q_T)},$$

$$\|\rho_{,x_i} f(x,\rho^*,\nabla\rho^*)q_{,x_j}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^{1}(0,T)}\|\nabla q\|_{W_2^{0,\frac{l}{2}}(Q_T)}.$$

Regarding the term $(\nabla^2 - \widetilde{\nabla}^2)\boldsymbol{u}$, we can use (5.15) to split it in several addends, of the type $f(x,\rho^*,\nabla\rho^*)$ times

$$\rho^*_{,x_j}\boldsymbol{u}_{,x_m}, \quad \rho^*_{,x_ix_j}\boldsymbol{u}_{,x_k}, \quad \text{or} \quad \rho^*_{,x_k}\boldsymbol{u}_{,x_ix_j}. \tag{5.16}$$

The terms of the first type are estimated as in (5.13),(5.14), setting

$$\rho^* g(x,\rho^*,\nabla\rho^*)v^j = f(x,\rho^*,\nabla\rho^*).$$

We estimate the $W_2^{l,0}(Q_T)$ norm of the other two terms through proposition 2.1.3 and lemma 5.1.2:

$$\|\rho^*_{,x_ix_j} f(x,\rho^*,\nabla\rho^*)\boldsymbol{u}_{,x_m}\|_{W_2^{l,0}(Q_T)} \le c\sup_{t<T}\|D^2\rho\|_{W_2^{l}(\mathcal{G})}\|\nabla\boldsymbol{u}\|_{W_2^{l+1,0}(Q_T)},$$

$$\|\rho^*_{,x_m} f(x,\rho^*,\nabla\rho^*)\boldsymbol{u}_{,x_ix_j}\|_{W_2^{l,0}(Q_T)} \le c\sup_{t<T}\|\nabla\rho\|_{W_2^{l+1}(\mathcal{G})}\|D^2\boldsymbol{u}\|_{W_2^{l,0}(Q_T)},$$

and we conclude using (5.2). Regarding the time derivative, the third term in (5.16) is estimated through (5.4):

$$\|\rho^*_{,x_m} f(x,\rho^*,\nabla\rho^*)\boldsymbol{u}_{,x_ix_j}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^{1}([0,T])}\|D^2\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}}(Q_T)}.$$

For the time derivative of the second term in (5.16) it suffice, by lemma 5.1.2, to estimate $\|\rho^*_{,x_ix_j}\boldsymbol{u}_{,x_m}\|_{W_2^{0,\frac{l}{2}}(Q_T)}$. Since

$$\begin{aligned}\|\rho^*_{,x_ix_j}\boldsymbol{u}_{,x_m}\|^2_{W_2^{0,\frac{l}{2}}(Q_T)} &= \int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\Delta_{-h}(\rho^*_{,x_ix_j}\boldsymbol{u}_{,x_m})\|^2_{L^2(\Omega_b)}dt \\ &\le c\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\boldsymbol{u}_{,x_m}\Delta_{-h}\rho^*_{,x_ix_j}\|^2_{L^2(\Omega_b)} + \|\rho^*_{,x_ix_j}\Delta_{-h}\boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)}dt\end{aligned}$$

we split the estimate into two parts. We have, by Proposition 2.1.3

$$\begin{aligned}
&\int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\rho^*_{,x_ix_j} \Delta_{-h} \boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)} dt \\
&\leq \int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\rho_{,x_ix_j}\|^2_{W_2^l(\mathcal{G})} \|\Delta_{-h} \boldsymbol{u}_{,x_m}\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)} dt \\
&\leq \sup_{t<T} \|\rho\|^2_{W_2^{2+l}(\mathcal{G})} \int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\Delta_{-h} \boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)} + \|\Delta_{-h} \nabla \boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)} dt \\
&\leq \sup_{t<T} \|\rho\|^2_{W_2^{2+l}(\mathcal{G})} \|\boldsymbol{u}\|^2_{W_2^{l+2,\frac{l}{2}+1}(Q_T)},
\end{aligned}$$

since $\frac{3}{2} - l < 1$, while, by Hölder inequality

$$\begin{aligned}
&\int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\boldsymbol{u}_{,x_m} \Delta_{-h} \rho^*_{,x_ix_j}\|^2_{L^2(\Omega_b)} dt \leq \\
&\qquad \leq \int_0^T \frac{dh}{h^{1+l}} \int_h^T \sup_{\Omega_b} |\nabla \boldsymbol{u}(\cdot,t)|^2 \| \int_0^h D^2 \rho_{\cdot,t}(t-\xi) d\xi \|^2_{L^2(\mathcal{G})} dt \\
&\qquad \leq c \int_0^T \frac{dh}{h^l} \int_h^T \|\boldsymbol{u}(\cdot,t)\|^2_{W_2^{l+2}(\Omega_b)} \int_0^h \|D^2 \rho_{,t}(\cdot,t-\xi)\|^2_{L^2(\mathcal{G})} d\xi dt \\
&\qquad \leq c \int_0^T \|D^2 \rho_{,t}(\cdot,s)\|^2_{L^2(\mathcal{G})} ds \int_0^T \|\boldsymbol{u}(\cdot,t)\|^2_{W^{l+2}(\Omega_b)} dt \int_0^T \frac{dh}{h^l} \\
&\qquad \leq c T^{1-l} \|\boldsymbol{u}\|^2_{W_2^{l+2,0}(Q_T)} \|\rho_{,t}\|^2_{W_2^{2,0}(G_T)}.
\end{aligned}$$

The addend $\rho^*_{,t}(\mathcal{L}^{-1} e_3 \cdot \nabla)\boldsymbol{u}$ in $\boldsymbol{l}_1$ is a linear combination of terms of the form $\rho^*_{,t} f(x,\rho^*,\nabla\rho^*) u^j_{,x_i}$, which are estimated as

$$\begin{aligned}
\|\rho^*_{,t} f(x,\rho^*,\nabla\rho^*) u^j_{,x_i}\|_{W_2^{l,0}(Q_T)} &\leq c \sup_{t<T} \|\rho_{,t}\|_{W_2^l(\mathcal{G})} \|\nabla \boldsymbol{u}\|_{W_2^{l+1,0}(Q_T)} \\
&\leq c \|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} \|\rho_{,t}\|_{W_2^{l+1,\frac{l}{2}+\frac{1}{2}}(G_T)},
\end{aligned}$$

and, by (5.4) and noting that $\frac{l}{2} + \frac{1}{4} > \frac{1}{2}$,

$$\begin{aligned}
\|\rho^*_{,t} f(x,\rho^*,\nabla\rho^*) u^j_{,x_i}\|_{W_2^{0,\frac{l}{2}}(Q_T)} &\leq c \sup_{\mathcal{G}} \|\rho_{,t}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}([0,T])} \|\nabla \boldsymbol{u}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \\
&\leq c \|\rho\|_l \|\boldsymbol{u}\|_{W_2^{l+1,\frac{l}{2}+\frac{1}{2}}(Q_T)}.
\end{aligned}$$

We now estimate the remaining terms in (3.17). Recall by (3.15), that $\mathcal{L}^{-T} - I - \delta\mathcal{L}^{-T}$ has entries of the form $\rho^*\rho^*_{,x_k} f^k(x,\rho^*)$. Looking at the explicit form (3.17) of $\widetilde{\boldsymbol{l}}_1$, we have to estimate terms of the form $f(x,\rho,\nabla\rho)$ times

$$\rho^*\rho^*_{,x_ix_j}, \qquad \rho^*_{,x_i}\rho^*_{,x_j}, \qquad \rho^*\rho^*_{,t}, \qquad \rho^*\rho^*_{,x_i}.$$

We estimate all these terms using (5.2) and (5.4): for the first type of terms

$$\|\rho^* D^2\rho^*\|_{W_2^{l,0}(Q_T)} \le c\sup_{t<T}\|\rho\|_{W_2^{l+1}(\mathcal{G})}\|D^2\rho\|_{W_2^l(\mathcal{G})} \le c\|\rho\|_l\|\rho\|_{W_2^{l+2,0}(G_T)},$$

$$\|\rho^* D^2\rho^*\|_{W_2^{0,\frac{l}{2}}(Q_T)} \le c\sup_{\mathcal{G}}\|\rho\|_{W_2^1(\mathcal{G})}\|D^2\rho\|_{W_2^{0,\frac{l}{2}}(G_T)} \le c\|\rho\|_l\|\rho\|_{W_2^{l+2,\frac{l}{2}+1}(G_T)},$$

and both are bounded by $c\|\rho\|_l^2$. Similarly for the second type of terms

$$\|\rho^*_{,x_i}\rho^*_{,x_j}\|_{W_2^{l,0}(Q_T)} \le c\sup_{t<T}\|\rho\|_{W_2^{l+2}(\mathcal{G})}\|\nabla\rho\|_{W_2^l(G_T)} \le c\|\rho\|_l^2,$$

$$\|\rho^*_{,x_i}\rho^*_{,x_j}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^1(0,T)}\|\rho\|_{W_2^{l+1,\frac{l}{2}+\frac{1}{2}}(G_T)} \le c\|\rho\|_l^2,$$

while for the third type it holds

$$\|\rho^*\rho^*_{,t}\|_{W_2^{l,0}(Q_T)} \le c\sup_{t<T}\|\rho\|_{W_2^{l+1}(\mathcal{G})}\|\rho_{,t}\|_{W_2^l(\mathcal{G})} \le c\|\rho\|_l^2,$$

$$\|\rho^*\rho^*_{,t}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \le \sup_{\mathcal{G}}\|\rho\|_{W_2^1(\mathcal{G})}\|\rho_{,t}\|_{W_2^{0,\frac{l}{2}}(G_T)} \le c\|\rho\|_l^2.$$

The fourth one is lower order and is estimated as before. Collecting these estimates we get

$$\|\widetilde{\boldsymbol{l}}_1(\boldsymbol{u},q,\rho)\|_{W_2^{l,\frac{l}{2}}(Q_T)} \le c\|\rho\|_l(\|\nabla q\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)} + \|\rho\|_l).$$

Estimates of $\|\widetilde{l}_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)}$ *and* $\|\boldsymbol{G}(\boldsymbol{u},\rho)\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}$.

We have that $\widetilde{l}_2$, as given in (3.22), is a linear combination of terms of the form $\rho^*_{,x_i}u^k_{,x_3}$ and $\rho^* u^k_{,x_i}$ and thus its $W_2^{l+1,0}(Q_T)$ norm is estimated as in (5.13). For the time derivative of $\boldsymbol{G}$, also given in (3.22), notice that its $W_2^{0,\frac{l}{2}}(Q_T)$ has been already estimated in (5.14). Therefore it suffice to estimate the $W_2^{0,\frac{l}{2}}(Q_T)$ of its time derivative, i.e.

$$(\nabla\rho^*_{,t}\cdot\boldsymbol{u})\boldsymbol{e}_3 + (\nabla\rho\cdot\boldsymbol{u}_{,t})\boldsymbol{e}_3 - \theta'(\rho_{,t}\boldsymbol{u} + \rho\boldsymbol{u}_{,t}).$$

To this end notice that, applying (5.5), one gets

$$\begin{aligned}\|\nabla\rho^*_{,t}\cdot\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}}(Q_T)} &\le c\sup_{\Omega_b}\|\boldsymbol{u}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}([0,T])}\|\nabla\rho_{,t}\|_{W_2^{0,\frac{l}{2}}(G_T)}\\ &\le c\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}\|\rho_{,t}\|_{W_2^{l+1,\frac{l}{2}+\frac{1}{2}}(G_T)}\\ &\le c\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}\|\rho\|_l,\end{aligned}$$

since $\frac{l}{2}+\frac{1}{4}>\frac{1}{2}$. Furthermore, by (5.4),

$$\begin{aligned}\|\nabla\rho^*\boldsymbol{u}_{,t}\|_{W_2^{0,\frac{l}{2}}(Q_T)} &\le c\sup_{\mathcal{G}}\big(\|\rho\|_{W_2^1([0,T])}+\|\nabla\rho\|_{W_2^1([0,T])}\big)\|\boldsymbol{u}_{,t}\|_{W_2^{0,\frac{l}{2}}(Q_T)}\\ &\le c\|\rho\|_l\|\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}.\end{aligned}$$

The same estimates holds for the terms in $\rho_{,t}\boldsymbol{u}$ and $\rho\boldsymbol{u}_{,t}$, and thus we have obtained

$$\|\widetilde{l}_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)}+\|\boldsymbol{G}(\boldsymbol{u},\rho)\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}\le c\|\rho\|_l\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}.$$

Estimate of $\|\widetilde{\boldsymbol{l}}_3\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)}$.

We look at $\|\Pi_b\boldsymbol{b}\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)}$ for $\boldsymbol{b}$ given in (3.26). First notice that

$$\|\Pi_b\boldsymbol{b}\|_{W_2^{\eta,\frac{\eta}{2}}(G_T)}\le c\|\boldsymbol{b}\|_{W_2^{\eta,\frac{\eta}{2}}(G_T)};$$

indeed, $\Pi_b\boldsymbol{b}=\boldsymbol{b}-\boldsymbol{NbNN}$ and since $\boldsymbol{N}$ is smooth and independent of t, the claim follows from

$$\|\boldsymbol{NbNN}\|_{W_2^{\eta,0}(G_T)}\le\|\boldsymbol{N}\|_{W_2^{\max\{1+\varepsilon,\eta\}}(\mathcal{G})}\|\boldsymbol{d}\|_{W_2^{\eta,0}(G_T)}.$$

The first addend in (3.26) is $\Pi_b\mathbb{D}(\boldsymbol{u})\boldsymbol{N}-\Pi\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n}$ which amounts to

$$\begin{aligned}&\mathbb{D}(\boldsymbol{u})\boldsymbol{N}-\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n}-\boldsymbol{N}\mathbb{D}(\boldsymbol{u})\boldsymbol{NN}+\boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{nn}=\\ &=\big(\mathbb{D}(\boldsymbol{u})-\widetilde{\mathbb{D}}(\boldsymbol{u})\big)\boldsymbol{N}-\widetilde{\mathbb{D}}(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N})+\boldsymbol{N}\big(\mathbb{D}(\boldsymbol{u})-\widetilde{\mathbb{D}}(\boldsymbol{u})\big)\boldsymbol{NN}+\\ &\quad+(\boldsymbol{N}-\boldsymbol{n})\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{NN}+\boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{u})(\boldsymbol{N}-\boldsymbol{n})\boldsymbol{N}+\boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n}(\boldsymbol{N}-\boldsymbol{n}).\end{aligned}$$

Each addend is a linear combination of terms of the form $\rho_{,x_i}f(x,\rho,\nabla\rho)u^k_{,x_j}$, which are estimated as

$$\begin{aligned}\|\rho_{,x_i}f(x,\rho,\nabla\rho)u^k_{,x_j}\|_{W_2^{l+\frac{1}{2},0}(G_T)} &\le c\sup_{t<T}\|\nabla\rho\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\|\nabla\boldsymbol{u}\|_{W_2^{l+\frac{1}{2},0}(G_T)}\\ &\le c\sup_{t<T}\|\rho\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)},\end{aligned}$$

since $l+\frac{1}{2}>1$, and, using now $1>\frac{l}{2}+\frac{1}{4}>\frac{1}{2}$, we have by (5.4)

$$\begin{aligned}\|\rho_{,x_i}f(x,\rho,\nabla\rho)u^k_{,x_j}\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(G_T)} &\le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^1(0,T)}\|\nabla\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(G_T)}\\ &\le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^1(0,T)}\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)},\end{aligned}$$

Regarding the second addend, it can be decomposed as a sum of several second order terms: calling $\mathbb{D}(\boldsymbol{v}_b) = bDb$ and $\widetilde{\mathbb{D}}(\boldsymbol{v}_b) = \widetilde{\mathbb{D}}_b$, it is

$$\big(\widetilde{\mathbb{D}}_b - \mathbb{D}_b\big)(\boldsymbol{n}-\boldsymbol{N}) + \big(\widetilde{\mathbb{D}}_b - \mathbb{D}_b - \delta\mathbb{D}_b\big)\boldsymbol{N} + \mathbb{D}_b(\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N}) +$$
$$\boldsymbol{n}\big(\widetilde{\mathbb{D}}_b - \mathbb{D}_b - \delta\mathbb{D}_b\big)\boldsymbol{n}\boldsymbol{n} + (\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N})\mathbb{D}_b\boldsymbol{n}\boldsymbol{n} + \boldsymbol{N}\delta\mathbb{D}_b(\boldsymbol{n}-\boldsymbol{N})\boldsymbol{n} +$$
$$\boldsymbol{N}\mathbb{D}_b(\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N})\boldsymbol{n} + \boldsymbol{N}\mathbb{D}_b\boldsymbol{N}(\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N}) + \boldsymbol{N}\mathbb{D}_b\delta\boldsymbol{N}(\boldsymbol{n}-\boldsymbol{N}) +$$
$$\delta\boldsymbol{N}\mathbb{D}_b(\boldsymbol{n}-\boldsymbol{N})\boldsymbol{n} + (\boldsymbol{n}-\boldsymbol{N})\delta\mathbb{D}_b\boldsymbol{n}\boldsymbol{n} + \delta\boldsymbol{N}\mathbb{D}_b\boldsymbol{N}(\boldsymbol{n}-\boldsymbol{N}) + \boldsymbol{N}\delta\mathbb{D}_b\boldsymbol{N}(\boldsymbol{n}-\boldsymbol{N}).$$

Each factor of the type $(\boldsymbol{n}-\boldsymbol{N})$, $\delta\boldsymbol{N}$ or $\delta\mathbb{D}_b$ is a linear combination of $\rho_{,x_i} g(x,\rho,\nabla\rho)$, as can be checked in (3.27), and these terms are always paired up in the decomposition above. For these addends, the $W_2^{l+\frac12,\frac{l}{2}+\frac14}(G_T)$-norm can be estimated as

$$\|\rho_{,x_i}\rho_{,x_j} f(\rho,\nabla\rho)\|_{W_2^{l+\frac12,0}(G_T)} \le c\sup_{t<T}\|\nabla\rho\|_{W_2^{l+\frac12}(\mathcal{G})}\|\rho\|_{W_2^{l+\frac32,0}(G_T)},$$
$$\begin{aligned}\|\rho_{,x_i}\rho_{,x_j} f(\rho,\nabla\rho)\|_{W_2^{0,\frac{l}{2}+\frac14}(G_T)} &\le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^1([0,T])}\|\nabla\rho\|_{W_2^{0,\frac{l}{2}+\frac14}(G_T)}\\ &\le c\|\rho\|_l\|\rho\|_{W_2^{l+\frac32,\frac{l}{2}+\frac34}(G_T)}.\end{aligned}$$

Regarding the terms involving $\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N}$, we see that

$$\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N} = \int_0^1 (1-s)\frac{d^2}{ds^2}\frac{\big(-\nabla(\phi_b+s\rho),1\big)}{\sqrt{1+|\nabla(\phi_b+s\rho)|^2}}ds = \rho_{,x_i}\rho_{,x_j}\int_0^1 \boldsymbol{A}^{ij}(s,\nabla\rho)ds,$$

for some smooth functions $\boldsymbol{A}^{ij}$, and thus one can proceed in the same way. The terms involving $\widetilde{S}_b - S_b - \delta S_b$ are of the form $\rho_{,x_i}\rho f(\rho)$, which are of lower order, and thus can again be bounded with $c\|\rho\|_l^2$. All in all we get

$$\|\widetilde{\boldsymbol{l}}_3\|_{W_2^{l+\frac12,\frac{l}{2}+\frac14}(G_T)} \le c\|\rho\|_l(\|\rho\| + \|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}).$$

Estimate of $\|\widetilde{l}_4\|_{W_2^{l+\frac12,\frac{l}{2}+\frac14}(G_T)}$.

We observe that, using $\boldsymbol{N}\mathbb{D}_b\delta\boldsymbol{N} = \delta\boldsymbol{N}\mathbb{D}_b\boldsymbol{N} = 0$, $\widetilde{l}_4$ can be written as

$$\begin{aligned}&(\boldsymbol{n}-\boldsymbol{N})\widetilde{\mathbb{D}}(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{N}\widetilde{\mathbb{D}}(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N}) + (\boldsymbol{n}-\boldsymbol{N})(\widetilde{\mathbb{D}}_b - \mathbb{D}_b)\boldsymbol{n}\\ &+ \boldsymbol{N}(\widetilde{\mathbb{D}}_b - \mathbb{D}_b)(\boldsymbol{n}-\boldsymbol{N}) + (\boldsymbol{n}-\boldsymbol{N})\mathbb{D}_b(\boldsymbol{n}-\boldsymbol{N}) + \boldsymbol{N}\mathbb{D}_b(\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N})\\ &+ (\boldsymbol{n}-\boldsymbol{N}-\delta\boldsymbol{N})\mathbb{D}_b\boldsymbol{N} + \boldsymbol{N}(\widetilde{\mathbb{D}}_b - \mathbb{D}_b - \delta\mathbb{D}_b)\boldsymbol{N} - \sigma\int_0^1 (1-s)\frac{d^2}{ds^2}H_s ds.\end{aligned}$$

Each term except the last one is of a form treated above, either

$$\rho_{,x_i}\rho_{,x_j} f(x,\rho,\nabla\rho), \qquad \rho_{,x_i} f(x,\rho,\nabla\rho)u^k_{,x_j} \quad \text{or} \qquad \rho\rho_{,x_i} f(x,\rho,\nabla\rho).$$

For the last term we have, recalling (3.6),

$$\int_0^1 (1-s)\frac{d^2}{ds^2}H_s ds = \rho_{x_i}\rho_{,x_j}\int_0^1 f^{ij}(s,x,\nabla\rho)ds + \rho_{,x_k}\rho_{,x_i x_j}\int_0^1 g^{ijk}(s,x,\nabla\rho)ds.$$

The first type of addend is treated as above, while for the second one we proceed as follows: for the spatial derivative

$$\|\rho_{,x_k}\rho_{,x_i x_j}h(x,\nabla\rho)\|_{W_2^{l+\frac{1}{2},0}(G_T)} \le c\sup_{t<T}\|\nabla\rho\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\|\rho\|_{W_2^{l+\frac{5}{2},0}(G_T)},$$

since $l+\frac{1}{2}>1$, and thus a bound of the form $c\|\rho\|_l^2$ for this term is obtained via (5.2). For the time derivative, using (5.4) and $\frac{l}{2}+\frac{1}{4}<1<\frac{l}{2}+\frac{3}{4}$

$$\begin{aligned}\|\rho_{,x_k}\rho_{,x_i x_j}h(x,\nabla\rho)\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(G_T)} &\le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^1(0,T)}\|\rho_{,x_i x_j}\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(G_T)}\\ &\le c\|\rho\|_l\|\rho\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)}.\end{aligned}$$

Estimate of $\|\widetilde{l}_5\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)}$.

The explicit formula for $\widetilde{l}_5$ is given in (3.13). To estimate the spatial derivative we use (2.8):

$$\|\nabla\rho\cdot\boldsymbol{u}\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} \le c\big(\|\nabla\rho\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + \|\nabla\rho\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\big),$$

since $l+\frac{1}{2}>1$. Now (5.2) and the standard restriction estimates for anisotropic Sobolev–Slobodetskii spaces give

$$\begin{aligned}\|\nabla\rho\cdot\boldsymbol{u}\|_{W_2^{l+\frac{3}{2},0}(G_T)} &\le c\big(\sup_{t<T}\|\rho\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+\frac{3}{2},0}(G_T)}+\\ &\quad+\sup_{t<T}\|\boldsymbol{u}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\|\rho\|_{W_2^{l+\frac{5}{2},0}(\mathcal{G})}\big)\\ &\le c\|\rho\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)}\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}.\end{aligned}\tag{5.17}$$

For the time derivative we use (5.4), obtaining

$$\begin{aligned}\|\nabla\rho\cdot\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+\frac{3}{4}}(G_T)} &\le c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)}\|\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+\frac{3}{4}}(G_T)}\\ &\le c\|\rho\|_l\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)},\end{aligned}$$

which is the last estimate needed for the proof of 5.1.1.

Remark 5.1.5 *As noted in the introduction, if a solution of the nonlinear problem* (1.2) *has free boundary function* $\rho \in W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)$ *and* $\boldsymbol{u} \in W_2^{l+2,\frac{l}{2}+1}(Q_T)$, *then* $\sup_{t<T}\|\rho\|_{W_2^{l+2}(\mathcal{G})}$ *is bounded. Indeed formula* (3.12) *shows that* $\rho_{,t} \in W_2^{l+\frac{3}{2},0}(G_T)$ *since the nonlinear term is estimated as in* (5.17), *and the lower order terms are in* $W_2^{l+\frac{3}{2},0}(G_T)$ *for sufficiently smooth* $\boldsymbol{v}_b$ *and* ϕ_b ($\boldsymbol{v}_b \in W_2^{l+2}(\Omega_b)$ *and* $\phi_b \in W_2^{l+\frac{5}{2}}(\mathcal{G})$ *suffice). Now* (5.2) *proves the claim.*

Finally we prove a continuity estimate for the nonlinear terms.

Theorem 5.1.6 *Let* $l \in (\frac{1}{2}, 1)$, *and* $\|\rho\|_l, \|\rho'\|_l \leq \mu$ *such that* (5.1) *holds. There exists* $c(\mu)$, *bounded for bounded* μ, *such that*

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{l}}_0(\boldsymbol{u},\rho) - \widetilde{\boldsymbol{l}}_0(\boldsymbol{u}',\rho')\|_{W_2^{l,\frac{l}{2}}(Q_T)} + \|\widetilde{\boldsymbol{l}}_1(\boldsymbol{u},p,\rho) - \widetilde{\boldsymbol{l}}_1(\boldsymbol{u}',p',\rho')\|_{W_2^{l,\frac{l}{2}}(Q_T)} \\
&+ \|\widetilde{l}_2(\boldsymbol{u},\rho) - \widetilde{l}_2(\boldsymbol{u}',\rho')\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{G}(\boldsymbol{u},\rho) - \boldsymbol{G}(\boldsymbol{u}',\rho')\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \\
&+ \|\widetilde{\boldsymbol{l}}_3(\boldsymbol{u},\rho) - \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}',\rho')\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|\widetilde{l}_4(\boldsymbol{u},\rho) - \widetilde{l}_4(\boldsymbol{u}',\rho')\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} \\
&+ \|\widetilde{l}_5(\boldsymbol{u},\rho) - \widetilde{l}_5(\boldsymbol{u}',\rho')\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \leq \\
&c(T,\mu)(\|(\boldsymbol{u},p,\rho)\|_{W,l,T} + \|(\boldsymbol{u}',p',\rho')\|_{W,l,T})\|(\boldsymbol{u}-\boldsymbol{u}',p-p',\rho-\rho')\|_{W,l,T}.
\end{aligned}
$$

Proof. This is a consequence of the structure of the nonlinear terms: as noted in the previous estimates, each nonlinear term is a linear combinations of products of the form

$$f(x,\rho,\nabla\rho)\pi(\boldsymbol{u},\nabla\boldsymbol{u},\rho,\rho_{,t},\nabla\rho,\nabla^2\rho,p,\nabla p)$$

where π stands for a monomial of total degree at least 1 in a certain subset of the arguments. Therefore, except for the term $f(x,\rho,\nabla\rho)$, each of these terms is separately linear in its arguments, and can be estimated as above, provided one can prove an estimate of the form

$$
\begin{aligned}
\|(f(x,\rho,\nabla\rho) - f(x,\rho',\nabla\rho'))g\|_{W_2^{\eta,0}(G_T)} &\leq c_f(\mu)\|\rho-\rho'\|_l\|g\|_{W_2^{\eta,0}(G_T)}, \\
\|(f(x,\rho,\nabla\rho) - f(x,\rho',\nabla\rho'))g\|_{W_2^{0,\eta'}(G_T)} &\leq c_f(\mu)\|\rho-\rho'\|_l\|g\|_{W_2^{0,\eta'}(G_T)},
\end{aligned}
\tag{5.18}
$$

(and analogous ones for ρ^* on Q_T), with $\eta \leq 1+l$ and $\eta' \leq 1$ (see remark 5.1.4 with regard to η'). Indeed it suffice to split the difference of the products following the algebraic formula

$$\Pi_{i=1}^N x_i - \Pi_{i=1}^N y_i = \sum_{i=1}^N y_1 \ldots \cdot y_{i-1}(x_i - y_i)x_{i+1}\ldots x_N, \tag{5.19}$$

and use estimates (5.18) for the terms containing the differences of the nonlinear terms f's. The particular structure of the various products π (for example being of degree at most one in $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$) ensures that the estimates of the previous proof carry over in this case. To prove (5.18) we notice that, as in the proof of lemma 5.1.2, it suffice to show that

$$\sup_{t<T}\|f(x,\rho,\nabla\rho)-f(x,\rho',\nabla\rho')\|_{W_2^{1+l}(\mathcal{G})}\le c_f(\mu)\|\rho-\rho'\|_l, \tag{5.20}$$

$$\sup_{\mathcal{G}}\|f(x,\rho,\nabla\rho)-f(x,\rho',\nabla\rho')\|_{W_2^{1}(0,T)}\le c_f(\mu)\|\rho-\rho'\|_l. \tag{5.21}$$

(this also implies estimates of the form (5.18) involving ρ^* on Q_T, see remark 5.1.3 in this regard). We sketch the proof of these two estimate, supposing for simplicity that $f=f(\nabla\rho)$, which is the higher order term. The smoothness of f together with (5.3) gives the Lipschitzianity w.r.t. the norm $\|\rho\|_l$ of $f(\nabla\rho)$ in C^0 and thus in L^2, i.e.

$$\sup_{G_T}|f(\nabla\rho)-f(\nabla\rho')|\le c_f(\mu)\|\rho-\rho'\|_l. \tag{5.22}$$

Thus it remains to estimate

$$\begin{aligned}\|f_p(\nabla\rho)D^2\rho-&f_p(\nabla\rho')D^2\rho'\|_{W_2^l(\mathcal{G})}\\&\le\|(f_p(\nabla\rho)-f_p(\nabla\rho'))D^2\rho\|_{W_2^l(\mathcal{G})}+\|f_p(\nabla\rho')D^2(\rho'-\rho)\|_{W_2^l(\mathcal{G})}.\end{aligned}$$

The second addend is treated through lemma 5.1.2 and (5.2), while for the first one, applying proposition 2.1.3, point 2, we get

$$\begin{aligned}\|(f_p(\nabla\rho)-&f_p(\nabla\rho'))D^2\rho\|_{W_2^l(\mathcal{G})}\\&\le c\mu(\|f_p(\nabla\rho)-f_p(\nabla\rho')\|_{W_2^1(\mathcal{G})}+\sup_{\mathcal{G}}|f_p(\nabla\rho)-f_p(\nabla\rho')|).\end{aligned}$$

Property (5.22) takes care of the second addend, and the L^2 part of the first one's norm. Therefore it remains to estimate $f_p(\nabla\rho)D^2\rho-f_p(\nabla\rho')D^2\rho$ in L^2, which can be splitted as before, and thus

$$\begin{aligned}\|f_p(\nabla\rho)D^2\rho-&f_p(\nabla\rho')D^2\rho\|_{L^2(\mathcal{G})}\\&\le c_f(\mu)\|\rho-\rho'\|_{W_2^2(\mathcal{G})}+\sup_{\mathcal{G}}|f_p(\nabla\rho)-f_p(\nabla\rho')|\|\rho\|_{W_2^2(\mathcal{G})}.\end{aligned}$$

Applying once again (5.22) proves (5.20). To prove of (5.21) it suffice again to estimate $f_p(\nabla\rho)\nabla\rho_{,t}-f_p(\nabla\rho')\nabla\rho'_{,t}$ in L^2, which can be done as before:

$$\begin{aligned}\|f_p(\nabla\rho)&\nabla\rho_{,t}-f_p(\nabla\rho')\nabla\rho'_{,t}\|_{L^2(0,T)}\\&\le\|(f_p(\nabla\rho)-f_p(\nabla\rho'))\nabla\rho_{,t}\|_{L^2(0,T)}+\|f_p(\nabla\rho')\nabla(\rho_{,t}-\rho'_{,t})\|_{L^2(0,T)}\\&\le c_f(\mu)(\|\rho-\rho'\|_l\|\nabla\rho\|_{W_2^1(0,T)}+\|\nabla(\rho-\rho')\|_{W_2^1(0,T)})\le c_f(\mu)\|\rho-\rho'\|_l.\end{aligned}$$

□

5.2 The abstract linearization principle

In this section we will prove a conditional stability result for some smooth, stationary solution $(\boldsymbol{v}_b, p_b, \phi_b)$ of problem (1.2). Our main hypothesis is that the solutions of the homogeneous linearized system

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q - \boldsymbol{\Phi}_1(\boldsymbol{u},\rho) = 0 & \text{in } \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = 0 & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \widetilde{\boldsymbol{\Phi}}(\rho) = 0 & \text{on } \mathcal{G}, \\ \rho_t + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = 0 & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma, \text{ for all } t\geq 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \rho(x',0) = \rho_0(x'), & \text{for } x\in\Omega_b,\ x'\in\Sigma, \end{cases} \tag{5.23}$$

subjected to the compatibility conditions

$$\begin{cases} \nabla\cdot\boldsymbol{u}_0 - \Phi_2(\rho_0) = 0, \\ \nu\Pi_b\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_0) = 0, \end{cases} \tag{5.24}$$

decay exponentially in time, i.e. there exists $\gamma < 0$ such that

$$\|e^{-\gamma t}(\boldsymbol{u},p,\rho)\|_{W,l,\infty} \leq c(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W^{l+2}(\mathcal{G})}), \tag{5.25}$$

with c independent of T. Under this condition we will prove the existence, uniqueness and exponential decay of a global in time solution of (3.10) if the initial data are sufficiently small, and thus of (1.2) if the initial data differs little from the stable stationary solution.

Theorem 5.2.1 *Let $l \in (\frac{1}{2}, 1)$, and suppose that for the periodic stationary solution $(\boldsymbol{v}_b, p_b, \rho_b)$ of* (1.2)*, the corresponding linearized system is exponentially stable, i.e. there exists $\gamma < 0$ such that* (5.25) *holds for any solution of* (5.23) *with periodic initial data satisfying* (5.24)*. Then, for any sufficiently small periodic initial data $\boldsymbol{u}_0$, ρ_0 (in the $W_2^{l+1}(\Omega_b)$ and $W_2^{l+2}(\mathcal{G})$ norms respectively) satisfying the compatibility conditions* (3.11)*, there exists a unique periodic solution $(\boldsymbol{u},p,\rho)$ of* (3.10)*, defined for all $t \geq 0$ and such that* (5.25) *holds for some $\gamma' < 0$.*

Proof. We will construct the solution as a sum $(\boldsymbol{u},p,\rho) = (\boldsymbol{u}_1 + \boldsymbol{u}_2, p_1 + p_2, \rho_1 + \rho_2)$, where $(\boldsymbol{u}_1, p_1, \rho_1)$ is a solution of (5.23) for some initial data $\boldsymbol{u}_1\lfloor_{t=0} := \boldsymbol{u}_1^0$, $\rho_1\lfloor_{t=0} = \rho_0$, and $(\boldsymbol{u}_2, p_2, \rho_2)$ solves a nonlinear problem with initial data $\boldsymbol{u}_2\lfloor_{t=0} = \boldsymbol{u}_0 - \boldsymbol{u}_1^0$, $\rho_2\lfloor_{t=0} = 0$. We split the proof into three steps.

Step 1: construction of $(\boldsymbol{u}_1, p_1, \rho_1)$.
We start by constructing $\boldsymbol{w}_0$ solving

$$\begin{cases} \nabla \cdot \boldsymbol{w}_0 = \widetilde{l}_2(\boldsymbol{u}_0, \rho_0) & \text{in } \Omega_b, \\ \nu \Pi_b \mathbb{D}(\boldsymbol{w}_0)\boldsymbol{N} = \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_0, \rho_0) & \text{on } \mathcal{G}, \\ \boldsymbol{v}_0 = 0 & \text{on } \Sigma, \end{cases} \tag{5.26}$$

with the estimate

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \le c(\|\widetilde{l}_2(\boldsymbol{u}_0, \rho_0)\|_{W_2^l(\Omega_b)} + \|\widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_0, \rho_0)\|_{W_2^{l-\frac{1}{2}}(\mathcal{G})}).$$

This can be done setting $\boldsymbol{w}_0 = \boldsymbol{w} + \nabla \times \boldsymbol{V}$, for a periodic $\boldsymbol{w}$ such that

$$\begin{cases} \nabla \cdot \boldsymbol{w} = \widetilde{l}_2(\boldsymbol{u}_0, \rho_0) & \text{in } \Omega_b, \\ \boldsymbol{w} = 0 & \text{on } \Sigma, \end{cases}$$

and a vector $\boldsymbol{V}$ such that

$$\begin{cases} \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{N}} = 0 & \text{on } \mathcal{G}, \\ \frac{\partial^2 \boldsymbol{V}}{\partial \boldsymbol{N}^2} = (\widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_0, \rho_0) - \nu \Pi_b \mathbb{D}(\boldsymbol{w})\boldsymbol{N}) \times \boldsymbol{N} & \text{on } \mathcal{G}, \\ \nabla \times \boldsymbol{V} = 0 & \text{on } \Sigma, \end{cases}$$

with $\boldsymbol{w}$ and $\boldsymbol{V}$ satisfying optimal regularity estimates. The vector $\boldsymbol{w}$ can be constructed, for example, through theorem 4.1.3, and the vector $\boldsymbol{V}$ applying theorem 2.1.6 on $\mathcal{G}$ and cutting off the resulting vector near Σ. To obtain an estimate of $\boldsymbol{w}_0$ in terms of $\boldsymbol{u}_0$ and ρ_0, we recall that $\widetilde{l}_2(\boldsymbol{u}, \rho)$ is a linear combination of terms of the form $\rho^*_{x_i} u^k_{x_3}$ and $\rho^* u^k_{x_i}$, and thus proposition 2.1.3, point 2 gives

$$\|\widetilde{l}_2(\boldsymbol{u}_0, \rho_0)\|_{W_2^l(\Omega_b)} \le c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}.$$

For the term involving $\widetilde{\boldsymbol{l}}_3$, as found in section 4, one has to consider addends of the following kind:

$$\rho_{,x_i} f(x, \rho, \nabla \rho) u^k_{,x_j}, \qquad \rho_{x_i} \rho_{,x_j} f(\rho, \nabla \rho), \qquad \text{or} \qquad \rho \rho_{,x_j} f(\rho, \nabla \rho).$$

As in the proof of lemma 5.1.2 for $\rho \equiv \rho_0$, one gets

$$\|\widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_0, \rho_0)\|_{W_2^{l-\frac{1}{2}}(\mathcal{G})} \le c(\|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})}).$$

All in all we get

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \le c(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)})^2, \tag{5.27}$$

and define

$$U_0 := \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)},$$

supposing it is sufficiently small, in a sense to be specified later. Now, since $(\boldsymbol{u}_0, \rho_0)$ satisfies (3.11), it is clear that defining $\boldsymbol{u}_1^0 := \boldsymbol{u}_0 - \boldsymbol{w}_0$, the couple $(\boldsymbol{u}_1^0, \rho_0)$ satisfies the compatibility conditions (5.24) for the homogeneous linear problem (5.23). We will then let $(\boldsymbol{u}_1, p_1, \rho_1)$ be the solution of (5.23) with such initial data. Recalling the notation (4.79), the stability hypothesis gives, for $\gamma < 0$ and $T \geq 1$

$$\begin{aligned} \|(\boldsymbol{u}_1, p_1, \rho_1)\|_{W,l,\infty} &\leq \|e^{-\gamma t}(\boldsymbol{u}_1, p_1, \rho_1)\|_{W,l,\infty} \\ &\leq c(\|\boldsymbol{u}_1^0\|_{W_2^{l+1}} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}) \leq c_1 U_0, \end{aligned} \tag{5.28}$$

and by standard restriction estimates for unbounded intervals

$$\begin{aligned} e^{-\gamma T}\big(\|\boldsymbol{u}_1(\cdot,T)\|_{W_2^{l+1}(\Omega_b)} &+ \|\rho_1(\cdot,T)\|_{W_2^{l+2}(\mathcal{G})}\big) \\ &\leq \|e^{-\gamma t}(\boldsymbol{u}_1, p_1, \rho_1)\|_{W,l,\infty} \leq c_1 U_0, \end{aligned} \tag{5.29}$$

with a constant $c_1 \geq 1$ independent of $T \geq 1$.

Step 2: construction of $(\boldsymbol{u}_2, p_2, \rho_2)$.

We seek for a solution of the nonlinear problem

$$\begin{cases} \boldsymbol{u}_{2,t} - \nu\Delta \boldsymbol{u}_2 + \nabla q_2 - \boldsymbol{\Phi}_1(\boldsymbol{u}_2, \rho_2) = (\widetilde{\boldsymbol{l}}_0 + \widetilde{\boldsymbol{l}}_1)(\boldsymbol{u}_1 + \boldsymbol{u}_2, q_1 + q_2, \rho_1 + \rho_2) & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{u}_2 - \Phi_2(\rho_2) = \widetilde{l}_2(\boldsymbol{u}_1 + \boldsymbol{u}_2, \rho_1 + \rho_2) = \nabla \cdot \boldsymbol{G}(\boldsymbol{u}_1 + \boldsymbol{u}_2, \rho_1 + \rho_2) & \text{in } \Omega_b, \\ \nu \Pi_b \mathbb{D}(\boldsymbol{u}_2)\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_2) = \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_1 + \boldsymbol{u}_2, \rho_1 + \rho_2) & \text{on } \mathcal{G}, \\ -q + \nu \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u}_2)\boldsymbol{N} + \sigma L \rho_2 - \Phi_4(\rho_2) = \widetilde{l}_4(\boldsymbol{u}_1 + \boldsymbol{u}_2, \rho_1 + \rho_2) & \text{on } \mathcal{G}, \\ \rho_{2,t} + \nabla' \phi_b \cdot \boldsymbol{u}_2 - u_2^3 + \nabla' \rho_2 \cdot \boldsymbol{v}_b = \widetilde{l}_5(\boldsymbol{u}_1 + \boldsymbol{u}_2, \rho_1 + \rho_2) & \text{on } \mathcal{G}, \\ \boldsymbol{u}_2 = 0 \quad \text{on } \Sigma, \text{ for all } t \geq 0, \\ \boldsymbol{u}_2(x,0) = \boldsymbol{w}_0(x), \quad \text{for } x \in \Omega_b, \qquad \rho_2(x',0) = 0 \quad \text{for } x' \in \Sigma. \end{cases} \tag{5.30}$$

To find the solution we apply the standard iteration scheme, defining a sequence of solutions of linear problems. We consider an extension $\boldsymbol{v}_0$ for $t \geq 0$ of $\boldsymbol{w}_0$ such that

$$\|\boldsymbol{v}_0\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} \leq c\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \leq c_3 U_0^2.$$

and start with the triple $(\boldsymbol{v}_0, 0, 0)$, supposing $c_3 U_0^2 \leq 1$. Then we iteratively define $(\boldsymbol{v}_{n+1}, p_{n+1}, \rho_{n+1})$ as the solution to problem (4.1) with right hand side, respectively

$$\boldsymbol{f}_n := (\widetilde{\boldsymbol{l}}_0 + \widetilde{\boldsymbol{l}}_1)(\boldsymbol{u}_1 + \boldsymbol{v}_n, q_1 + q_n, \rho_1 + \rho_n), \qquad h_n := \widetilde{l}_2(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n),$$

$$\boldsymbol{d}_n := \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_1+\boldsymbol{u}_n, \rho_1+\rho_n)+\boldsymbol{N}\widetilde{l}_4(\boldsymbol{u}_1+\boldsymbol{u}_n, \rho_1+\rho_n), \qquad g_n := l_5(\boldsymbol{u}_1+\boldsymbol{u}_n, \rho_1+\rho_n).$$

and initial data $\boldsymbol{v}_{n+1}(0) = \boldsymbol{w}_0$ and $\rho_{n+1}(0) = 0$. The compatibility conditions (4.81) for this problem are satisfied at each stage by (5.26). The coercive estimate (4.82), together with theorem 5.1.1, (5.27) and (5.28), gives

$$\begin{aligned}\|(\boldsymbol{v}_{n+1}, q_{n+1},\rho_{n+1})\|_{W,l,T}& \\ &\le c(\mu,T)(\|(\boldsymbol{v}_n, q_n, \rho_n)\|^2_{W,l,T} + \|(\boldsymbol{u}_1, p_1, \rho_1)\|^2_{W,l,T} + cU_0^2) \\ &\le c_2(\mu,T)(\|(\boldsymbol{v}_n, q_n, \rho_n)\|^2_{W,l,T} + U_0^2),\end{aligned}$$

if

$$\max\{\|(\boldsymbol{v}_n, q_n, \rho_n)\|_{W,l,T}, \|(\boldsymbol{u}_1, p_1, \rho_1)\|_{W,l,T}\} \le \mu, \tag{5.31}$$

for μ satisfying (5.1). We can thus fix $\mu = \mu(\theta) < 1$ so that (5.1) holds, set $c_2(\mu, T) = c_2(T)$ and we can suppose $c_2(T) \ge \max\{\mu^{-1}, c_1, c_3\}$. Then we choose U_0 so small that

$$U_0 \le \frac{1}{8c_1c_2(T)} =: \varepsilon(T) \le \frac{1}{8c_2(T)}. \tag{5.32}$$

With this choice of U_0 one can prove by induction that (5.31) holds (it holds for $(\boldsymbol{v}_0, 0, 0)$ by construction). More precisely,

$$\|(\boldsymbol{v}_n, q_n, \rho_n)\|_{W,l,T} \le \frac{2c_2(T)U_0^2}{1+\sqrt{1-4c_2(T)^2U_0^2}} \le 2c_2(T)U_0^2 \le \frac{U_0}{4} \le \mu. \tag{5.33}$$

It remains to prove that $(\boldsymbol{v}_n, q_n, \rho_n)$ strongly converges to a solution of (5.30). To this end, consider $(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n) := (\boldsymbol{v}_{n+1} - \boldsymbol{v}_n, p_{n+1} - p_n, \rho_{n+1} - \rho_n)$. They satisfy a linear system of the type (4.1) with right hand sides

$$\begin{aligned}\boldsymbol{f}_n &:= (\widetilde{\boldsymbol{l}}_0 + \widetilde{\boldsymbol{l}}_1)(\boldsymbol{u}_1 + \boldsymbol{v}_n, q_1 + q_n, \rho_1 + \rho_n) \\ &\quad - (\widetilde{\boldsymbol{l}}_0 + \widetilde{\boldsymbol{l}}_1)(\boldsymbol{u}_1 + \boldsymbol{v}_{n-1}, q_1 + q_{n-1}, \rho_1 + \rho_{n-1}), \\ h_n &:= \widetilde{l}_2(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - \widetilde{l}_2(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1}) \\ &\quad = \nabla \cdot \boldsymbol{G}(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n), \\ \boldsymbol{d}_n &:= \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1}) + \\ &\quad + \boldsymbol{N}(\widetilde{l}_4(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - \widetilde{l}_4(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1})), \\ g_n &:= \widetilde{l}_5(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - \widetilde{l}_5(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1}),\end{aligned}$$

and zero initial data. Since we can safely suppose that the constant $c(1, T)$ in theorem 5.1.6 is equal to $c_2(T)$, from (4.82), (5.28), (5.33) and theorem 5.1.6, we get

$$\begin{aligned}\|(\widehat{\boldsymbol{v}}_{n+1}, \widehat{p}_{n+1},\widehat{\rho}_{n+1})\|_{W,l,T} &\le c_2(T)\big(\|(\boldsymbol{v}_{n+1}, p_{n+1}, \rho_{n+1})\|_{W,l,T} + \|(\boldsymbol{v}_n, p_n, \rho_n)\|_{W,l,T} \\ &\quad + 2\|(\boldsymbol{u}_1, p_1, \rho_1)\|_{W,l,T}\big)\|(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)\|_{W,l,T} \le \frac{1}{2}\|(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)\|_{W,l,T},\end{aligned}$$

by (5.32) and (5.28). This in turn gives strong convergence of the sequence $(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)$. Finally the continuity estimate of theorem 5.1.6 ensures that the nonlinear terms converge too, and thus the limit solves (5.30). Clearly (5.33) holds for the solution.

Step 3: construction of the global solution
We chose T_0 so large that $c_1 e^{\gamma T_0} < \frac{1}{4}$ in (5.28), then choose $\varepsilon_0 := \varepsilon(T_0)$ as in (5.32). If $U_0 \leq \varepsilon_0$ we have a global solution in $[0, T_0]$ of (3.10) defined as the sum $(\boldsymbol{u}, p, \rho) := (\boldsymbol{u}_1 + \boldsymbol{u}_2, p_1 + p_2, \rho_1 + \rho_2)$. From (5.29) and (5.33) we obtain that

$$\begin{aligned}
&\|\boldsymbol{u}(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho(T_0)\|_{W_2^{l+2}(\mathcal{G})} \\
&\leq \|\boldsymbol{u}_1(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho_1(T)\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_2(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho_2(T_0)\|_{W_2^{l+2}(\mathcal{G})} \\
&\leq \frac{U_0}{4} + c_1\|(\boldsymbol{u}_2, p_2, \rho_2)\|_{W,l,T} \leq \frac{U_0}{4} + 2c_1c_2(T_0)U_0^2 \leq \frac{U_0}{2} \leq \frac{\varepsilon_0}{2}.
\end{aligned}$$

Setting $U_1 = \|\boldsymbol{u}(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho(T_0)\|_{W_2^{l+2}(\mathcal{G})}$, (5.32) thus holds for U_1 with $\varepsilon_1 = \varepsilon_0/2$, and we can solve system (3.10) in $[T_0, 2T_0]$ with the same procedure as above, and initial data $\boldsymbol{u}(T_0)$, $\rho(T_0)$. Proceeding in this way we obtain a global solution $(\boldsymbol{u}, p, \rho)$, which satisfies

$$U_k := \|\boldsymbol{u}(kT_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho(kT_0)\|_{W_2^{l+2}(\mathcal{G})} \leq \frac{U_0}{2^k},$$

If between kT_0 and $(k+1)T_0$ we denote the solution of the linear system as $(\boldsymbol{u}_1^{(k)}, p_1^{(k)}, \rho_1^{(k)})$ and the solution of the nonlinear one as $(\boldsymbol{u}_2^{(k)}, p_2^{(k)}, \rho_2^{(k)})$, then (5.29) and (5.33) hold with U_k at each step. With obvious meaning, we then have

$$\begin{aligned}
\|(\boldsymbol{u}_1^{(k)}, p_1^{(k)}, \rho_1^{(k)})\|_{W,l,[kT_0,(k+1)T_0]} &+ \|(\boldsymbol{u}_2^{(k)}, p_2^{(k)}, \rho_2^{(k)})\|_{W,l,[kT_0,(k+1)T_0]} \\
&\leq c_1 U_k + \frac{U_k}{4} \leq c\frac{U_0}{2^k}.
\end{aligned}$$

Since T_0 is bounded away from zero we can safely split the norms over $[0, +\infty)$ as a sum of the norms over $[kT_0, (k+1)T_0)$, and from the previous inequality and

$$\|e^{-\gamma' t}(\boldsymbol{u}, p, \rho)\|_{W,l,[kT_0,(k+1)T_0]} \leq ce^{-\gamma'(k+1)T_0}\|(\boldsymbol{u}, p, \rho)\|_{W,l,[kT_0,(k+1)T_0]},$$

we get (5.25) for $0 < -\gamma' < \log 2/T_0$. The uniqueness statement follows from uniqueness for small times, which will be proved in theorem 5.4.6.

□

5.3 Exponential stability of the rest state

We apply this linearization principle to obtain nonlinear stability of the rest state

$$\boldsymbol{v}_b = 0, \qquad p_b = p_{atm} + g(h - x_3), \qquad \phi_b = h,$$

corresponding to a layer of fluid subjected to the gravitational force $\boldsymbol{f} = -g\boldsymbol{e}_3 = \nabla p_b$. Bringing the force into the pressure term, this corresponds to the stability of zero state (for both $\boldsymbol{v}_b$ and p_b) in the layer $\{0 \le x_3 \le h \equiv \phi_b\}$. The corresponding linearized problem is

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q = 0 & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} + \sigma\Delta'\rho\boldsymbol{N} = 0 & \text{on } \mathcal{G}, \\ \rho_{,t} - u^3 = 0 & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma, \text{ for all } t \ge 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \rho(x',0) = \rho_0(x'), & \text{for } x \in \Omega_b,\ x' \in \Sigma, \end{cases} \tag{5.34}$$

with the compatibility conditions

$$\nabla \cdot \boldsymbol{u}_0 = 0, \qquad \Pi_b\mathbb{D}(\boldsymbol{u}_0) = 0, \qquad \int_\Sigma \rho dx' = 0.$$

We now want to study the exponential stability of this problem.

Let $\mathcal{J}$ be the set of Σ-periodic, square summable solenoidal vector fields with third component vanishing on Σ. More precisely, letting $L^2(\Omega_{b\#})$ be the the set of Σ-periodic vector fields on the Σ periodic extension $\Omega_{b\#}$ of Ω_b, we have

$$\mathcal{J} := \{\boldsymbol{v} \in L^2_{loc}(\Omega_{b\#}) : \int_{\Omega_b} \boldsymbol{v} \cdot \nabla\eta dx = 0,\ \forall\eta \in W^1_2(\Omega_{b\#})\ s.t.\ \eta\lfloor_{\mathcal{G}}= 0\}.$$

We denote by P the orthogonal projection on this space. Given a periodic vector field $\boldsymbol{w}$, it can be splitted as $\boldsymbol{w} = P\boldsymbol{w} + (I-P)\boldsymbol{w}$, where $(I-P)\boldsymbol{w} = \nabla\varphi_{\boldsymbol{w}}$ and $\varphi_{\boldsymbol{w}}$ is the periodic weak solution to

$$\begin{cases} \Delta\varphi_{\boldsymbol{w}} = \nabla \cdot \boldsymbol{w} & \text{in } \Omega_b, \\ \varphi_{\boldsymbol{w}} = 0 & \text{on } \mathcal{G}, \\ \frac{\partial\varphi_{\boldsymbol{w}}}{\partial x_3} = w^3 & \text{on } \Sigma. \end{cases} \tag{5.35}$$

As has been proved before, the operator P is continuous in $W^\eta_2(\Omega_b)$, $\eta \ge 0$. Projecting the first equation of (5.34) onto $\mathcal{J}$ gives

$$\boldsymbol{u}_{,t} - \nu P\Delta\boldsymbol{u} + \nabla\chi = 0,$$

where, using (5.35), χ is a Σ-periodic function such that

$$\begin{cases} \Delta\chi = 0 & \text{in } \Omega_b, \\ \chi = \nu \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u})\boldsymbol{N} + \sigma\Delta'\rho & \text{on } \mathcal{G}, \\ \frac{\partial \chi}{\partial x_3} = 0 & \text{on } \Sigma. \end{cases}$$

It can be splitted as $\chi = \chi_{\boldsymbol{u}} + \chi_\rho$ where $\chi_{\boldsymbol{u}}$ and χ_ρ are two Σ-periodic harmonic functions with vanishing normal derivative on Σ and

$$\chi_{\boldsymbol{u}}\lfloor_{\mathcal{G}} = \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u})\boldsymbol{N}, \qquad \chi_\rho\lfloor_{\mathcal{G}} = \sigma\Delta'\rho.$$

We then define a linear operator $\mathcal{A}$ on the Hilbert space

$$X := \mathcal{J} \times \{\rho \in L^2(\Sigma) : \int_\Sigma \rho dx' = 0\},$$

equipped with the norm

$$\|(\boldsymbol{u}, \rho)\|_X = \left(\|\boldsymbol{u}\|^2_{L^2(\Omega_b)} + \|\rho\|^2_{L^2(\Sigma)}\right)^{\frac{1}{2}},$$

and corresponding standard inner product. We let $\mathcal{A} = (A_{ij})_{i,j=1,2}$, where

$$A_{11}(\boldsymbol{u}) = \nu P\Delta\boldsymbol{u} - \nabla\chi_{\boldsymbol{u}}, \qquad A_{12}(\rho) = \nabla\chi_\rho, \qquad A_{21} = u^3, \qquad A_{22} = 0.$$

The linear operator $\mathcal{A}$ will have domain $Y =: D(\mathcal{A})$ defined as

$$Y := \{\boldsymbol{u} \in W^2_2(\Omega_b) \cap \mathcal{J} : \boldsymbol{u}\lfloor_\Sigma = \Pi_b\mathbb{D}(\boldsymbol{u})\lfloor_{\mathcal{G}} = 0\} \times \{\rho \in W_2^{\frac{5}{2}}(\Sigma) : \int_\Sigma \rho dx' = 0\},$$

with norm

$$\|(\boldsymbol{u}, \rho)\|_Y = \left(\|\boldsymbol{u}\|^2_{W^2_2(\Omega_b)} + \|\rho\|^2_{W_2^{\frac{5}{2}}(\Sigma)}\right)^{\frac{1}{2}}.$$

A resolvent estimate for $\mathcal{A}$ when $\mathrm{Re}\,\lambda$ is sufficiently large has been proved in theorem 4.2.4 (for $l = 0$), and this gives that $\lambda - \mathcal{A}$ is coercive (and thus closed) for $\mathrm{Re}\,\lambda$ sufficiently large. Thus $\mathcal{A}$ is closed and since $D(\mathcal{A})$ is compactly embedded in X, its spectrum consists of a countable number of eigenvalues with the only accumulation point at infinity.

We can look at problem (5.34) as the evolutionary problem

$$\boldsymbol{U}_{,t} - \mathcal{A}\boldsymbol{U} = 0, \qquad \boldsymbol{U}(0) = \boldsymbol{U}_0 = (\boldsymbol{u}_0, \rho_0),$$

whose exponential stability follows from classical results once one can show positivity of the real part of the spectrum of $\mathcal{A}$. To this end suppose $(\boldsymbol{u}, \rho) \in D(\mathcal{A})$ is a solution of the complex eigenvalue problem

$$\begin{cases} \lambda \boldsymbol{u} - \nu P \Delta \boldsymbol{u} + \nabla \chi = 0 & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} + \sigma \Delta' \rho \boldsymbol{N} = 0 & \text{on } \mathcal{G}, \\ \lambda \rho - u^3 = 0 & \text{on } \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on } \Sigma, \end{cases}$$

for some $\lambda \in \mathbb{C}$. Taking the scalar product with $\boldsymbol{u}$ in the first equation and integrating by parts gives

$$\begin{aligned} 0 &= \int_{\Omega_b} \lambda |\boldsymbol{u}|^2 + \frac{\nu}{2} |\mathbb{D}(\boldsymbol{u})|^2 dx + \int_{\mathcal{G}} \mathbb{D}(\boldsymbol{u})\boldsymbol{N} \cdot \boldsymbol{u} dS - \int_{\mathcal{G}} \chi \boldsymbol{u} \cdot \boldsymbol{N} dS \\ &= \int_{\Omega_b} \lambda |\boldsymbol{u}|^2 + \frac{\nu}{2} |\mathbb{D}(\boldsymbol{u})|^2 dx + \int_{\mathcal{G}} \mathbb{D}(\boldsymbol{u})\boldsymbol{N} \cdot \boldsymbol{u} - \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u})\boldsymbol{N} \boldsymbol{u} \cdot \boldsymbol{N} dS \\ &\quad - \sigma \int_{\Sigma} \Delta' \rho \lambda \rho dS = \lambda \left(\int_{\Omega_b} |\boldsymbol{u}|^2 dx + \sigma \int_{\Sigma} |\nabla' \rho|^2 dx \right) + \frac{\nu}{2} \int_{\Omega_b} |\mathbb{D}(\boldsymbol{u})|^2 dx, \end{aligned}$$

where we used the fact that $\boldsymbol{u} \cdot \boldsymbol{N} = \lambda \rho$ and $\Pi_b \mathbb{D}(\boldsymbol{u})\boldsymbol{N} = 0$ to cancel out the boundary terms containing $\mathbb{D}(\boldsymbol{u})$. This clearly implies that the spectrum is real and $\lambda < 0$ for any eigenvalue of $\mathcal{A}$ and thus exponential stability of the associated linear problem. More precisely, by standard classical results (see [12] for example), it holds

$$\begin{aligned} \|\boldsymbol{U}(t)\|_X &\le c(\gamma) e^{\gamma t} \|\boldsymbol{U}_0\|_X, \\ \int_0^T e^{-2\gamma t} \|\boldsymbol{U}(t)\|_X^2 dt &\le c(\gamma) \|\boldsymbol{U}_0\|_X^2, \end{aligned} \tag{5.36}$$

for $0 > \gamma > \sup\{\lambda : \lambda \in \sigma(\mathcal{A})\}$, with $c(\gamma)$ independent of T. Now, suppose $(\boldsymbol{u}, p, \rho)$ solves (5.23), and consider the equation satisfied by $e^{-\gamma t}(\boldsymbol{u}, p, \rho) =: (\boldsymbol{u}_\gamma, p_\gamma, \rho_\gamma)$: the system is the same except for a forcing term $-\gamma \boldsymbol{u}_\gamma$ in the equation for $\boldsymbol{u}_{\gamma,t}$ and $-\gamma \rho_\gamma$ in the one for $\rho_{\gamma,t}$. Applying (4.83) to $(\boldsymbol{u}_\gamma, p_\gamma, \rho_\gamma)$ we get

$$\begin{aligned} \|e^{-\gamma t}(\boldsymbol{u}, p, \rho)\|_{W,l,\infty} \le c(&\|\boldsymbol{u}_\gamma\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} + \|\rho_\gamma\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} \\ &+ \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_\gamma\|_{L^2(Q_\infty)} + \|\rho_\gamma\|_{L^2(G_\infty)}). \end{aligned}$$

The interpolation inequalities

$$\|\rho_\gamma\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} \le \varepsilon \|\rho_\gamma\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_\infty)} + c(\varepsilon) \|\rho_\gamma\|_{L^2(G_\infty)},$$

$$\|\boldsymbol{u}_\gamma\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} \leq \varepsilon\|\boldsymbol{u}_\gamma\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} + c(\varepsilon)\|\boldsymbol{u}_\gamma\|_{L^2(Q_\infty)},$$

hold. Using these inequalities and (5.36), we finally get

$$\|(\boldsymbol{u}_\gamma, p_\gamma, \rho_\gamma)\|_l \leq c(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}),$$

which is (5.25). Applying theorem 5.2.1 thus concludes the proof of the nonlinear exponential stability of the rest state.

5.4 Local solvability in time

In this section we consider the solvability for small times of problem (1.2). We adopt a semi-linearization argument, following [19].

We consider problem (1.2) and choose a smooth ϕ_b sufficiently near (in a sense to be specified) to ϕ_0. We set $\rho = \phi - \phi_b$ and use Hanzawa transformation to reduce problem (1.2) to problem (3.7) in Ω_b, as in section 3. We then modify the resulting problem, fixing $\boldsymbol{v}_b$ sufficiently near (in a sense to be specified) to $\boldsymbol{v}_0$ and writing the equation for $\rho_{,t}$

$$\rho_{,t} + \nabla'\phi_b \cdot \boldsymbol{v} - v^3 = -\nabla'\rho \cdot \boldsymbol{v}$$

as

$$\rho_{,t} + \nabla'\phi_b \cdot \boldsymbol{v} - v^3 + \nabla'\rho \cdot \boldsymbol{v}_b = \nabla'\rho \cdot (\boldsymbol{v}_b - \boldsymbol{v}) =: l_5(\boldsymbol{v}, \rho). \tag{5.37}$$

We are then reduced to the following problem

$$\begin{cases} \boldsymbol{v}_{,t} - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{l}_1(\boldsymbol{v}, p, \rho) + \boldsymbol{l}_0(\boldsymbol{v}, \rho) & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{v} = l_2(\boldsymbol{v}, \rho) = \nabla \cdot \boldsymbol{G} & \text{in } \Omega_b, \\ \nu\Pi_{\mathcal{G}} S(\boldsymbol{v})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{v}, \rho) & \text{on } \mathcal{G}, \\ -p + \nu\boldsymbol{N} \cdot S(\boldsymbol{v})\boldsymbol{N} + \sigma L\rho = l_4(\boldsymbol{v}, \rho) - \sigma H_b(y) & \text{on } \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b \cdot \boldsymbol{v} - v^3 + \nabla'\rho \cdot \boldsymbol{v}_b = l_5(\boldsymbol{v}, \rho) & \text{on } \mathcal{G}, \\ \boldsymbol{v}(x, 0) = \widetilde{\boldsymbol{v}}_0(x), \quad \text{in } \Omega_b, \qquad \rho(x, 0) = \rho_0(x), \quad \text{on } \mathcal{G}, \\ \boldsymbol{v}(x', t) = \boldsymbol{\alpha}(x', t) \text{ for } t \geq 0,\ x' \in \Sigma, \end{cases} \tag{5.38}$$

where l_i with $i = 0, \ldots 4$ are given in (3.8) and l_5 in (5.37), with compatibility conditions

$$\begin{cases} \nabla \cdot \widetilde{\boldsymbol{v}}_0 = l_2(\widetilde{\boldsymbol{v}}_0, \rho_0), \\ \nu\Pi_{\mathcal{G}} S(\widetilde{\boldsymbol{v}}_0)\boldsymbol{N} = \boldsymbol{l}_3(\widetilde{\boldsymbol{v}}_0, \rho_0), \\ \widetilde{\boldsymbol{v}}_0\lfloor_\Sigma = \boldsymbol{\alpha}(\cdot, 0). \end{cases} \tag{5.39}$$

The linear part of (5.38) is not exactly of the type (4.1) since we are not truly linearizing around $(\boldsymbol{v}_b, \phi_b)$, however adding the term $\nabla'\rho \cdot \boldsymbol{v}_b$ on one hand doesn't affect the coercive estimates, while the nonlinear term l_5 is much smaller than the original one, and thus, more comfortably estimated. Regarding the linear problem, it is easy to check that the proof of theorem 4.3.2 still holds, and is in fact easier in this case. Indeed the Laplace transform of the homogeneous linear problem associated to (5.38) directly gives a problem of the type (4.43), for which theorem 4.2.4 holds. The reduction to homogenous initial data is simpler in this case: we won't fill in the details, since it suffices to follow the proof of theorem 4.1 assuming all the Φ_i being zero.

We thus can assume that the following theorem holds true.

Theorem 5.4.1 *Let $l \in (\frac{1}{2}, 1)$ and $T \leq 1$. For any Σ-periodic choice of $\boldsymbol{f} \in W_2^{l,\frac{l}{2}}(Q_T)$, $h \in W_2^{l+1,0}(Q_T)$, $\boldsymbol{F} \in W_2^{0,\frac{l}{2}+1}(Q_T)$ with $F^3\lfloor_\Sigma = 0$, $\boldsymbol{d} \in W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)$, $g \in W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)$, $\boldsymbol{a} \in W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma)$ with $a^3 \equiv 0$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$ and $\boldsymbol{v}_0 \in W_2^{l+1}(\Omega)$ such that*

$$\begin{cases} \nabla \cdot \boldsymbol{v}_0(x) = \nabla \cdot \boldsymbol{F}(x,0) & \text{for } x \in \Omega_b, \\ \nu \Pi_b S(\boldsymbol{v}_0)(x)\boldsymbol{N}(x) = \Pi_b \boldsymbol{d}(x,0) & \text{for } x \in \mathcal{G}, \\ \boldsymbol{v}_0\lfloor_\Sigma = \boldsymbol{a}(\cdot,0), \end{cases} \tag{5.40}$$

there exists a unique solution to

$$\begin{cases} \boldsymbol{v}_{,t} - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} & \text{in } \Omega_b, \\ \nabla \cdot \boldsymbol{v} = h = \nabla \cdot \boldsymbol{F} & \text{in } \Omega_b, \\ T(p,\boldsymbol{v})\boldsymbol{N} + \sigma L\rho\boldsymbol{N} = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b \cdot \boldsymbol{v} - v^3 + \nabla'\rho \cdot \boldsymbol{v}_b = g & \text{on } \mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad \text{in } \Omega_b, \qquad \rho(x,0) = \rho_0(x), \quad \text{on } \mathcal{G}, \\ \boldsymbol{v}(x',t) = \boldsymbol{\alpha}(x',t) \text{ for } t \geq 0,\ x' \in \Sigma, \end{cases} \tag{5.41}$$

and it holds

$$\begin{aligned} \|(\boldsymbol{v},p,\rho)\|_{H,l,T} \leq c\big(&\|\boldsymbol{f}\|_{H_2^{l,\frac{l}{2}}(Q_T)} + \|h\|_{H_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)} \\ &+ \|\boldsymbol{d}\|_{H_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|\boldsymbol{a}\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(\Sigma_T)} + \|g\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \\ &+ \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\big) \end{aligned} \tag{5.42}$$

with constant independent of $T \leq 1$.

Notice that the particular form of the Hanzawa transformation we used (and thus, the choice of θ in the definition of the transformation (3.1)) only affects the nonlinearities, and not the linear part of the system.

We modify the norms used in the following, defining

$$\|\rho\|_{l,T}^2 := \|\rho\|^2_{W_2^{l+\frac{5}{2},0}(G_T)} + \|\rho\|^2_{\widehat{W}_2^{0,\frac{l}{2}+\frac{5}{4}}(G_T)} + \sup_{t<T}\|\rho\|^2_{W_2^{l+2}(\mathcal{G})} + \sup_{t<T}\|\rho_{,t}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}.$$

Clearly

$$\|\rho\|_{H,l,T} = \|\rho\|_{H_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_T)} + \|\rho_{,t}\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \le c\|\rho\|_{l,T},$$

with a constant independent of T. We now show that an inequality of the opposite type holds, thus proving the equivalence of the two norms. We construct an extension $C(\rho)$ of ρ to $t \ge 0$ such that

$$\|C(\rho)\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_\infty)} + \|C(\rho)_{,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} \le \|\rho\|_{H,l,T}. \tag{5.43}$$

This can be done as in the proof of theorem 4.3.2, namely as in(4.87): there exists $\rho_1 : \mathcal{G} \times [T, +\infty) \to \mathbb{R}$ such that

$$\rho_1\lfloor_{t=T} = \rho\lfloor_{t=T}, \qquad \rho_{1,t}\lfloor_{t=T} = \rho_{,t}\lfloor_{t=T}$$

and

$$\|\rho_1\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_{T,+\infty})} + \|\rho_{1,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_{T,+\infty})} \le c(\|\rho(\cdot,T)\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_{,t}(\cdot,T)\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}).$$

Performing again the calculations done in theorem 2.3.3 to prove (2.30), we obtain (5.43). This gives the claim, since theorem 2.2.3 now applies for the unbounded interval $[0, +\infty)$, and thus with constant independent of T. Thus (5.42) holds with $\|\rho\|_{l,T}$ instead of $\|\rho\|_{H,l,T}$ in the left hand side.

We now look at the analogue of (5.4). Using the norm $\|\rho\|_{l,T}$ allows to obtain a similar chain of inequalities with constant independent of $T \le 1$. We employ theorem 2.3.3 to construct an extension $C(\rho_{,t})$ of $\rho_{,t}$ to $\mathbb{R}_+$ such that $\|C(\rho_{,t})\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} \le c\|\rho\|_{l,T}$.

$$\begin{aligned}\sup_{\mathcal{G}}\|\rho\|_{W_2^{\frac{l}{2}+\frac{5}{4}}(0,T)} &\le \sqrt{T}\sup_{G_T}|\rho| + \sup_{\mathcal{G}}\|\rho_{,t}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)} \\ &\le c\sqrt{T}\sup_{t<T}\|\rho\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + \sup_{\mathcal{G}}\|C(\rho_{,t})\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(\mathbb{R}_+)} \\ &\le c\sqrt{T}\|\rho\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} + \|C(\rho_{,t})\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)} \\ &\le c\|\rho\|_{l,T}\end{aligned} \tag{5.44}$$

Similarly one has

$$\sup_{\mathcal{G}}\|\nabla'\rho_{,t}\|_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)}+\sup_{\mathcal{G}}\|\nabla'\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)}\leq c\|\rho\|_{l,T}, \tag{5.45}$$

and thus (5.4) follows, with constants independent of T. Notice that a similar chain of inequalities does not a priori hold on Ω, due to the fact that the term $\nabla\rho^*$ contains a factor $\rho\nabla\theta$ which blows up for $\varepsilon \to 0$. Another useful and immediate estimate is

$$\sup_{Q_T}|\rho^*_{\varepsilon,t}|=\sup_{G_T}|\rho_{,t}|\leq c\sup_{t<T}\|\rho_{,t}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\leq c\|\rho\|_{l,T}. \tag{5.46}$$

Let us discuss our main hypotheses. Since $\rho(x,t)=\phi(x,t)-\phi_b(x)$, where ϕ is the free boundary function of the original problem (1.2), we assume that, if $\rho_0=\rho(\cdot,0)$,

$$\|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\leq\delta\ll 1. \tag{5.47}$$

Notice that this in turn gives, for $\rho: G_T\to\mathbb{R}$ and any $t\leq T$,

$$\|\rho(\cdot,t)\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\leq\|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}+\int_0^t\|\rho_{,t}(\cdot,t)\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}dt\leq\delta+\sqrt{T}\|\rho_{,t}\|_{W_2^{l+\frac{3}{2},0}(G_T)}.$$

Therefore, as soon as the quantity $\|\rho\|_{l,T}$ is bounded, for sufficiently small T depending only on δ it holds

$$\sup_{G_T}|\rho|+|\nabla'\rho|\leq c\sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\leq c\delta(1+\|\rho\|_{l,T}). \tag{5.48}$$

Since our extension of ρ from $\mathcal{G}$ (or, equivalently, Σ) to Ω_b is not optimal, we will need the following result.

Lemma 5.4.2 *For any $\varepsilon>0$ there exists a smooth $\theta:\Omega_b\to\mathbb{R}$ such that $\theta=1$ in a neighbourhood of $\mathcal{G}$, $\theta=0$ in a neighbourhood of Σ and for any function $\rho:\mathcal{G}\to\mathbb{R}$ and $\eta\geq 0$ it holds*

$$\|\theta\rho\|_{W_2^{\eta}(\Omega_b)}\leq c\varepsilon\|\rho\|_{W_2^{\eta}(\mathcal{G})}+c_\eta(\varepsilon)\|\rho\|_{L^2(G_T)}.$$

Proof. We consider a smooth diffeomorphism $\Psi:\Omega\to\Sigma\times[0,1]$, bringing $\mathcal{G}$ to $\Sigma^1:=\Sigma\times\{1\}$ and Σ to $\Sigma_0:=\Sigma\times\{0\}$. We then define a smooth $\widetilde{\theta}:\Sigma\times[0,1]\to\mathbb{R}$ such that $\widetilde{\theta}\leq 1$, $\widetilde{\theta}=1$ for $x_3>1-2\varepsilon$ and $\widetilde{\theta}=0$ for $x_3\leq 1-\varepsilon$. By lemma 2.1.4, for any $\rho:\Sigma^1\to\mathbb{R}$ it holds

$$\|\widetilde{\theta}\rho\|_{W_2^{\eta}(\Sigma\times[0,1])}\leq c\varepsilon\|\rho\|_{W_2^{\eta}(\Sigma^1)}+c_\eta(\varepsilon)\|\rho\|_{L^2(\Sigma^1)}.$$

It is easy to check that

$$\|f\|_{W_2^\eta(\Omega_b)} \le c\|f \circ \Psi^{-1}\|_{W_2^\eta(\Sigma\times[0,1])},$$

and a similar estimate for the norm $\|f\|_{W_2^\eta(\Sigma^1)}$, with a constant depending on $\|\Psi\|_{W_2^{\eta'}(\Omega_b)}$, for $\eta' > \frac{5}{2}$, $\eta' \ge \eta$. Therefore, for $\theta := \widetilde{\theta} \circ \Psi$ it holds

$$\|\theta\rho\|_{W_2^\eta(\Omega_b)} \le c\varepsilon\|\rho\|_{W_2^\eta(\mathcal{G})} + c_\eta(\varepsilon)\|\rho\|_{L^2(\mathcal{G})}.$$

□

It is easy to see that there is a bound of the form

$$\sup_\Omega |D^k\theta| \le \frac{c}{\varepsilon^k}, \qquad c(\varepsilon) \le c\varepsilon^{-\gamma},$$

in the previous lemma, where, in the case $\eta < 3$, one can take $\gamma = \frac{5}{2}$. For any $\varepsilon > 0$ we can define the extension $\rho_\varepsilon^* := \theta\rho$ using the function θ given above. Any norm of ρ_ε^* can be bounded by the same norm of ρ, with a suitable constant which depends on ε (and usually blows up as $\varepsilon \to 0$). Assuming (5.47) however, allows in some cases to keep the constant bounded with suitable choices of the parameter δ and T. For example, it holds

$$\begin{aligned}\sup_{t<T}\|\rho_\varepsilon^*\|_{W_2^{l+2}(\Omega_b)} &\le c\varepsilon\|\rho\|_{l,T} + c(\varepsilon)\sup_{t<T}\|\rho\|_{L^2(\mathcal{G})}\\ &\le c\varepsilon\|\rho\|_{l,T} + c(\varepsilon)\sup_{t<T}\|\rho\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\\ &\le c(\varepsilon + c(\varepsilon)\delta)(1 + \|\rho\|_{l,T}),\end{aligned} \tag{5.49}$$

for any sufficiently small T depending only on δ. We will suppose that δ is such that $\varepsilon + \delta c(\varepsilon)$ is arbitrarily small. Notice that in the Hanzawa transformation (3.1), in order to obtain a well defined diffeomorphism, we also require that

$$\sup_\Omega |\theta_{,x_3}\rho| \le c\frac{\delta}{\varepsilon} \ll 1,$$

for δ sufficiently small compared to ε. To fix ideas, we may think that $\varepsilon \simeq \delta^{\frac{2}{7}}$, which allows all the previous assumptions to hold. Clearly from (5.49) it follows

$$\sup_{\Omega_b} |\rho_\varepsilon^*| + |\nabla\rho_\varepsilon^*| \le c(\varepsilon + c(\varepsilon)\delta)(1 + \|\rho\|_{l,T}), \tag{5.50}$$

by standard embedding theorems on Ω_b.

We will only give the details of the estimates for $\boldsymbol{l}_0$, $\boldsymbol{l}_1$ and l_2 which are the only ones affected by the choice of the extension, since the other terms are estimated as in [19].

For $\boldsymbol{l}_0$ and $\boldsymbol{l}_1$, we will need a tool to get rid of the "spurious" factor in the estimate of nonlinear terms of the form $f(\nabla\rho^*)\boldsymbol{m}$, similar to lemma 5.1.2 (there is no such type of factor in l_2 and $\boldsymbol{G}$). For the spatial derivatives, notice that (5.6) holds with a constant independent of T, δ and ε, if they are all sufficiently small. Indeed, it is enough to look at (5.8): its proof was based on (5.2), whose analogue (5.49) holds true with constant independent of T. Moreover, the constant $c_f(\mu)$ depends only on the C^k norm of f in a set bounded by $\sup_{Q_T}|\rho^*_\varepsilon| + |\nabla\rho^*_\varepsilon|$, which, by (5.50), is bounded by $\|\rho\|_{l,T}$ if $\delta c(\varepsilon) \ll 1$, as we are assuming. Thus, for any ρ such that $\|\rho\|_{l,T} \le \mu$ and $\eta \le l+1$ it holds

$$\|f(x,\rho^*_\varepsilon,\nabla\rho^*_\varepsilon)g\|_{H_2^{\eta,0}(Q_T)} \le c_f(\mu)\|g\|_{H_2^{\eta,0}(Q_T)}.$$

Regarding the estimates of the time derivative, we will need the following variant of lemma 5.1.2.

Lemma 5.4.3 *Let $T \le 1$, $l \le 1$ and $\|\rho\|_{l,T} \le \mu$. It holds the inequality*

$$\|f(\nabla\rho^*)g\|_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)} \le c_f(\mu)(1+\frac{\sqrt{T}}{\varepsilon})\|\rho\|_{l,T}\|g\|_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)}.$$

Proof. As already noted, $f(\nabla\rho^*)$ is bounded by a constant depending only on f and μ, and this takes care of the L^2 term. One can check that in the proof of 2.1.3, given $\eta > \frac{1}{2}$, the inequality

$$\|uv\|_{W_2^{\frac{l}{2}}(0,T)} \le c(\sup_{[0,T]}|u| + \|u\|_{W_2^{\eta}(0,T)})\|v\|_{W_2^{\frac{l}{2}}(0,T)},$$

holds, with a constant independent of $T \le 1$. Then it suffice to prove the inequality

$$\sup_{\Omega}\|f(\nabla\rho^*)\|_{\mathring{W}_2^{1}(0,T)} \le c_f(\mu)(1+\frac{\sqrt{T}}{\varepsilon})\|\rho\|_{l,T}.$$

To this end, for any fixed $x=(x',x_3)$ in Ω we have, using the fact that θ does not depend on t, $|\theta| \le 1$ and $|\nabla\theta| \le c/\varepsilon$,

$$\begin{aligned}\|f'(\nabla\rho^*(x,\cdot))\nabla\rho^*_{,t}(x,\cdot)\|_{L^2(0,T)} &\le c_f(\mu)\big(\|\nabla'\rho_{,t}(x',\cdot)\|_{L^2(0,T)} + \frac{1}{\varepsilon}\|\rho_{,t}(x',\cdot)\|_{L^2(0,T)}\big)\\ &\le c_f(\mu)\big(\sup_{\mathcal{G}}\|\nabla'\rho_{,t}\|_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} + \frac{\sqrt{T}}{\varepsilon}\sup_{G_T}|\rho_{,t}|\big),\end{aligned}$$

which gives the claim by (5.45) and (5.46). □

We now sketch the proof of the estimates of the nonlinear terms which are different from the ones contained in [19].

For $\boldsymbol{l}_1$ and $\boldsymbol{l}_0$ the troublesome terms are

$$\nabla\rho^*\nabla p, \qquad D^2\rho^*\nabla\boldsymbol{v}, \qquad \nabla\rho^* D^2\boldsymbol{v}, \qquad \rho^*_{,t}\nabla\boldsymbol{v}.$$

Notice that we didn't consider factors of the form $f(\nabla\rho^*)$ for smooth f's due to the previous discussion, and only considered the higher order terms. The first two type of terms are estimated in the $H_2^{l,\frac{l}{2}}(Q_T)$ norm through the following procedure, where $\boldsymbol{m}\in H_2^{l,\frac{l}{2}}(Q_T)$ is to be understood as ∇p or $D^2\boldsymbol{v}$. According to (5.49) and (5.50) we can set (for sufficiently small T), $\delta_1=(\varepsilon+\delta c(\varepsilon))\ll 1$ and obtain

$$\begin{aligned}\|\nabla\rho^*\boldsymbol{m}\|_{W_2^{l,0}(Q_T)} &\le c\sup_{t<T}\|\nabla\rho^*\|_{W_2^{l+1}(\Omega_b)}\|\boldsymbol{m}\|_{W_2^{l,0}(Q_T)}\\ &\le c\delta_1(1+\|\rho\|_{l,T})\|\boldsymbol{m}\|_{W_2^{l,0}(Q_T)},\end{aligned}$$

$$\frac{1}{T^{\frac{l}{2}}}\|\nabla\rho^*\boldsymbol{m}\|_{L^2(Q_T))}\le c\sup_{Q_T}|\nabla\rho^*|\|\boldsymbol{m}\|_{\widehat{W}_2^{\frac{l}{2}}(Q_T)}\le c\delta_1(1+\|\rho\|_{l,T})\|\boldsymbol{m}\|_{\widehat{W}_2^{\frac{l}{2}}(Q_T)};$$

for the $\mathring{W}_2^{0,l}(Q_T)$ norm, we use

$$\Delta_{-h}(\nabla\rho^*_{,t}\boldsymbol{m})=\nabla\rho^*_{-h}\Delta_{-h}\boldsymbol{m}+\boldsymbol{m}\Delta_{-h}\nabla\rho^*, \tag{5.51}$$

where $f_{-h}(t):=f(t-h)$. Splitting the estimate according to this formula, we have

$$\begin{aligned}\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\nabla\rho^*_{-h}\Delta_{-h}\boldsymbol{m}\|^2_{L^2(\Omega_b)}dt &\le \sup_{Q_T}|\nabla\rho^*|^2\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\Delta_{-h}\boldsymbol{m}\|^2_{L^2(\Omega_b)}dt\\ &\le c\delta_1^2(1+\|\rho\|_{l,T})^2\|\boldsymbol{m}\|^2_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)}\end{aligned} \tag{5.52}$$

and, using proposition 2.1.3, point 1 and lemma 5.4.2,

$$\begin{aligned}&\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\boldsymbol{m}\Delta_{-h}\nabla\rho^*\|^2_{L^2(\Omega_b)}dt\\ &\le\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\boldsymbol{m}(\cdot,t)\|^2_{W_2^l(\Omega_b)}\|\Delta_{-h}\nabla\rho^*(\cdot,t)\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)}dt\\ &\le\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\boldsymbol{m}(\cdot,t)\|^2_{W_2^l(\Omega_b)}\Big\|\int_0^h\nabla\rho^*_{,t}(\cdot,t-\xi)d\xi\Big\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)}dt\\ &\le\int_0^T\frac{dh}{h^l}\int_h^T\|\boldsymbol{m}(\cdot,t)\|^2_{W_2^l(\Omega_b)}\int_0^h\|\nabla\rho^*_{,t}(\cdot,t-\xi)\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)}d\xi dt\\ &\le\int_0^T\|\rho^*_{,t}(\cdot,\xi)\|^2_{W_2^{\frac{5}{2}-l}(\Omega_b)}d\xi\int_0^T\|\boldsymbol{m}(\cdot,t)\|^2_{W_2^l(\Omega_b)}dt\int_0^T\frac{dh}{h^l}\\ &\le cT^{1-l}\|\rho^*_{,t}\|^2_{W_2^{l+\frac{3}{2},0}(Q_T)}\|\boldsymbol{m}\|^2_{W_2^{l,0}(Q_T)}\le cT^{1-l}c(\varepsilon)\|\rho_{,t}\|^2_{W_2^{l+\frac{3}{2}}(G_T)}\|\boldsymbol{m}\|^2_{W_2^{l,0}(Q_T)}.\end{aligned} \tag{5.53}$$

For the terms of the form $D^2\rho^*\nabla\boldsymbol{v}$ we have

$$\begin{aligned}\|D^2\rho^*\nabla\boldsymbol{v}\|_{W_2^{l,0}(Q_T)} &\leq \sup_{t<T}\|D^2\rho^*\|_{W_2^l(\Omega_b)}\|\nabla\boldsymbol{v}\|_{W_2^{l+1}(Q_T)}\\ &\leq c\delta_1(1+\|\rho\|_{l,T})\|\boldsymbol{v}\|_{W_2^{l+2,0}(Q_T)},\end{aligned}$$

and using $\|D^2\rho^*\nabla\boldsymbol{v}\|_{L^2(\Omega_b)} \leq c\|D^2\rho^*\|_{W_2^l(\Omega_b)}\|\nabla\boldsymbol{v}\|_{W_2^{\frac{3}{2}-l}(\Omega_b)}$,

$$\begin{aligned}\frac{1}{T^{\frac{l}{2}}}\|D^2\rho^*\nabla\boldsymbol{v}\|_{L^2(Q_T)} &\leq c\sup_{t<T}\|D^2\rho^*\|_{W_2^l(\Omega_b)}\|\nabla\boldsymbol{v}\|_{\widehat{W}^{\frac{l}{2}}(0,T;W_2^{\frac{3}{2}-l}(\Omega_b)}\\ &\leq c\delta_1(1+\|\rho\|_{l,T})\|\nabla\boldsymbol{v}\|_{H_2^{\frac{3}{2},\frac{3}{4}}(Q_T)}\\ &\leq c\delta_1(1+\|\rho\|_{l,T})\|\boldsymbol{v}\|_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}\end{aligned}$$

Finally, splitting the estimate for $\mathring{W}_2^{0,l}(Q_T)$ according to (5.51),

$$\begin{aligned}&\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|D^2\rho^*_{-h}\Delta_{-h}\nabla\boldsymbol{v}\|^2_{L^2(\Omega_b)}dt\\ &\qquad\leq c\sup_{t<T}\|D^2\rho^*\|^2_{W_2^l(\Omega_b)}\|\nabla\boldsymbol{v}\|^2_{\widehat{W}_2^{\frac{l}{2}}(0,T;W_2^{\frac{3}{2}-l}(\Omega_b))}\\ &\qquad\leq c\delta_1(1+\|\rho\|_{l,T})\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(Q_T)},\end{aligned}$$

and, proceeding as in (5.53),

$$\begin{aligned}&\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\nabla\boldsymbol{v}\Delta_{-h}D^2\rho^*\|^2_{L^2(\Omega_b)}dt\\ &\qquad\leq c\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\nabla\boldsymbol{v}(\cdot,t)\|^2_{W_2^{2-l}(\Omega_b)}\|\Delta_{-h}D^2\rho^*(\cdot,t)\|^2_{W_2^{l-\frac{1}{2}}(\Omega_b)}dt\\ &\qquad\leq c(\varepsilon)\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\boldsymbol{v}(\cdot,t)\|^2_{W_2^{l+2}(\Omega_b)}\|\rho(\cdot,t)\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})}dt\\ &\qquad\leq c(\varepsilon)\int_0^T \frac{dh}{h^l}\int_h^T \|\boldsymbol{v}(\cdot,t)\|^2_{W_2^{l+2}(\Omega_b)}\int_0^h \|\rho_{,t}(\cdot,t-\xi)\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})}d\xi dt\\ &\qquad\leq c(\varepsilon)T^{1-l}\|\rho_{,t}\|^2_{W_2^{l+\frac{3}{2},0}(G_T)}\|\boldsymbol{v}\|^2_{W_2^{l+2,0}(Q_T)}.\end{aligned}$$

We now estimate the term $\rho^*_{,t}\nabla\boldsymbol{v}$. Again from proposition 2.1.3 and lemma

5.4.2

$$\begin{aligned}\|\rho^*_{,t}\nabla \boldsymbol{v}\|_{W_2^{l,0}(Q_T)} &\le c\sup_{t<T}\|\nabla \boldsymbol{v}\|_{W_2^l(\Omega_b)}\|\rho^*_{,t}\|_{W_2^{l+1,0}(\Omega_b)}\\ &\le c\|\boldsymbol{v}\|_{H_2^{l,\frac{l}{2}}(Q_T)}\left(\varepsilon\|\rho_{,t}\|_{W_2^{l+1,0}(G_T)}+c(\varepsilon)\|\rho_{,t}\|_{L^2(G_T)}\right)\\ &\le c\|\boldsymbol{v}\|_{H_2^{l,\frac{l}{2}}(Q_T)}\left(\varepsilon\|\rho_{,t}\|_{W_2^{l+\frac{3}{2},0}(G_T)}+c(\varepsilon)\sqrt{T}\sup_{t<T}\|\rho_{,t}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\right)\\ &\le c(\varepsilon+c(\varepsilon)\sqrt{T})\|\boldsymbol{v}\|_{H_2^{l,\frac{l}{2}}(Q_T)}\|\rho\|_{l,T}.\end{aligned}$$

Moreover

$$\begin{aligned}\frac{1}{T^{\frac{l}{2}}}\|\rho^*_{,t}\nabla \boldsymbol{v}\|_{L^2(Q_T)} &\le cT^{\frac{1}{2}-\frac{l}{2}}\sup_{t<T}\|\rho^*_{,t}\|_{W_2^{\frac{3}{2}-l}(\Omega_b)}\sup_{t<T}\|\nabla \boldsymbol{v}\|_{W_2^l(\Omega_b)}\\ &\le cT^{\frac{1}{2}-\frac{l}{2}}\sup_{t<T}\|\rho^*_{,t}\|_{W_2^{l+\frac{1}{2}}(\Omega_b)}\sup_{t<T}\|\boldsymbol{v}\|_{W_2^{l+1}(\Omega_b)}\\ &\le cT^{\frac{1}{2}-\frac{l}{2}}c(\varepsilon)\sup_{t<T}\|\rho_{,t}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\sup_{t<T}\|\boldsymbol{v}\|_{W_2^{l+1}(\Omega_b)}\\ &\le cT^{\frac{1}{2}-\frac{l}{2}}c(\varepsilon)\|\rho\|_{l,T}\|\boldsymbol{v}\|_{H_2^{l,\frac{l}{2}}(Q_T)}.\end{aligned}$$

Finally, for the norm $\mathring{W}_2^l(Q_T)$, we split the estimate according to formula (5.51). For the term $\rho^*_{-h,t}\Delta_{-h}\nabla\boldsymbol{v}$ notice that for any $t\le T$,

$$\sup_{\Omega_b}|\rho^*_{,t}(\cdot,t)|\le\sup_{\mathcal{G}}|\rho_{,t}(\cdot,t)|\le c\|\rho_{,t}(\cdot,t)\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\le c\|\rho_{,t}\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)},$$

with a constant independent of ε, since θ does not depend on t and is not greater than 1. Hence

$$\begin{aligned}\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\rho^*_{-h,t}\Delta_{-h}\nabla\boldsymbol{v}\|^2_{L^2(\Omega_b)}dt &\le\sup_{Q_T}|\rho^*_{,t}|^2\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|\Delta_{-h}\nabla\boldsymbol{v}\|^2_{L^2(\Omega_b)}dt\\ &\le cT\|\rho\|^2_{l,T}\int_0^T\frac{dh}{h^{1+2(\frac{l}{2}+\frac{1}{2})}}\int_h^T\|\Delta_{-h}\nabla\boldsymbol{v}\|^2_{L^2(\Omega_b)}dt\\ &\le cT\|\rho\|^2_{l,T}\|\nabla\boldsymbol{v}\|^2_{W_2^{0,\frac{l}{2}+\frac{1}{2}}(Q_T)}\le cT\|\rho\|^2_{l,T}\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}.\end{aligned}$$

Finally, proceeding as in (5.53),

$$\int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\nabla \boldsymbol{v} \Delta_{-h} \rho^*_{,t}\|^2_{L^2(\Omega_b)} dt$$
$$\le c \int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\nabla \boldsymbol{v}\|^2_{W_2^{2-l}(\Omega_b)} \|\Delta_{-h}\rho^*_{,t}\|^2_{W^{l-\frac{1}{2}}(\Omega_b)} dt$$
$$\le c \int_0^T \frac{dh}{h^l} \int_h^T \|\nabla \boldsymbol{v}(\cdot,t)\|^2_{W_2^{1+l}(\Omega_b)} dt \int_0^h \|\rho^*_{,tt}(\cdot, t-\xi)\|^2_{W^{l-\frac{1}{2}}(\Omega_b)} d\xi$$
$$\le cT^{1-l} c(\varepsilon) \|\rho_{,tt}\|^2_{W_2^{l-\frac{1}{2},0}(G_T)} \|\nabla \boldsymbol{v}\|^2_{W_2^{l+1,0}(Q_T)}$$
$$\le cT^{1-l} c(\varepsilon) \|\rho_{,t}\|^2_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \|\boldsymbol{v}\|^2_{W_2^{l+2,0}(Q_T)}.$$

We finally estimate the terms in l_2 and $\boldsymbol{G}$, since the others are unaffected by our choice of extension. As calculated in (3.22), l_2 only have "pure" terms of the form $\nabla\rho^*\nabla\boldsymbol{v}$, whose $W^{l+1,0}(Q_T)$ norm is estimated using (5.49)

$$\|\nabla\rho^*\nabla\boldsymbol{v}\|_{W_2^{l+1,0}(Q_T)} \le c \sup_{t<T} \|\nabla\rho^*\|_{W_2^{l+1}(\Omega_b)} \|\nabla\boldsymbol{v}\|_{W_2^{l+1,0}(Q_T)}$$
$$\le c\delta_1(1+\|\rho\|_{l,T}) \|\boldsymbol{v}\|_{H_2^{l+2,\frac{l}{2}+1}(Q_T)},$$

and similarly

$$\sup_{t<T} \|\nabla\rho^*\nabla\boldsymbol{v}\|_{W_2^l(\Omega_b)} \le c\delta_1(1+\|\rho\|_{l,T}) \|\boldsymbol{v}\|_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}.$$

The time derivative of $\boldsymbol{G}$, also given in (3.22), can be estimated as

$$\|\boldsymbol{G}_{,t}\|_{L^2(Q_T)} \le \sqrt{T} \sup_{t<T} \|\nabla\rho^*_{,t}\|_{L^2(\Omega_b)} \sup_{Q_T} |\boldsymbol{v}| + \sup_{Q_T} |\nabla\rho^*| \|\boldsymbol{v}_{,t}\|_{L^2(Q_T)}$$
$$\le c\sqrt{T} \sup_{t<T} \|\rho^*_{,t}\|_{W^{l+\frac{1}{2}}(\Omega_b)} \sup_{t<T} \|\boldsymbol{v}\|_{W_2^{l+1}(\Omega_b)} + \delta_1(1+\|\rho\|_{l,T}) \|\boldsymbol{v}_{,t}\|_{L^2(Q_T)},$$

and thus

$$\frac{1}{T^{\frac{l}{2}}} \|\boldsymbol{G}_{,t}\|_{L^2(Q_T)} \le c(\delta_1 + T^{\frac{1}{2}-\frac{l}{2}} c(\varepsilon))(1+\|\rho\|_{l,T}) \|\boldsymbol{v}\|_{H_2^{l+2,\frac{l}{2}+1}(Q_(T)}.$$

Finally, for the $W_2^{0,\frac{l}{2}}$ norm of $\boldsymbol{G}_{,t}$, which is of the type $\boldsymbol{v}_{,t}\nabla\rho^* + \boldsymbol{v}\nabla\rho^*_{,t}$, the term $\boldsymbol{v}_{,t}\nabla\rho^*$ is of the form $\nabla\rho^*\boldsymbol{m}$ with $\boldsymbol{m} \in W_2^{l,\frac{l}{2}}(Q_T)$, and can be estimated as in (5.52), (5.53). For the term $\boldsymbol{v}\nabla\rho_{,t}$, we split the $\mathring{W}_2^{0,\frac{l}{2}}(Q_T)$ according to

(5.51) and obtain

$$\begin{aligned}\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\boldsymbol{v}\Delta_{-h}\nabla\rho^*_{,t}\|^2_{L^2(\Omega_b)}dt &\le \sqrt{T}\sup_{Q_T}|\boldsymbol{v}|^2\int_0^T \frac{dh}{h^{\frac{3}{2}+l}}\int_h^T \|\Delta_{-h}\nabla\rho^*_{,t}\|^2_{L^2(\Omega_b)}dt\\ &\le c\sqrt{T}\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}\|\nabla\rho^*_{,t}\|^2_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(Q_T)}\\ &\le c\sqrt{T}c(\varepsilon)\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}\|\nabla\rho_{,t}\|^2_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(G_T)}\\ &\le c\sqrt{T}c(\varepsilon)\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}\|\rho\|^2_{l,T},\end{aligned}$$

while for the other term, proceeding as in (5.53), we have

$$\begin{aligned}&\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\nabla\rho^*_{,t}\Delta_{-h}\boldsymbol{v}\|^2_{L^2(\Omega_b)}dt\\ &\quad\le c\int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\nabla\rho^*_{,t}\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)}\|\Delta_{-h}\boldsymbol{v}\|^2_{W_2^l(\Omega_b)}dt\\ &\quad\le c\int_0^T \frac{dh}{h^{l}}\int_h^T \|\nabla\rho^*_{,t}(\cdot,t)\|^2_{W_2^{\frac{1}{2}+l}(\Omega_b)}\int_0^h \|\boldsymbol{v}_{,t}(\cdot,t-\xi)\|^2_{W_2^l(\Omega_b)}d\xi dt\\ &\quad\le cT^{1-l}\|\nabla\rho^*_{,t}\|^2_{W_2^{\frac{1}{2}+l,0}(Q_T)}\|\boldsymbol{v}_{,t}\|^2_{W_2^{l,0}(Q_T)}\\ &\quad\le c(\varepsilon)T^{1-l}\|\rho\|^2_{l,T}\|\boldsymbol{v}\|^2_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}.\end{aligned}$$

All in all we have obtained the following estimate:

$$\begin{aligned}&\|\boldsymbol{l}_0+\boldsymbol{l}_1\|_{W_2^{l,\frac{l}{2}}(Q_T)}+\|l_2\|_{W_2^{l+1}(Q_T)}+\sup_{t<T}\|l_2\|_{W_2^l(\Omega_b)}+\|\boldsymbol{G}\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)}\\ &\qquad\le c(\delta_1+c(\varepsilon)T^{\frac{1}{2}-\frac{l}{2}})(1+\|\rho\|_{l,T})(\|\boldsymbol{v}\|_{H_2^{l+2,\frac{l}{2}+1}(Q_T)}+\|\nabla p\|_{H_2^{l,\frac{l}{2}}(Q_T)}),\end{aligned}$$

and it is clear that the coefficient $\delta_1+c(\varepsilon)T^{\frac{l}{2}-\frac{1}{2}}$ can be made arbitrarily small for suitably small ε, δ and T. The same type of estimate holds true for the other nonlinear terms, and thus we have the following result.

Proposition 5.4.4 *Let $l\in(\frac{1}{2},1)$,*

$$\|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}+\|\widetilde{\boldsymbol{v}}_0-\boldsymbol{v}_b\|_{W_2^{l+\frac{1}{2}}(\Omega_b)}\le\delta, \tag{5.54}$$

and ρ^*_ε be the extension $\rho^*_\varepsilon=\theta\rho$ defined in lemma 5.4.2. For any $\delta_2>0$, the inequality

$$\begin{aligned}&\|\boldsymbol{l}_0+\boldsymbol{l}_1\|_{W_2^{l,\frac{l}{2}}(Q_T)}+\|l_2\|_{H_2^{l+1,0}(Q_T)}+\|\boldsymbol{G}\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)}+\|\boldsymbol{l}_3+l_4\boldsymbol{N}\|_{H_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)}\\ &+\|l_5\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)}\le\delta_2\sum_{k=1}^{3}\|(\boldsymbol{v},p,\rho)\|^k_{H,l,T},\end{aligned} \tag{5.55}$$

holds for any sufficiently small ε, $\delta(\varepsilon)$ and $T(\delta,\varepsilon)$.

We will also need a continuity estimate for the nonlinear terms, analogue to the one obtained in theorem 5.1.6.

Theorem 5.4.5 *Let $l \in (\frac{1}{2}, 1)$, and ρ_1 and ρ_2 satisfy*

$$\|\rho_1\|_{l,T} + \|\rho_2\|_{l,T} \le \mu, \qquad \rho(\cdot,0) = \rho'(\cdot,0) = \rho_0,$$

$$\|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} \le \delta \ll 1.$$

There exists $c(\mu)$, bounded for bounded μ, such that

$$\begin{aligned}
&\|\widetilde{\boldsymbol{l}}_0(\boldsymbol{u}_1,\rho_1) - \widetilde{\boldsymbol{l}}_0(\boldsymbol{u}_1,\rho_2)\|_{H_2^{l,\frac{l}{2}}(Q_T)} + \|\widetilde{\boldsymbol{l}}_1(\boldsymbol{u}_1,p_1,\rho_1) - \widetilde{\boldsymbol{l}}_1(\boldsymbol{u}_2,p_2,\rho_2)\|_{H_2^{l,\frac{l}{2}}(Q_T)}\\
&+ \|\widetilde{l}_2(\boldsymbol{u}_1,\rho_1) - \widetilde{l}_2(\boldsymbol{u}_2,\rho_2)\|_{H_2^{l+1,0}(Q_T)} + \|\boldsymbol{G}(\boldsymbol{u}_1,\rho_1) - \boldsymbol{G}(\boldsymbol{u}_2,\rho_2)\|_{\widehat{W}_2^{0,\frac{l}{2}+1}(Q_T)}\\
&+ \|\widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_1,\rho_1) - \widetilde{\boldsymbol{l}}_3(\boldsymbol{u}_2,\rho_2)\|_{H_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)} + \|\widetilde{l}_4(\boldsymbol{u}_1,\rho_1) - \widetilde{l}_4(\boldsymbol{u}_2,\rho_2)\|_{H_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)}\\
&+ \|\widetilde{l}_5(\boldsymbol{u}_1,\rho_1) - \widetilde{l}_5(\boldsymbol{u}_2,\rho_2)\|_{H_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_T)} \le\\
&c\delta_1(\|(\boldsymbol{u}_1,p_1,\rho_1)\|_{H,l,T} + \|(\boldsymbol{u}_2,p_2,\rho_2)\|_{H,l,T})\|(\boldsymbol{u}_1-\boldsymbol{u}_2,p_1-p_2,\rho_1-\rho_2)\|_{H,l,T},
\end{aligned}$$

for a constant δ_1 which is arbitrary small depending on ε, $\delta(\varepsilon)$ and $T(\varepsilon,\delta)$.

Proof. We only sketch the proof for the terms $\boldsymbol{l}_0$, $\boldsymbol{l}_1$, and $l_2 = \nabla\cdot\boldsymbol{G}$, since the other terms are treated as in [19]. Following the argument of theorem 5.1.6, it is enough to prove an estimate of the form

$$\begin{aligned}
&\|(f(\nabla\rho_1^*) - f(\nabla\rho_2^*))g\|_{H_2^{l,0}(Q_T)} \le c(\mu)\delta_1\|\rho_1-\rho_2\|_{l,T}\|g\|_{H_2^{l,0}(Q_T)}\\
&\|(f(\nabla\rho_1^*) - f(\nabla\rho_2^*))g\|_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)} \le c(\mu)\delta_1\|\rho_1-\rho_2\|_{l,T}\|g\|_{H_2^{l,\frac{l}{2}}(Q_T)},
\end{aligned} \tag{5.56}$$

with constants independent of T (notice that in l_2 and $\boldsymbol{G}$ there are no such nonlinear factors as $f(\nabla\rho^*)$). Notice first that from $\rho_1(\cdot,0) = \rho_2(\cdot,0)$ we get

$$\sup_{t<T}\|\rho_1-\rho_2\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} \le \int_0^T \|\rho_{1,t}-\rho_{2,t}\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}dt \le \sqrt{T}\|\rho_1-\rho_2\|_{l,T},$$

which implies by lemma 5.4.2

$$\sup_{t<T}\|\rho_1^*-\rho_2^*\|_{W_2^{l+2}(\mathcal{G})} \le c(\varepsilon + c(\varepsilon)\sqrt{T})\|\rho_1-\rho_2\|_{l,T}. \tag{5.57}$$

Setting $\delta_1 = c(\varepsilon + c(\varepsilon)\sqrt{T})$ (which is arbitrarily small depending on ε and T), we thus have

$$\begin{aligned}\sup_{Q_T}|f(\nabla\rho_1^*) - f(\nabla\rho_2^*)| &\le c_f(\mu)\sup_{Q_T}|\nabla(\rho_1^* - \rho_2^*)| \le c_f(\mu)\sup_{t<T}\|\rho_1^* - \rho_2^*\|_{W_2^{l+2}(\Omega_b)} \\ &\le c_f(\mu)\delta_1\|\rho_1 - \rho_2\|_{l,T}.\end{aligned} \tag{5.58}$$

To prove the first inequality in (5.56), we use proposition 2.1.3 and prove

$$\sup_{t<T}\|f(\nabla\rho_1^*) - f(\nabla\rho_2^*)\|_{W_2^{l+1}(\Omega_b)} \le c_f(\mu)\delta_1\|\rho_1 - \rho_2\|_{l,T}. \tag{5.59}$$

For any $t \le T$, inequality (5.57) takes care of the $L^2(\Omega_b)$ term in the W_2^{l+1} norm. From

$$\begin{aligned}&\|f'(\nabla\rho_1^*)D^2\rho_1^* - f'(\nabla\rho_2^*)D^2\rho_2^*\|_{H_2^l(\Omega_b)} \\ &\qquad \le \|f'(\nabla\rho_1^*)D^2(\rho_1^* - \rho_2^*)\|_{W_2^l(\Omega_b)} + \|(f'(\nabla\rho_2^*) - f'(\nabla\rho_1^*))D^2\rho_2^*\|_{H_2^l(\Omega_b)},\end{aligned}$$

we proceed splitting the estimate on the two terms. For the first one, as already noted, $\|f'(\nabla\rho^*)\|_{W^{l+1}(\Omega_b)}$ is bounded by $c(\mu)$, and thus proposition 2.1.3 and (5.57) gives the desired inequality. In the same way we can estimate the L^2 part of the norm of the second term, and thus it only remains its $\mathring{W}_l^W 2(\Omega_b)$ norm. Splitting the Δ_z operator as in (5.51) and using (5.58) allows to estimate just one term:

$$\begin{aligned}&\int_{|z|\le 1}\|D^2\rho_2^*\Delta_z(f'(\nabla\rho_2^*) - f'(\nabla\rho_1^*))\|^2_{L^2(\Omega_b)}\frac{dz}{|z|^{3+2l}} \\ &\qquad \le \|D^2\rho_2^*\|^2_{W_2^l(\Omega_b)}\int_{|z|\le 1}\|\Delta_z(f'(\nabla\rho_2^*) - f'(\nabla\rho_1^*))\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)}\frac{dz}{|z|^{3+2l}} \\ &\qquad \le c_f(\mu)\int_{|z|\le 1}\|\Delta_z(\nabla\rho_1^* - \nabla\rho_2^*)\|^2_{W_2^1(\Omega_b)}\frac{dz}{|z|^{3+2l}} \\ &\qquad \le c_f(\mu)(\|\nabla(\rho_1^* - \rho_2^*)\|^2_{W_2^l(\Omega_b)} + \|D^2(\rho_1^* - \rho_2^*)\|^2_{W_2^l(\Omega_b)}) \\ &\qquad \le c_f(\mu)\delta_1^2\|\rho_1 - \rho_2\|^2_{l,T},\end{aligned}$$

by (5.57), which concludes the proof of (5.59) and thus of first inequality in (5.56). For the second one, the L^2 term is again estimated through (5.58), and for the $\mathring{W}_2^{\frac{l}{2}}$ norm we split the finite difference operator as in (5.51) and proceed as in (5.52):

$$\int_0^T\frac{dh}{h^{1+l}}\int_h^T\|f(\nabla\rho_1^*) - f(\nabla\rho_2^*)\Delta_{-h}g\|^2_{L^2(\Omega_b)}dt \le c\delta_1^2\|\rho_1 - \rho_2\|^2_{l,T}\|g\|^2_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)},$$

by (5.58). Moreover, as in (5.53),

$$\int_0^T \frac{dh}{h^{1+l}} \int_h^T \|g\Delta_{-h} f(\nabla\rho_1^*) - f(\nabla\rho_2^*)\|^2_{L^2(\Omega_b)} dt \\ \leq cT^{1-l} \|f'(\nabla\rho_1^*)\nabla\rho_{1,t}^* - f'(\nabla\rho_2^*)\nabla\rho_{2,t}^*\|^2_{W_2^{\frac{3}{2}-l,0}(Q_T)} \|g\|^2_{W_2^{l,0}(Q_T)},$$

and, using $\frac{3}{2} - l < 1 < l + \frac{1}{2}$ and (5.59),

$$\|f'(\nabla\rho_1^*)\nabla\rho_{1,t}^* - f'(\nabla\rho_2^*)\nabla\rho_{2,t}^*\|_{W_2^{\frac{3}{2}-l,0}(Q_T)} \\ \leq \|(f'(\nabla\rho_1^*) - f'(\nabla\rho_2^*))\nabla\rho_{1,t}^*\|_{W_2^{l+\frac{1}{2},0}(Q_T)} + \|f'(\nabla\rho_2^*)(\nabla\rho_{1,t}^* - \nabla\rho_{2,t}^*)\|_{W_2^{1,0}(Q_T)} \\ \leq c_f(\mu)\delta_1\|\rho_1 - \rho_2\|_{l,T}\|\nabla\rho_{1,t}^*\|_{W_2^{l+\frac{1}{2},0}(Q_T)} + c_f(\mu)\|\nabla\rho_{1,t}^* - \nabla\rho_{2,t}^*\|_{W_2^{l+\frac{1}{2},0}(Q_T)} \\ \leq c_f(\mu)c(\varepsilon)\|\rho_1 - \rho_2\|_{l,T}.$$

All in all we obtained

$$\|(f(\nabla\rho_1^*) - f(\nabla\rho_2^*))g\|_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)} \\ \leq c_f(\mu)(\delta_1\|g\|_{\widehat{W}_2^{0,\frac{l}{2}}(Q_T)} + c(\varepsilon)T^{\frac{1-l}{2}}\|g\|_{W_2^{l,0}(Q_T)})\|\rho_1 - \rho_2\|_{l,T},$$

which gives the claim for sufficiently small T.

□

We can now prove the main result on the existence for small time of solutions to (1.2).

Theorem 5.4.6 *Let $l \in (\frac{1}{2}, 1)$. For any $\phi_0 \in W_2^{l+2}(\Sigma)$ and $\boldsymbol{v}_0 \in W_2^{l+1}(\Omega_0)$ such that $\nabla \cdot \boldsymbol{v}_0 = 0$ in Ω_0 , there exists a smooth ϕ_b, defining $\Omega_b = \{(x', x_3) : x' \in \Sigma, 0 < x_3 < \phi_b(x_3)\}$ and $\mathcal{G} = \{(x', x_3) : x' \in \Sigma, x_3 = \phi_b(x')\}$, such that for sufficiently small T, there exists a unique Σ-periodic solution in $[0, T)$ of problem (1.2), and it satisfies the inequality*

$$\|(\boldsymbol{v}, p, \phi - \phi_b)\|_{H,l,T} \leq c(\|\phi_0 - \phi_b\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_0)}).$$

Proof. For any $\delta \ll 1$ and $\varepsilon \ll 1$ to be chosen later, we choose a smooth ϕ_b in such a way that (5.54) holds, and perform the Hanzawa transformation in Ω_b so that the extension ρ^* is defined through lemma 5.4.2. For any δ_2 we chose $\overline{\delta}$, $\overline{\varepsilon}$ and $\overline{T}$ so that (5.55) holds for any $\delta < \overline{\delta}$, $\varepsilon < \overline{\varepsilon}$ and $T \leq T_0$. We set $\widetilde{\boldsymbol{v}}_0(x) = \boldsymbol{v}_0(e_{\rho_0(x)})$ as in (3.1), and choose $\boldsymbol{v}_b$ in such a way that (5.54) holds. It is easy to check that the compatibility conditions (5.39) are satisfied for $\rho_0 = \phi_0 - \phi_b$ and $\widetilde{\boldsymbol{v}}_0$.and define $\boldsymbol{v}^{(0)}$, $\rho^{(0)}$ as the extension of $\widetilde{\boldsymbol{v}}_0$ and ρ_0 to $t \geq 0$ such that, recalling (2.29)

$$\|(\boldsymbol{v}^{(0)}, 0, \rho^{(0)})\|_{H,l,T} \leq c\big(\|\boldsymbol{v}^{(0)}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_\infty)} + \|\rho^{(0)}\|_{W_2^{l+\frac{5}{2},\frac{l}{2}+\frac{5}{4}}(G_\infty)} + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2},\frac{l}{2}+\frac{3}{4}}(G_\infty)}\big) \leq c(\|\widetilde{\boldsymbol{v}}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}).$$

This can be done in a standard way for $\boldsymbol{v}^{(0)}$ and as in (4.87) for $\rho^{(0)}$. Notice that by lemma 5.4.2,

$$\|e_{\rho_0}\|_{W_2^{l+2}(\Omega_b)} \leq c\varepsilon\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + c(\varepsilon)\|\rho_0\|_{L^2(\mathcal{G})} \leq c(\varepsilon + c(\varepsilon)\delta)\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} \leq \|\rho_0\|_{W_2^{l+2}(\mathcal{G})},$$

for sufficiently small ε and $\delta(\varepsilon)$. Thus we can suppose

$$\|\widetilde{\boldsymbol{v}}_0\|_{W_2^{l+1}(\Omega_b)} \leq c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_0)},$$

independently of δ and ε, if those are sufficiently small. Then we iteratively define $(\boldsymbol{v}^{(n+1)}, p^{(n+1)}, \rho^{(n+1)})$ as the solution of (5.41) with right hand sides

$$\boldsymbol{f}_n = (\boldsymbol{l}_1 + \boldsymbol{l}_0)(\boldsymbol{v}^{(n)}, p^{(n)}, \rho^{(n)}), \quad h_n = l_2(\boldsymbol{v}^{(n)}, \rho^{(n)}),$$
$$\boldsymbol{d}_n = (\boldsymbol{l}_3 + l_4\boldsymbol{N})(\boldsymbol{v}^{(n)}, \rho^{(n)}), \quad g_n = l_5(\boldsymbol{v}^{(n)}, \rho^{(n)}),$$

and initial data $\widetilde{\boldsymbol{v}}_0$, ρ_0. Theorem 5.4.1 applied to the corresponding linear system, together with (5.55) give

$$\|(\boldsymbol{v}^{(n+1)}, p^{(n+1)}, \rho^{(n+1)})\|_{H,l,T} \leq c\delta_2 \sum_{k=1}^{3} \|(\boldsymbol{v}^{(n)}, p^{(n)}, \rho^{(n)})\|_{H,l,T}^k + c\big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \sigma\|H_b\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)}\big).$$

We let

$$c\big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \sigma\|H_b\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)}\big) \leq c\big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+2}(\Sigma)} + \sqrt{T}\|\phi_b\|_{W_2^{l+\frac{5}{2}}(\Sigma)} + \|\phi_b\|_{W_2^{l+\frac{3}{2}}(\Sigma)}\big) =: N_T,$$

and choose δ_2 so small that

$$c\delta_2 \sum_{k=1}^{3} (2N_T)^k \leq N_T,$$

and consequently small ε, δ and T such that (5.55) holds. Now

$$\|(\boldsymbol{v}^{(0)}, 0, \rho^{(0)})\|_{H,l,T} \leq 2N_T,$$

by construction, and by induction $\|(\boldsymbol{v}^{(n)}, p^{(n)}, \rho^{(n)})\|_{H,l,T} \leq 2N_T$ for all $n \geq 1$. To prove convergence, we proceeding as in the proof of theorem 5.2.1, defining

$$(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n) := (\boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)}, p^{(n+1)} - p^{(n)}, \rho^{(n+1)} - \rho^{(n)}).$$

By theorem 5.4.5, for sufficiently small ε, T and δ, it holds

$$\|(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)\|_{H,l,T} \leq \frac{1}{2}\|(\widehat{\boldsymbol{v}}_{n-1}, \widehat{p}_{n-1}, \widehat{\rho}_{n-1})\|_{H,l,T},$$

and thus strong convergence of $(\boldsymbol{v}^{(n)}, p^{(n)}, \rho^{(n)})$ to a solution $(\boldsymbol{v}, p, \rho)$ of (5.38), for which it holds the estimate

$$\|(\boldsymbol{v}, p, \rho)\|_{H,l,T} \leq 2N_T.$$

It is clear that, eventually decreasing T, ϕ_b can be chosen in such a way that

$$\sqrt{T}\|\phi_b\|_{W_2^{l+\frac{5}{2}}(\Sigma)} + \|\phi_b\|_{W_2^{l+\frac{3}{2}}(\Sigma)} \leq c\|\rho_0\|_{W_2^{l+2}(\Sigma)},$$

and this gives the claimed estimate.

We now prove that the global solution so obtained is unique. If $(\boldsymbol{v}', p', \rho')$ is another solution, then $(\widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\rho}) := (\boldsymbol{v} - \boldsymbol{v}', p - p', \rho - \rho')$ satisfies the linear problem (4.1) with right hand sides

$$\begin{aligned}
\widehat{f} &:= (\boldsymbol{l}_1 + \boldsymbol{l}_0)(\boldsymbol{v}, p, \rho) - (\boldsymbol{l}_1 + \boldsymbol{l}_0)(\boldsymbol{v}', p', \rho'), \\
\widehat{h} &:= l_2(\boldsymbol{v}, \rho) - l_2(\boldsymbol{v}', \rho'), \\
\widehat{\boldsymbol{d}} &:= \boldsymbol{l}_3(\boldsymbol{v}, \rho) - \boldsymbol{l}_3(\boldsymbol{v}', \rho') + \big(l_4(\boldsymbol{v}, \rho) - l_4(\boldsymbol{v}', \rho')\big)\boldsymbol{N}, \\
\widehat{g} &:= l_5(\boldsymbol{v}, \rho) - l_5(\boldsymbol{v}', \rho'),
\end{aligned}$$

and zero initial data. By the coercive estimate for the associated linear problem and theorem 5.4.5, for sufficiently small ε, δ and T it holds

$$\|(\widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\rho})\|_{H,l,T} \leq \frac{1}{2}\|(\widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\rho})\|_{H,l,T},$$

and thus the claimed uniqueness for sufficiently small time. □

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