The Bipolar Choquet Integral Representation

Fabio Rindone

Faculty of Economics

University of Catania, Italy

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Randomness in Economic Theory

Surprisingly, risk and uncertainty have a rather short history in economics. The formal incorporation of these concepts into economic theory was only accomplished when Von Neumann and Morgenstern (1944) published their *Theory of Games and Economic Behavior* - although the exceptional effort of Ramsey (1926) and Keynes (1921) must be mentioned as an antecedent. Indeed, the very idea that risk and uncertainty might be relevant for economic analysis was only really suggested by Frank Knight (1921) in his formidable treatise, *Risk, Uncertainty and Profit.* What makes this lateness even more surprising is that the very concept of marginal utility, the foundation stone of Neoclassical economics, was introduced by Bernoulli (1738)[1] in the context of choice under risk. Bernoulli's notion of expected utility which decomposed the valuation of a risky venture as the sum of utilities from outcomes weighted by the probabilities of outcomes, was generally not appealed to by economists. Part of the problem was that it did not seem sensible for rational agents to maximize expected utility and not some-

[1] reprinted in Bernoulli (1954)

thing else. Specifically, Bernoulli's assumption of diminishing marginal utility seemed to imply that, in a gamble, a gain would increase utility less than a decline would reduce it. Consequently, many concluded, the willingness to take on risk must be "irrational", and thus the issue of choice under risk or uncertainty was viewed suspiciously, or at least considered to be outside the realm of an economic theory which assumed rational actors. The great task of Von Neumann and Morgenstern (1944) was to lay a rational foundation for decision-making under risk according to expected utility rules. However the novelty of using the axiomatic method - combining sparse explanation with often obtuse axioms - ensured that most economists of the time would find their contribution inaccessible. Restatements and re-axiomatizations by Marschak (1950); Samuelson (1952); Herstein and Milnor (1953) did much to improve the situation. A second revolution occurred soon afterwards. The expected utility hypothesis was given a celebrated subjectivist twist by Savage (1954) in his classic *Foundations of Statistics.* Inspired by the work of Ramsey (1926) and De Finetti (1931, 1937), Savage derived the expected utility hypothesis without imposing objective probabilities but rather by allowing subjective probabilities to be determined jointly. Savage's brilliant performance was followed up by Anscombe and Aumann (1963). In some regards, the Savage-Anscome-Aumann subjective approach to expected utility has been considered more general than the older von Neumann-Morgenstern concept. Another "subjectivist" revolution was initiated with the "state-preference" approach to uncertainty of Arrow (1964) and Debreu (1959). Although not necessarily opposed to the expected utility hypothesis, the state-preference approach does not involve the assignment of mathematical probabilities, whether objective or subjective, although it often might be useful to do so. The structure of the state-preference approach is more amenable to Walrasian general equilibrium theory where *payoffs* are not merely money amounts but actual bundles of goods. In economic theory, utility is usually understood as a numerical representation of a *preference relation*; preferences are assumed to satisfy certain condition of internal consistency, *axioms*, which ensure that a utility representation exists for preferences and that choosing consistently with one's preferences can be represented as the maximization of utility. Expected utility theory imposes a particular set of consistency conditions, which imply that choice under uncertainty can be represented as the maximization of the mathematical expectation of the utility of consequences. Famous paradoxes such as those in Shackle (1952); Allais (1953); Ellsberg (1961), have been presented to critic this set of axioms underling the expected utility theory, see Camerer (1995) for an overview. On the other hand, influential experimental studies, such as those by Kahneman and Tversky (1979), have revealed a range of systematic patterns of behavior which appear to contravene expected utility theory. These motivations reinforced the need to rethink and to re-axiomatize much of the theory of choice under risk and uncertainty. In the last thirty years, an enormous amount of work has been done to develop new decision theories, the so called *non-expected utility theories*, which can accommodate at least some of the patterns of choice that contravene expected utility theory. Non expected utility theories impose different - usually weaker - consistency conditions with respect to those imposed by utility theory, which represent choice under uncertainty as the maximization of some other function rather than the mathematical expectation of the utility of consequences. Over these alternative theories, are worthy to be mentioned: weighted expected utility (e.g. Allais and Hagen (1979); Chew and MacCrimmon (1979); Fishburn (1983)), rank-dependent expected utility (Quiggin (1982); Yaari (1987)), non-linear expected utility (e.g. Machina (1982)), regret theory (Loomes and Sugden (1982), non-additive expected utility (Shackle (1949); Schmeidler (1989)) and state-dependent preferences (Karni (1985)). Prospect Theory (PT) of Kahneman and Tversky (1979) merits a special mention. Due to the great descriptive powerful of the theory, the two authors (psychologists) imposed initially prospect theory as a descriptive model, without an axiomatic foundation. The modern version Cumulative Prospect Theory (CPT), Tversky and Kahneman (1992), is nowadays considered one of the most suitable generalization of classical expected utility. With CPT the authors generalized PT, and summarized the most relevant ideas contained in the other non-expected utility theories. Since CPT is nowadays considered the reference model of choice under risk and uncertainty, most of empirical studies are designed to test it. In very recent years some critiques have been advanced against CPT, particularly regarding the *gain loss separability*, i.e. the fact that in the model gains and losses contained within a *lottery* are evaluated separately, and then summed to obtain an overall evaluation. These critiques have been formalized in many studies, the most relevant of which is surely Wu and Markle (2008). With this thesis we aim to generalize CPT, allowing gains and losses within a mixed prospect to be evaluated conjointly, in this bypassing the critique regarding the gain loss separability but retaining the main features of CPT. We call our model the bipolar Cumulative Prospect

Theory (bCPT), to underline that we retain it the most natural extension of CPT. At the end we wish to point out that this thesis does not provide a full preference foundation of bCPT, i.e. the model is not elicited from a set of axioms, but we use an utility-based method. We think most theorist of choice under uncertainty will acknowledge that, in their actual practice, they use axiomatic and utility-based methods in parallel. Some new theories have been developed out of the consideration of alternative axioms about preferences, but in many cases, theories were initially developed in the language of utility and the equivalent axiomatic form were discovered later. Expected utility theory itself provides an extreme example of this process: the utility-based form of this theory preceded its axiomatic form by about two hundred years. In recent years PT (utility-based) preceded CPT by thirteen years. It is not obvious, then, that the axiom-based versions of theories are more fundamental than the utility-based versions.

Chapter 1

Historical Background

1.1 The St. Petersburg paradox

During the development of probability theory in the 17th century, mathematicians such as Blaise Pascal and Pierre de Fermat assumed that the attractiveness of a gamble offering the payoffs $(x_1, \ldots, x_n)$ with probabilities $(p_1, \ldots, p_n)$ was given by its expected value $\sum_i x_i p_i$.

Nicholas Bernoulli proposed the following St. Petersburg game in 1713, which was resolved independently by his cousin Daniel Bernoulli (in 1738) and Gabriel Cramer (in 1728). Suppose someone offers to toss a fair coin repeatedly until it comes up heads. If the first head appears at the nth toss, the payoff is $\$2^{n-1}$. What is the largest sure gain you would be willing to forgot in order to undertake a single play of this game? Typically the gamble is represented as

$$\mathcal{G} = (\$2^0, 2^{-1}; \$2^1, 2^{-2}; \cdots; \$2^{n-1}, 2^{-n}; \cdots) \tag{1.1}$$

Since its expected value is $\sum_i 2^{-i}2^{i-1} = \sum_i 1/2 = \infty$, a person would be willing to pay any sum to play the game, yet, real-world people is willing to pay only a moderate amount of money. This is the so-called *St. Petersburg paradox.* Daniel Bernoulli's solution involved two ideas that have since revolutionized economics: firstly, that people's utility from wealth, $u(w)$, is not linearly related to wealth w but rather increases at a decreasing rate - the famous idea of *diminishing marginal utility*; secondly that a person's evaluation of a risky venture is not the expected return of that venture, but rather the expected utility from that venture.

In general, by Bernoulli's logic, the valuation of any risky venture takes the expected utility form:

$$E(u, p, X) = \sum_{x \in X} p(x)u(x)$$

where X is the set of possible outcomes, $p(x)$ is the probability of a particular outcome $x \in X$ and $u : X \rightarrow \Re$ is a utility function over outcomes.
In the St. Petersburg case, the sure gain λ which would yield the same utility as the St. Petersburg gamble, i.e., the certainty equivalent λ of this gamble, is determined by the following equation

$$u(\omega + \lambda) = \frac{1}{2}u(\omega + 1) + \frac{1}{4}u(\omega + 2) + \cdots$$

where ω is the person's initial wealth. For example, when $u(x) = ln(x)$ and $\omega = \$50000$, λ is about \$9 even thought the gamble has an infinite expected value. Finally we note that as Menger (1934) later pointed out, placing an ironical twist on all this, Bernoulli's hypothesis of diminishing marginal utility is actually not enough to solve all St. Petersburg-type Paradoxes. To see this, note that we can always find a sequence of payoffs $x_1, x_2, x_3, \cdots$ which yield infinite expected value, and then propose, say, that $u(x_n) = 2^n$ so that expected utility is also infinite. The Menger game is nowadays called the *super St. Petersburg paradox.* Thus, Menger proposed that utility must also be bounded above for paradoxes of this type to be resolved.

1.2 von Neumann and Morgenstern

While Bernoulli theory - the first statement of EUT - solved the St. Petersburg puzzle, it did not find much favor with modern economists until the 1950s. This is partly explained by the fact that, in the form presented by Bernoulli, the theory presupposes the existence of a cardinal utility scale; an assumption that did not sit well with the drive towards ordinalization during the first half of the twentieth century. Interest in the theory was revived when Von Neumann and Morgenstern (1944) showed that the expected utility hypothesis could be derived from a set of apparently appealing axioms on preference. Since then, numerous alternative axiomatizations have been developed, some of which seem highly appealing, some might even say compelling, from a normative point of view. To the extent that its axioms can be justified as sound principles of rational choice to which any reasonable person would subscribe, they provide grounds for interpreting EUT normatively (as a model of how people ought to choose) and prescriptively (as a practical aid to choice). Our concern, however, is with how people actually choose, whether or not such choices conform with a priori notions of rationality. Consequently, we will not be delayed by questions about whether particular axioms can or cannot be defended as sound principles of rational choice, and I will start from the presumption that evidence relating to actual behavior should not be discounted purely on the basis that it falls foul of conventional axioms of choice.

In the von Neumann-Morgenstern (vNM) hypothesis, probabilities are assumed to be "objective" or exogenously given by "Nature" and thus cannot be influenced by the agent. The problem of an agent under objective uncertainty (another name commonly used in the literature is "risk") is to choose among lotteries. vNM's original formulation involved decision trees in which compound lotteries were explicitly modeled. We use here the more compact formulation of Fishburn (1970), which implicitly assumes that compound lotteries are simplified according to Bayes's formula. Thus, lotteries are defined by their distributions, and the notion of "mixture" implicitly supposes that the decision maker is quite sophisticated in terms of her probability calculations. Let X be a set of alternatives. There is no additional structure imposed on it. X can be a familiar topological and linear space, but it can also be anything you wish. In particular, X need not be restricted to a space of product-bundles such as $\Re^l_+$ and it may include outcomes such as, God forbid, death. The objects of choice are lotteries with finite support. Formally, define

$$\mathcal{L} = \left\{ P : X \to [0,1] \mid \# \{x \mid P(x) > 0\} < \infty \;\wedge\; \sum_{x \in X} P(x) = 1 \right\}$$

Observe that the expression $\sum_{x \in X} P(x) = 1$ is well-defined thanks to the finite support condition that precedes it. A mixing operation is performed on $\mathcal{L}$, defined for every $P, Q \in \mathcal{L}$ and every $\alpha \in [0,1]$ as follows: $\alpha P + (1-\alpha)Q \in \mathcal{L}$ is given by

$$(\alpha P + (1-\alpha)Q)(x) = \alpha P(x) + (1-\alpha)Q(x)$$

for every $x \in X$. The intuition behind this operation is of conditional probabilities: assume that we offer you a compound lottery that will give you the lottery P with probability α and the lottery Q with probability $(1-\alpha)$. If you know probability theory, you can ask yourself what is the probability to obtain a certain outcome x, and observe that it is, indeed, α times the conditional probability of x if you get P plus $(1-\alpha)$ times the conditional probability of x is you get Q.

Since the objects of choice are lotteries, the observable choices are modeled by a binary relation, $\succsim$, on $\mathcal{L}$, i.e. $\succsim \subseteq \mathcal{L} \times \mathcal{L}$, where $P \succsim Q$ means that lottery P is considered at least as "good" as lottery Q and the strict preference $\succ$ and indifference $\sim$ are defined as usually, i.e. $\succ$ is the asymmetric part of $\succsim$ and $\sim$ is its symmetric part.

The vNM's axioms are:

V1. Weak order: $\succsim$ is complete and transitive.

V2. Continuity: For every $P, Q, R \in \mathcal{L}$ if $P \succ Q \succ R$, there exist $\alpha, \beta \in (0,1)$ such that $\alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R$.

V3. Independence: For every $P, Q, R \in \mathcal{L}$ and every $\alpha \in (0,1)$, $P \succsim Q$ iff $\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$.

The weak order axiom is not very different from the same assumption in consumer theory or in choice under certainty and allows lotteries to be ordered. Continuity may be viewed as a "technical" condition needed for the mathematical representation and for the proof to work. Indeed, one cannot design a real-life experiment in which it could be violated, since its violation would require infinitely many observations. But continuity can be hypothetically "tested"by some thought experiments (Gedankenexperiments). For instance, you can imagine very small, but positive probabilities, and try to speculate what your preferences would be between lotteries involving such probabilities. If you are willing to engage in such an exercise, consider the following example, supposedly challenging continuity: assume that P guarantees \$1, Q \$0, and R death. You are likely to prefer \$1 to nothing, that is, to exhibit preferences $P \succ Q \succ R$. The axiom then demands that, for a high enough $\alpha < 1$, you will also exhibit the preference

$$\alpha P + (1 - \alpha)R \succ Q,$$

namely, that you will be willing to risk your life with probability $(1 - \alpha)$ in order to gain \$1. The point of the example is that you are supposed to say that no matter how small is the probability of death $(1 - \alpha)$, you will not risk your life for a dollar. A counter-argument to this example (suggested by Raiffa) was that we often do indeed take such risks. For instance, suppose that you are about to buy a newspaper, which costs \$1. But you see that it is handed free on the other side of the street. Would you cross the street to get it for free? If you answer yes, you are willing to accept a certain risk, albeit very small, of losing your life in order to save one dollar. This counter-argument can be challenged in several ways. For instance, you may argue that even if you do not cross the street your life is not guaranteed with probability 1. Indeed, a truck driver who falls asleep may hit you anyway. In this case, we are not comparing death with probability 0 to death with probability $(1-\alpha$, and, the argument goes, it is possible that if you had true certainty on your side of the street, you would have not crossed the street, thereby violating the axiom. In any event, we understand the continuity axiom, and we are willing to accept it as a reasonable assumption for most applications.

The independence axiom is related to dynamic consistency. However, it involves several steps, each of which could be and indeed has been challenged in the literature (see Karni and Schmeidler (1991)). Consider the following four choice situations: 1. You are asked to make a choice between P and Q.
2. Nature will first decide whether, with probability $(1-\alpha)$, you get R, and then you have no choice to make. Alternatively, with probability α, nature will let you choose between P and Q.
3. The choices are as in (2), but you have to commit to making your choice before you observe Nature's move.
4. You have to choose between two branches. In one, Nature will first decide whether, with probability $(1-\alpha)$, you get R, or, with probability α, you get P. The second branch is identical, with Q replacing P.
Clearly, (4) is the choice between $\alpha P+(1-\alpha)R$ and $\alpha Q+(1-\alpha)R$. To relate the choice in (1) to that in (4), we can use (2) and (3) as intermediary steps, as follows. Compare (1) and (2). In (2), if you are called upon to act, you are choosing between P and Q. At that point R will be a counterfactual

world. Why would it be relevant? Hence, it is argued, you can ignore the possibility that did not happen, R, in your choice, and make your decision in (2) identical to that in (1). The distinction between (2) and (3) has to do only with the timing of your decision. Should you make different choices in these scenarios, you would not be dynamically consistent: it is as if you plan (in (3)) to make a given choice, but then, when you get the chance to make it, you do (or would like to do) something else (in (2)). Observe that when you make a choice in problem (3) you know that this choice is conditional on getting to the decision node. Hence, the additional information you have should not change this conditional choice. Finally, the alleged equivalence between (3) and (4) relies on changing the order of your move (to which you already committed) and Nature's move. As such, this is an axiom of reduction of compound lotteries, assuming that the order of the draws does not matter, as long as the distributions on outcomes, induced by your choices, are the same. Whether you find the independence axiom compelling or not, we suppose that its meaning if clear. We can finally state the theorem:

Theorem 1 (von Neumann-Morgenstern) *The preference relation, $\succsim$, satisfies axioms V1-V3 if and only if there exists $u : X \to \mathfrak{R}$ such that, for every $P, Q \in \mathcal{L}$*

$$P \succsim Q \qquad \textit{if and only if} \qquad \sum_{x \in X} P(x)u(x) \geq \sum_{x \in X} Q(x)u(x)$$

moreover, in this case u is unique up to a positive linear transformation.

1.3 Descriptive Limitations of Expected Utility Theory

Empirical studies dating from the early 1950s have revealed a variety of patterns in choice behavior that appear inconsistent with EUT. With hindsight, it seems that violations of EUT fall under two broad headings: those which have possible explanations in terms of some "conventional" theory of preferences and those which apparently do not. The former category consists primarily of a series of observed violations of the independence axiom of EUT; the latter of evidence that seems to challenge the assumption that choices derive from well-defined preferences. Let us begin with the former.

1.3.1 Allais paradoxes

There is now a large body of evidence indicating that actual choice behavior may *systematically* violate the independence axiom. Two examples of such phenomena, first discovered by Maurice Allais (1953), have played a particularly important role in stimulating and shaping theoretical developments in non-EU theory. These are the so-called *common consequence effects* and *common ratio effects*. The first sighting of such effect came in the form of the following pair of hypothetical choice problems. In the first you have to imagine choosing between the two prospects: $P = (\$1M,\ 1)$ or $Q = (\$5M,\ 0.1;\ \$1M,\ 0.89;\ 0,\ 0.01)$. The first option gives one million of dollars for sure; the second gives five million with a probability of 0.1, one million with a probability of 0.89, otherwise nothing. What would you choose? Now consider a second problem where you have to choose between the two prospects: $P' = (\$1M,\ 0.11;\ 0,\ 0.89)$ or $Q' = (\$5M,\ 0.1;\ 0,\ 0.9)$. What would you do if you really faced this choice? Allais believed that EUT was not an adequate characterization of individual risk preferences and he designed these problems as a counterexample. A person with expected utility preferences would either choose both "P" options, or choose both "Q" options across this pair of problems. In fact rewriting the prospect P and confronting it with Q

$$P = (\$1M,\ 0.11;\ \$1M,\ 0.89)$$

$$Q = (\$5M,\ 0.1;\ \$1M,\ 0.89;\ 0, 0.01).$$

the preference over P and Q should be independent from the common consequence $\$1M$ with probability 0.89, so that replacing it with 0 in both the prospects (i.e. P' and Q') should not reverse the preference. Allais expected that people faced with these choices might opt for P in the first problem, lured by the certainty of becoming a millionaire, and select Q' in the second choice where the odds of winning seem very similar, but the prizes very different. Evidence quickly emerged that many people did respond to these problems as Allais had predicted. This is the famous "Allais paradox" and it is one example of the more general *common consequence effect.* Allais was the first who discovered this phenomenon, however, numerous studies have found that choices between prospects with this basic structure are systematically influenced by the value of the common consequence. A closely related phenomenon, also discovered by Allais, is the so-called *common ratio effect.*

Suppose you had to make a choice between \$3000 for sure, or entering a gamble with an 80 percent chance of getting \$4000 (otherwise nothing). What would you choose? Now think about what you would do if you had to choose either a 25 percent chance of gaining \$3000 or a 20 percent chance of gaining \$4000. A good deal of evidence suggests that many people would opt for the certainty of \$3000 in the first choice and opt for the 20 percent chance of \$4000 in the second. Such a pattern of choice, however, is inconsistent with EUT and would constitute one example of the common ratio effect. More generally, this phenomenon is observed in choices among pairs of problems with the following form: $P'' = (y,\ p;\ 0,\ 1-p)$ and $Q'' = (x,\ \lambda p;\ 0,\ 1-\lambda p)$ where $x \succ y$ and $\lambda \in (0,1)$. Assume that the ratio of "winning"probabilities, λ, is constant, then for pairs of prospects of this structure, EUT implies that preferences should not depend on the value of p. In fact calculating the vNM's expectation of the two prospects $V(P'') = u(y)p$ and $V(Q'') = u(x)\lambda p$, that is $P'' \succ Q''$ iff $u(y)p > u(x)\lambda p$, i.e. iff $u(y)/u(x) > \lambda$, independly by p. Yet numerous studies (e.g. Loomes and Sugden (1987); Starmer and Sugden (1989)) reveal a tendency for individuals to switch their choice from P'' to Q'' as p falls.

1.3.2 The early evidence

It would, of course, be unrealistic to expect any theory of human behavior to predict accurately one hundred percent of the time. Perhaps the most one could reasonably expect is that departures from such a theory is equally probable in each direction. These phenomena, however, involve systematic (i.e., predictable) directions in majority choice. As evidence against the independence axiom accumulated, it seemed natural to wonder whether its assorted violations might be revealing some underlying feature of preferences that, if properly understood, could form the basis of a unified explanation. Consequently, a wave of theories designed to explain the evidence began to emerge at the end of the 1970s. Most of these theories have the following features in common: (i)preferences are represented by some function $V(\cdot)$ defined over individual prospects; (ii) the function satisfies ordering and continuity; and (iii) while $V(\cdot)$ is designed to permit observed violations of the independence axiom, the principle of monotonicity is retained. We will call theories with these properties *conventional theories.* The general spirit of the approach is to seek "well behaved" theories of preference consistent with observed violations of independence; this general approach can be called *the conventional strategy.* There is evidence to suggest that failures of EUT may run deeper than violations of independence. Two assumptions implicit in any conventional theory are: *procedure invariance*, i.e. preferences over prospects are independent of the method used to elicit them; and *description invariance*, i.e. preferences over prospects are purely a function of the probability distributions of consequences implied by prospects and do not depend on how those given distributions are described. While these assumptions probably seem natural to most economists, so natural that they are rarely even discussed when stating formal theories, there is ample evidence that, in practice, both assumptions fail. One well-known phenomenon, often interpreted as a failure of procedure invariance, is *preference reversal.* The classic preference reversal experiment requires individuals to carry out two distinct tasks (usually separated by some other intervening tasks). The first task requires the subject to choose between two prospects: one prospect (often called the $-bet) offers a small chance of winning a "good" prize; the other (the "P-bet") offers a larger chance of winning a smaller prize. The second task requires the subject to assign monetary value - usually minimum selling prices denoted $M(\$)$ and $M(P)$ - to the two prospects. Repeated studies (Tversky and Thaler (1990); Hausman (1992); Tammi (1997)) have revealed a tendency for individuals to chose the P-bet (i.e., reveal $P \succ \$$) while placing a higher value on the $-bet (i.e., $M(\$) > M(P)$). This is the so-called *preference reversal phenomenon* first observed by psychologists Lichtenstein and Slovic (1971); Lindman (1971). It presents a puzzle for economics because, viewed from the standard theoretical perspective, both tasks constitute ways of asking essentially the same question, that is, "which of these two prospects do you prefer?" In these experiments, however, the ordering revealed appears to depend upon the elicitation procedure. One explanation for preference reversal suggests that choice and valuation tasks may invoke different mental processes which in turn generate different orderings of a given pair of prospects (see Slovic *et al.* (1995)). Consequently, the rankings observed in choice and valuation tasks cannot be explained with reference to a single preference ordering. An alternative interpretation explains preference reversal as a failure of transitivity (see Loomes and Sugden (1983)): assuming that the valuation task reveals true monetary valuations, (i.e., $M(\$) \sim \$$; $M(P) \sim P$), preference reversal implies $P \succ \$ \sim M(\$) \succ M(P) \sim P$; which involves a violation of transitivity (assuming that more money is preferred to less). Although attempts have been made to explain the evidence in ways which preserve conventional assumptions - see for example Holt (1986); Karni and Safra (1987); Segal (1988) - the weight of evidence suggests that failures of transitivity and procedure invariance both contribute to the phenomenon (Loomes *et al.* (1989); Tversky *et al.* (1990)). There is also widespread evidence that very minor changes in the presentation or "framing" of prospects can have dramatic impacts upon the choices of decision makers: such effects are failures of description invariance. Here is one famous example due to Tversky and Kahneman (1981) in which two groups of subjects-call them groups I and II-were presented with the following cover story: *"Imagine that the U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimate of the consequences of the programs are as follows:..."* Each group then faced a choice between two policy options. Options presented to group I: *"If program A is adopted, 200 people will be saved. If program B is adopted, there is a 1/3 probability that 600 people will be saved, and a 2/3 probability that no people will be saved."* Options presented to group II: *"If program C is adopted, 400 people will die. If program D is adopted, there is a 1/3 probability that nobody will die, and a 2/3 probability that 600 people will die."*

The two pairs of options are stochastically equivalent. The only difference is that the group I description presents the information in terms of lives saved while the information presented to group II is in terms of lives lost. Tversky and Kahneman found a very striking difference in responses to these two presentations: 72 percent of subjects preferred option A to option B while only 22 percent of subjects preferred C to D. Similar patterns of response were found amongst groups of undergraduate students, university faculty, and practicing physicians. Failures of procedure invariance and description invariance appear, on the face of it, to challenge the very idea that choices can, in general, be represented by any well behaved preference function. If that is right, they lie outside the explanatory scope of the conventional strategy. Some might even be tempted to say they lie outside the scope of economic theory altogether. That stronger claim, however, is controversial, and we will not be content to put away such challenging evidence so swiftly. For present purposes, suffice it to make two observations. First, whether or not we have adequate economic theories of such phenomenon, the "Asian disease" example is clearly suggestive that framing effects have a bearing on issues of genuine economic relevance. Second, there are at least some theories of choice that predict phenomena like preference reversal and framing effects, and some of these models have been widely discussed in the economics literature. Although most of these theories draw on ideas about preference to explain choices, they do so in unorthodox ways, and many draw on concepts more familiar to psychologists than economists. The one feature common to this otherwise heterodox bunch of theories is that none of them can be reduced to, or expressed purely in terms of, a single preference function $V(\cdot)$ defined over individual prospects. We will call such models *non-conventional theories*. For further more details see the survey of Starmer (2000).

1.4 The Axioms for Subjective Probability

1.4.1 Introduction

As we have seen in the vNM's approach the probabilities are objective. First to describe the model of Subjective Expected Utility of Savage (1954) we need some preliminaries about the concept of *subjective probability.* The most influential contributes on this field are due to Ramsey (1926); De Finetti (1931, 1937); Savage (1954).

The axioms of subjective probability refer to assumed properties of a binary relation "is more probable than", on a set of propositions or events. This relation often referred to as a *qualitative or comparative probability relation*, can be taken either as an undefined primitive (intuitive views, Koopman and Good) or as a relation derived from a preference relation (decision-oriented approach, Ramsey, de Finetti and Savage).

Definition 1 *Let S be a non-empty set. A Boolean algebra $\mathcal{A}$ for S is a non-empty collection of subsets of S such that it is closed under complementation and finite unions.*

A probability measure μ on $\mathcal{A}$ satisfies:

(a) $\mu(X) \geq 0$ *for all* $X \in \mathcal{A}$*;*

(b) $\mu(X \cup Y) = \mu(X) + \mu(Y)$ *whenever X and Y are disjoint elements in $\mathcal{A}$;*

(c) $\mu_(S) = 1$.

Definition 2 *A $\sigma-$Algebra $\mathcal{A}$ for S is a Boolean Algebra which satisfies*

$$X_i \in \mathcal{A} \text{ for } i = 1, 2, \cdots \Rightarrow \bigcup_{i=1}^{\infty} X_i \in \mathcal{A}.$$

Definition 3 *A probability measure μ on a $\sigma-$Algebra $\mathcal{A}$ is countable additive (or σ-additive) if*

$$\mu\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mu(X_i)$$

whenever $X_i \in \mathcal{A}$ for $i = 1, 2, \cdots$ and $X_i \cap X_j = \varnothing$ for $i \neq j$.

1.4.2 De Finetti's axioms

Let S be the set of *states* and $\mathcal{A}$ be an *algebra* of subsets of S.. We refer to each $A \in \mathcal{A}$ as an *event*. We take the binary relation $\succsim$ on $\mathcal{A}$ as basic. Read $A \succsim B$ as "event A is at least as probable as event B.". As usually the asymmetric part of this relation is denoted with $\succ$ and the symmetric part with $\sim$; so we will read $A \succ B$ as "event A is more probable than event B" and $A \sim B$ as "event A and B are equiprobable".

Definition 4 *A probability measure p on $\mathcal{A}$ agrees with $\succsim$ if for all $A, B \in \mathcal{A}$*

$$A \succsim B \text{ iff } p(A) \geq p(B) \tag{1.2}$$

Savage (1954) defines $\succsim$ as a qualitative probability when it satisfies the following axioms proposed by de Finetti, which are clearly necessary condition on $\succsim$ for there to be a representation by a probability measure as in (1.2).

- (S1) weak order: $\succsim$ on $\mathcal{A}$ is a weak order, i.e it is complete and transitive;
- (S2) non-triviality: $S \succ \varnothing$;
- (S3) non-negativity: $A \succsim \varnothing$ for all $A \in \mathcal{A}$;

- (S4) additivity or independence: If $A - C = \varnothing = B - C$, then $A \succ B$ iff $A \cup C \succ B \cup C$;

The question of whether de Finetti's axioms, (S1)-(S4) are sufficient for agreement remained open until it was settled in the negative by Kraft *et al.* (1959). In the following we assume that $\mathcal{A}$ is finite. Without loss of generality, let $\mathcal{A}$ be the family of all subsets of $S = \{1, 2, \cdots, n\}$, that is $\mathcal{A} = 2^S$. For convenience, let $p_i = p(i), i = 1, \cdots, n$. Then p agrees with $\succsim$, if for all $A, B \in \mathcal{A}$:

$$A \succsim B \;\Leftrightarrow\; \sum_{i \in A} p_i \geq \sum_{i \in B} p_i \tag{1.3}$$

Example (Kraft et al. *(1959)).* Let $n = 5$, with the following comparisons.
$\{1,3,4\} \succ \{2,5\} \succ \{1,2,4\} \succ \{1,5\} \succ \{3,4\} \succ \{2,4\} \succ \{1,2,3\} \succ \{5\} \succ \{2,3\} \succ \{1,4\} \succ \{4\} \succ \{1,3\} \succ \{1,2\} \succ \{3\} \succ \{2\} \succ \{1\} \succ \varnothing$.
The rest of $\succ$ is given by complementarity ($A \succ B \;\Leftrightarrow\; B^c \; A^c$) and additivity, hence

$$\{4\} \succ \{1,3\},$$

$$\{2,3\} \succ \{1,4\},$$

$$\{1,5\} \succ \{3,4\},$$

$$\{1,3,4\} \succ \{2,5\}.$$

If (1.3) holds, then

$$p_4 > p_1 + p_3,$$

$$p_2 + p_3 > p_1 + p_4,$$

$$p_1 + p_5 > p_3 + p_4,$$

$$p_1 + p_3 + p_4 > p_2 + p_5.$$

Addition and cancellation leave us $0 > 0$, a contradiction.

Let $(A_1, \ldots, A_m) =_0 (B_1, \ldots, B_m)$ mean that the A_j and B_j are in $\mathcal{A}$ and, for each $i = 1, \ldots, n$ the number of A_j that contain i equals the number of B_j that contain i. In other words, the sums of the indicator functions over the two event sequences are identical. Clearly, for any real numbers $p_1 \ldots p_n$, we have

$$(A_1, \ldots, A_m) =_0 (B_1, \ldots, B_m) \Rightarrow \sum_{j=1}^{m} \sum_{i \in A_j} p_i = \sum_{j=1}^{m} \sum_{i \in B_j} p_i$$

Definition 5 *We say that* $\succsim$ *on* $\mathcal{A}$ *is strongly additive if, for all* $m \geq 2$ *and all* A_j *and* B_j*,*

$$\{(A_1, \ldots, A_m) =_0 (B_1, \ldots, B_m),\ A_j \succsim B_j,\ j < m\} \to not(A_m \succ B_m).$$

Remark 1 *Strong additivity implies weak order as well as additivity.*

Strong additivity is actually much stronger. Namely, $\succsim$ on $\mathcal{A}$ is strongly additive if and only if there are real numbers $p_1, \ldots, p_n$ that satisfy (1.3). In the special case of subjective probability with $p_i \geq 0$ and $\sum_i p_i = 1$, it is enough to assume that $\succsim$ is non-trivial and non-negative as well as strongly additive. This fact is stated by the following theorem which solves the question for finite agreement.

Theorem 2 *Suppose that S is non-empty and finite with $\mathcal{A} = 2^S$. Then there is a probability measure p on $\mathcal{A}$ for which for all $A, B \in \mathcal{A}$:*

$$A \succsim B \iff \sum_{i \in A} p_i \geq \sum_{i \in B} p_i$$

if and only if (S2) and (S3) hold for $\succsim$ along with strong additivity.

For infinite Algebra we need to add Savage's Archimedean axiom.

- (S5) $A \succ B$ implies that there is a finite partition $\{C1, \ldots, C_m\}$ of S such that $A \succ (B \cup C_i)$ for $i = 1, \ldots, m$

Theorem 3 *Suppose that $A = 2^S$ and $\succsim$ on $\mathcal{A}$ satisfies (S1) - (S5). Then there is a unique probability measure $p : \mathcal{A} \to [0, 1]$ such that*
(a) for all $A, B \in \mathcal{A}$: $A \succsim B$ iff $p(A) \geq p(B)$
(b) for every $A \in \mathcal{A}$ with $p(A) > 0$ and every $0 < a < 1$, there is a $B \subset A$ for which $p(B) = ap(A)$.

1.5 Savage's Theorem

1.5.1 States, outcomes, and acts

The model of Savage (1954) includes two primitive concepts: *states and outcomes.* The set of states, S, should be thought as an exhaustive list of all scenarios that might unfold. A state, in Savage's words, "resolves all uncertainty": it should specify the answer to any question you might be interested in. The answer should be deterministic. If, for instance, in a given state an act leads to a toss of a coin, you should further split the state to two possible states, each consistent with the original one, but also resolving the additional uncertainty about the toss of the coin. The following chapter elaborates on this notion. Observe that Savage considers a one-shot decision problem. If the real problem extends over many period, the decision problem considered should be thought of as a choice of a strategy in a game. The game can be long or even infinite. You think of yourself as choosing a strategy a-priori, and assuming that you will stick to it with no difficulties of dynamic consistency, unforeseen contingencies, and so forth. This is symmetric to the conception of a state as Nature's strategy in this game. It specifies all the choices that Nature might have to make as the game unfolds. An event is any subset $A \subseteq S$. There are no measurability constraints, and S is not endowed with an algebra of measurable events. If you wish to be more formal about it, you can define the set of event to be maximal $\sigma-$algebra $\Sigma = 2^S$, with respect to which all subsets are measurable. The set of outcomes will be denoted by X. An outcome x is assumed to specify all that is relevant to your well-being, insomuch as it may be relevant to your decision. In this sense, Savage's model does not differ from utility maximization under certainty (as in consumer theory) or from vNM's model. In all of these we may obtain rather counter-intuitive results if certain determinants of utility are left outside of the description of the outcomes. The objects of choice are *acts*, which are defined as functions from states to outcomes, and denoted by F. That is,

$$F = X^S = \{f | f : S \to X\} .$$

Choosing an act f, you typically do not know the outcome you will experience. But if you do know both your choice f and the state s, you know that the outcome will be $x = f(s)$. The reason is that a state s should resolve all uncertainty, including what is the outcome of the act f. Acts that do not depend on the state of the world s are constant functions in F. We will abuse notation and denote them by the outcome they result in. Thus, $x \in X$ is also understood as $x \in F$ with $x(s) = x$. There are many confusing things in the world, but this is not one of them. Since the objects of choice are acts, Savage assumes a binary relation $\succsim \subseteq F \times F$. The relation will have its symmetric and asymmetric parts, $\sim$ and $\succ$ defined as usual. It will also be extended to X with the natural convention. Specifically, for two outcomes $x, y \in X$, we say that $x \succsim y$ if and only if the constant function that yields always x is related by $\succsim$ to the constant function that yields always y. Before we go on, it is worthwhile to note what does not appear in the model. If you're taking notes, and you know that you're going to see a theorem resulting in integrals of real-valued functions with respective to probability measures, you might be prepared to leave some space for the mathematical apparatus. You may be ready now for a page of some measure theory, describing the σ-algebra of events. You can leave half a page blank for the details of the linear structure on X, and maybe a few lines for the topology on X. Or maybe a page or so to discuss the topology on F. But none of it is needed. Savage does not assume any such linear, measure-theoretic, or topological structures. If you go back to the beginning of this sub-section, you will find that it says only, There are two sets, S and X, and a relation on S to X^S.

1.5.2 Axioms

The axioms are given here in their original names, P1-P7. They do have nicknames, but these are sometimes subject to debate and open to different interpretations. Before to list the axioms, it will be useful to have a bit more notation. As mentioned above, the objects of choice are simply functions from S to X. What operations can we perform on $F = X^S$ if we have no additional structure on X? The operation we used in the statement of *(P2)* involves "splicing" functions, that is, taking two functions and generating a third one from them, by using one function on a certain sub-domain, and the other on the complement. Formally, for two acts $f, g \in F$ and an event $A \subseteq S$, define an act f_A^g by

$$f_A^g(s) = \begin{cases} g(s) & s \in A \\ f(s) & s \in A^c \end{cases},$$

that is, f_A^g is "f, where on A we replaced it by g.

It is also useful to have a definition that captures the intuitive notion that an event is considered a practical impossibility, roughly, what we mean by a zero-probability event when a probability is given. One natural definition is to say that

Definition 6 *an event A is null if, for every $f, g \in F$, with $f(s) = g(s)$ for all $s \in A^c$, then $f \sim g$.*

That is, if you know that f and g yield the same outcomes if A does not occur, you consider them equivalent. Now we can enunciate the famous seven Savage's axioms.

P 1 (Ordering) *$\succsim$ on F is a weak order, i.e. it is complete and transitive.*

P 2 (Sure-Thing principle) *For every $f, g, h, h' \in F$, and every $A \subset S$,*

$$f^h_{A^c} \succsim g^h_{A^c} \quad \Leftrightarrow \quad f^{h'}_{A^c} \succsim g^{h'}_{A^c}$$

P 3 (Eventwise Monotonicity) *For every $f \in F$, non-null event $A \subset S$ and $x, y \in X$,*

$$x \succsim y \;\Leftrightarrow\; f^x_A \succsim f^y_A$$

P 4 (Comparative Probability) *For every $A, B \subseteq S$ and for every $x, y,$ $z, w \in X$ with $x \succ y$ and $z \succ w$,*

$$y^x_A \succsim y^x_B \;\Leftrightarrow\; w^z_A \succsim w^z_B$$

P 5 (Non-triviality) *There are $f, g \in F$ such that $f \succ g$*

P 6 (Small Event Continuity) *For every $f, g, h \in F$ with $f \succ G$, there exists a partition of S, $\{A_1, A_2 \ldots, A_n\}$ such that for every $i \leq n$,*

$$f^h_{A_i} \succ g \text{ and } f \succ g^h_{A_i}$$

P 7 (Dominance) *Consider acts $f, g \in F$ and an event $A \subseteq S$. If it is the case that, for every $s \in A$, $f \succsim_A g(s)$, then $f \succsim_A g$, and if, for every $s \in A$, $g(s) \succsim_A f$, then $g \succsim_A f$*

In the following we discuss the meaning and some common interpretations of the seven axioms of Savage.

(P1)(Ordering). The basic idea is very familiar, as are the descriptive and normative justifications. At the same time, completeness is a demanding axiom. Observe that all functions in F are assumed to be comparable. Implicitly, this suggests that choices between every pair of such functions can indeed be observed, or at least meaningfully imagined.

(P2)(Sure-Thing Principle). This axiom says that the preference between two acts, f and g, should only depend on the values of f and g when they differ. Assume that f and g differ only on an event A (or even a subset thereof). That is, if A does not occur, f and g result in the same outcomes exactly. Then, when comparing them, we can focus on this event, A, and ignore A^c. Observe that we do not need to know that f and g are constants outside of A. Thus, *(P2)* is akin to requiring that you will have conditional preferences, namely that you have well-defined preferences between f and g conditional on A occurring, and that these conditional preferences determine your choice between f and g if they are equal in case A does not occur. More technically, we can define the *conditional preference* $\succsim_A$ as follows

$$f \succsim_A g \;\Leftrightarrow\; f^h_{A^c} \succsim g^h_{A^c}$$

and we can state that under $(P1)$ and $(P2)$, $\succsim_A$ is a weak order.

(P3)(Eventwise Monotonicity). Axiom *(P3)* states, roughly, the following. If you take an act, which guarantees an outcome x on an event A, and you change it, on A, from x to another outcome y, the preference between the two acts should follow the preference between the two outcomes x and y (when understood as constant acts). There are two main interpretations

of *(P3)*. One, which appears less demanding, holds that *(P3)* is simply an axiom of monotonicity. The interpretation of *(P3)* as monotonicity is quite convincing when the outcomes are monetary payoffs. The other interpretation of *(P3)*, which highlights this issue is the following. The game we play is to try to derive utility and probability from observed behavior. That is, we can think of utility and probability as intentional concepts related to desires and wants on the one hand, and to knowledge and belief on the other. These concepts are measured by observed choices. In particular, if we wish to find out whether the decision maker prefers x to y, we can ask her whether she prefers to get x or y when an event A occurs, i.e., to compare f_A^x to f_A^y. If she says that she prefers f_A^x, we will conclude that she values x more than y.

(P4)(Comparative Probability). This axiom is the counterpart of *(P3)* under the second interpretation. Let us continue with the same line of reasoning. We wish to measure not only the ranking of outcomes, but also of events. Specifically, suppose that we wish to find out whether you think that event A is more likely than event B. Let we take two outcomes x, y such that $\succ y$. For example, x will denote \$100 and y \$0. We now intend to ask you whether you prefer to get the better outcome x if A occurs (and otherwise y), or to get the better outcome if B occurs (again, with y being the alternative). Suppose that you prefer the first, namely, $y_A^x \succsim y_B^x$, then it seems reasonable to conclude that A is more likely in your eyes than is B. Clearly axioms (P4) avoids contradictory conclusions on this event's ranking.

(P5)(Non-triviality). If *(P5)* does not hold, we get $f \sim g$ for every $f, g \in F$. This relation is representable by expected utility maximization: you can choose any probability measure and any constant utility function. Moreover,

the utility function will be unique up to a positive linear transformation, which boils down to an additive shift by a constant. But the probability measure will be very far from unique. But since a major goal of the axiomatization is to define *subjective probability*, and we want to pinpoint a unique probability measure, which will be "the subjective probability of the decision maker", *P5* appears as an explicit axiom. Not only in the mathematical sense, namely that *P5* follows from the representation, but in the sense that *P5* is necessary for the elicitation program to succeed. Someone who is incapable of expressing preferences cannot be ascribed subjective probabilities by the reverse engineering program of Ramsey-de Finetti-Savage.

(P6)(Small Event Continuity). This axiom has a flavor of continuity, but it also has an Archimedean twist. Let us assume that we start with strict preferences between two acts, $f \succ g$, and we wish to state some notion of continuity. We cannot require that, say $f' \succ g$ whenever $|f(s) - f'(s)| < \epsilon$, because we have no metric over X. We also cannot say that $f' \succ g$ whenever $P(\{s | f(s) \neq f'(s)\}) < \epsilon$, because we have no measure P on S. How can we say that f' is "close" to f? One attempt to state closeness in the absence of a measure P is the following. Assume that we had such a measure P and that we could split the state space into events $A_1, \ldots, A_n$ such that $P(A_i) < \epsilon$ for every $i \leq n$. Not every measure P allows such a partition, but assume we found one. Then we can say that, if f' and f differ only on one of the events $A_1, \ldots, A_n$, then f' is close enough to f and therefore $f' \succ g$. This last condition can be stated without reference to the measure P. And this is roughly what *(P6)* requires. Finally, observe that it combines two types of constraints: first, it has a flavor of continuity: changing f (or g) on a

"small enough" event does not change strict preferences. Second, it has an Archimedean ingredient, because the way to formalize the notion of a "small enough" event is captured by saying "any of finitely many events in a partition". *(P6)* thus requires that the entire state space not be too large; we have to be able to partition it into finitely many events, each of which is not too significant. The fact that we need infinitely many states, and that, moreover, the probability measure Savage derives as no atoms is certainly a constraint. The standard way to defend this requirement is to say that, given any state space S, we can always add to it another source of uncertainty, say, infinitely many tosses of a coin.

(P7)(Dominance). If there is a "technical" axiom, this is it. Formally, it is easy to state what it says. (P7) requires that if f is weakly preferred to any particular outcome that g may obtain, than f should be weakly preferred to g itself. But it is hard to explain what it does, or what it rules out. It is, in fact, very surprising that Savage needs it, especially if you were already told that he does not need it for the case of a finite X. But Savage does prove that the axiom is necessary. That is, he provides an example, in which axioms *(P1)-(P6)* hold, X is infinite, but there is no representation of $\succsim$ by an expected utility formula. Let us first assume that we only discuss $A = S$. If we did not have the axioms *(P1)-(P6)*, one can generate weird preferences that do not satisfy this condition. But we have *(P1)-(P6)*. Moreover, restricting attention to finitely many outcomes, we already hinted that there is a representation of preferences by an expected utility formula. How wild can preferences be, so as to violate (P7) nevertheless? We will regard *(P7)* as another type of continuity condition, one imposed on the outcome space.

1.5.3 The theorem

Let us remember that a probability measure, μ, such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$ is finite additive; ; μ is σ-additive if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ whenever $i \neq j \rightarrow A_i \cap A_j = \emptyset$. Having a finitely additive measure, σ-additivity is an additional constraint of continuity: define $B_n = \bigcup_i^n A_i$ and $B = \bigcup_i^{\infty} A_i$. Then $B_n \rightarrow B$ and σ-additivity means

$$\mu\left(\lim_{n\rightarrow\infty} B_n\right) = \mu\left(\bigcup_i^{\infty} A_i\right) = \sum_i^{\infty} \mu(A_i) = \lim_{n\rightarrow\infty} \mu(B_n)$$

that is, σ-additivity of μ is equivalent to say that the measure of the limit is the limit of the measure. As such, σ-additivity is a desirable constraint. Lebesgue, who pioneered measure theory, observed that the notion of length of intervals cannot be extended to all the subsets of the real line (or of $[0, 1]$) if σ-additivity is to be retained. We can relax this continuity assumption, but some nice theorems (e.g. Fubini's theorem) do not hold for finitely additive measures. De Finetti, Savage, and other probabilist in the 20th century had a penchant, or perhaps an ideological bias for finitely additive probability measures. If probability is to capture our subjective intuition, it does indeed seem much more natural to require finite additivity, rather than sophisticated mathematical condition such as σ-additivity. Given this background, it should come as no surprise to us that Savage's theorem yields a measure that is (only) finitely additive.

The discussion of (P6) above made references to "atoms" of measures. For a σ-additive measure an event A is an atom of μ if (i) $\mu(A) > 0$; (ii) for every event $B \subset A$, $\mu(B) = 0$ or $\mu(B) = \mu(A)$. That is, an atom cannot split, in terms of its probability, trying to split it to B and $A \setminus B$, we find either that all the probability is on B or on $A \setminus B$. A measure that has no atoms is called *non-atomic*. A more demanding definition of non-atomicity is: that for every event A with $\mu(A) > 0$, and for every $r \in [0,1]$ there be an event $B \subset A$ such that $\mu(B) = r\mu(A)$. In the case of a σ-additive μ, the two definitions coincide. But this is not true for finite additivity. Moreover, the condition that Savage needs, and the condition that turns out to follow from (P6), is the strongest, hence, we will adopt it as definition of non-atomicity. The Savage's theorem for a finite and for a general outcome set is

Theorem 4 (Savage) *Assume that X is finite. Then $\succsim$ satisfies (P1)-(P6) if and only if there exist a non-atomic finitely additive probability measure p on $(S, 2^S)$ and a non-constant function $u : X \to \mathfrak{R}$ such that, for every $f, g \in F$*

$$f \succsim g \quad \text{iff} \quad \int_S u(f(s))dp(s) \geq \int_S u(g(s))dp(s)$$

Furthermore, in this case p is unique, and u is unique up to positive linear transformations.

Theorem 5 (Savage) *$\succsim$ satisfies (P1)-(P7) if and only if there exist a non-atomic finitely additive probability measure p on $(S, 2^S)$ and a non-constant bounded function $u : X \to \mathfrak{R}$ such that, for every $f, g \in F$*

$$f \succsim g \quad \text{iff} \quad \int_S u(f(s))dp(s) \geq \int_S u(g(s))dp(s)$$

Furthermore, in this case p is unique, and u is unique up to positive linear transformations.

1.6 A critique of Savage

Savage's "technical" axioms, P6 and P7 have been discussed above. They are not presented as canons of rationality, but as mathematical conditions needed for the proof. We therefore do not discuss them any further. There is also little to add regarding axiom P5. It is worth emphasizing its role in the program of behavioral derivations of subjective probabilities, but it is hardly objectionable. By contrast, P1-P4 have been, and still are a subject of heated debates, based on their reasonableness from a conceptual viewpoint.

1.6.1 Critique of P3 and P4

The main difficulty with both P3 and P4 is that they assume a *separation of tastes from beliefs.* That is, they both rely on an implicit assumption that an outcome x is just as desirable, no matter at which state s it is experienced. The best way to see this is, perhaps, by a counterexample. The classical one, mentioned above, is considering a swimsuit, x, versus an umbrella, y. You will probably prefer y to x in the event A, in which it rains, but x to y in the event B, in which it does not rain. This is a violation of P3. Similarly, the same example can be used to construct a violation of P4.

1.6.2 Critique of P1 and P2

The basic problem with axioms P1 and P2, as well as with Savage's Theorem and with the entire Bayesian approach is the following: for many problems of interest, there is no sufficient information based on which one can define probabilities. Subjective probabilities are not a solution to the problem: subjectivity may save us needless arguments, but it does not give us a reason to choose one probability over another. This difficulty can manifest itself in violations of P1 and/or of P2. If one takes a strict view of rationality, and asks what preferences can be justified based on evidence and reasoning, one is likely to end up with incomplete preferences. One problem of the Bayesian approach is that it does not distinguish between choices that are justified by reasoning (and evidence) and choices that are not. David Schmeidler was bothered by these issues in the early '80s. He suggested the following mind experiment: you are asked to bet on a flip of a coin. You have a coin in your pocket, which you have tested often, and found to have a relative frequency of Head of about 50%. I also have a coin in my pocket, but you know nothing about my coin. If you wish to be Bayesian, you have to assign probabilities to each of these coins coming up Head. The coin for which relative frequencies are known should probably be assigned the probability .5. About the other coin your ignorance is symmetric. You have no reason to prefer one side to the other. So you assign the probability of .5, based on symmetry considerations alone. Now that probabilities were assigned, the two coins have the same probability. But Schmeidler intuition was that they feel very different. There is some sense that .5 assigned based on empirical frequencies is not the same as .5 that was assigned based on default. Schmeidler's work (i.e. Schmeidler (1986, 1989)) started with this intuition. As we will see below, this intuition has a behavioral manifestation in Ellsberg's paradox.

1.6.3 Ellsberg's urns

Ellsberg (1961) suggested two experiments. His original paper does not report laboratory experiment, only replies obtained from economists. The two-urn experiment is very similar to the two coin example.

Ellsberg's two-urn paradox. There are two urns, each containing 100 balls. Urn I contains 50 red balls and 50 black balls. Urn II contains 100 balls, each of which is known to be either red or black, but you have no information about how many of the balls are red and how many are black. A red bet is a bet that the ball drawn at random is red and a black bet is the bet that it is black. In either case, winning the bet, namely, guessing the color of the ball correctly, yields $100. First, you are asked, for each of the urns, if you prefer a red bet or a black bet. For each urn separately, most people say that they are indifferent between the red and the black bet. Then you are asked whether you prefer a red bet on urn I or a red bet on urn II. Many people say that they would strictly prefer to bet on urn I, the urn with known composition. The same pattern of preferences in exhibited for black bets (as, indeed, would follow from transitivity of preferences given that one is indifferent between betting on the two colors in each urn). That is, people seem to prefer betting on an outcome with a known probability of 50% than on an outcome whose probability can be anywhere between 0 and 100%. It is easy to see that the pattern of choices described above cannot be explained by expected utility maximization for any specification of subjective probabilities. Such probabilities would have to reflect the belief that it is more likely that a red ball will be drawn from urn I than from urn II, and that it is more likely that a black ball will be drawn from urn I than from urn II. This is impossible because in each urn the probabilities of the two colors have to add up to one. P2 is violated in this example, (to see this we should embed it in a state space). Thus, Ellsberg's findings suggest that many people are not subjective expected utility maximizers. Moreover, the assumption that comes under attack is not the expected utility hypothesis per se: any rule that employs probabilities in a reasonable way would also be at odds with Ellsberg's results. The questionable assumption here is the basic tenet of Bayesianism, namely, that all uncertainty can be quantified in a probabilistic way. Exhibiting preferences for known vs. unknown probabilities is incompatible with this tenet.

Ellsberg's single-urn paradox. The two urn example is intuitively very clear, but we need to work a bit to define the states of the world for it. By contrast, the single-urn example is more straightforward in terms of the analysis. This time there are 90 balls in an urn. We know that 30 balls are red, and that the other 60 balls are blue or yellow, but we do have any additional information about their distribution. There is going to be one draw from the urn. Assume first that you have to guess what color the ball will be. Do you prefer to bet on the color being red (with a known probability of 1/3) or being blue (with a probability that could be anything from 0 to 2/3)? The modal response here is to prefer betting on red, namely, to prefer the known probability over the unknown one. Next, with the same urn, assume that you have to bet on the ball not being red, that is being blue or yellow, versus not being blue, which means red or yellow. This time your chances are better. You know that the probability of the ball not being red is 2/3, and the probability of not-blue is anywhere from 1/3 to 1. Here, for similar reasons, the modal response if to prefer not red, again, where the probabilities are known. (Moreover, many participants simultaneously prefer red in the first choice situations and not-red in the second.) Writing down the *decision matrix* we obtain

	R	B	Y
red	1	0	0
$blue$	0	1	0
$not-red$	0	1	1
$not-blue$	1	0	1

With this simply notation we have indicate the sates with R, B and Y; the outcomes with 0 and 1; and the acts with red, blue, not-red and not-blue. It is readily seen that red and blue are equal on Y. If P2 holds, changing their value from 0 to 1 on Y should not change preferences between them. But when we make this change, red becomes not-blue and blue becomes not-red. That is, P2 implies

$$\textit{red} \gtrsim \textit{blue} \qquad \textit{iff} \qquad \textit{not-blue} \gtrsim \textit{not-red}$$

whereas model preferences are

$$\textit{red} \succ \textit{blue} \qquad \textit{iff} \qquad \textit{not-blue} \prec \textit{not-red}$$

1.7 Concluding remarks

The first attempts to develop a utility theory for choice situations under risk were undertaken by Cramer (1728) and Bernoulli (1738) to face the St. Petersburg paradox. To solve this “puzzle” Bernoulli (1738) proposed that the expected monetary value has to be replaced by the expected utility (“moral expectation”) as the relevant criterion for decision making under risk. Not until two centuries later, did Von Neumann and Morgenstern (1944) prove that if the preferences of the Decision Maker satisfy certain assumptions they can be represented by the expected value of a real-valued utility function defined on the set of consequences. Only a few years later, Savage (1954), building upon the works of Ramsey (1926); De Finetti (1931, 1937), developed a model of expected utility for choice situations under uncertainty. In Savage’s subjective approach, not only the utility function but also subjective probabilities have to be derived from the preferences of the Decision Maker. For the last fifty years the Expected Utility model has been the dominant framework for analyzing decision problems under risk and uncertainty. According to Machina (1982) this is due to “the simplicity and normative appeal of its axioms,the familiarity of the notions it employs (utility functions and mathematical expectation), the elegance of its characterizations of various types of behavior in terms of properties of the utility function...and the large number of results it has produced”. Since the well known paradoxes of Allais (1953); Ellsberg (1961), however, a large body of experimental evidence has been gathered which indicates that individuals tend to violate the assumptions underlying the expected Utility Model systematically. This empirical evidence has motivated researchers to develop alternative theories of choice under risk and uncertainty able to accommodate the observed patterns of behavior. These models are usually termed “non-expected utility” or “generalizations of expected utility”(for a review see Starmer (2000)). Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992), the moder version of Prospect Theory (PT) (Kahneman and Tversky (1979)) is surely considered one of the most valid non-expected utility theory, due to its great descriptive power. In the next chapters we will describe in detail CPT and, next, we will present a generalization of the model, called “bipolar CPT”, which is able to cover some situations that CPT cannot, for its nature, accommodate.

Chapter 2

Bipolar Cumulative Prospect Theory

2.1 Introduction

Since its publication, the Prospect Theory (PT) of Kahneman and Tversky (1979) have had an enormous impact on the decisions theory, and when the Cumulative Prospect Theory (CPT) (Tversky and Kahneman (1992)) appeared, this model has become the most widely used alternative to the classical Expected Utility Theory (EUT) of Von Neumann and Morgenstern (1944). This mainly depends on two factors of great importance. The first is that CPT has preserved the mathematical tractability and the descriptive power of the original PT, enlarging its applicability field to prospects with any number of outcomes (infinitely many or even continuous outcomes too) and to the uncertainty, whereas PT was thought only for risky prospects with only two outcomes. The second factor is that CPT has captured the fundamental idea of the Rank Dependent Utility (RDU) of Quiggin (1982) and of the Choquet Expected Utility (CEU) of Schmeidler (1986, 1989) and Gilboa (1987) solving some imperfections within the original model. As was pointed out from Fishburn (1978) and Kahneman and Tversky (1979) it was possible, under PT, to violate the axiom of first-order stochastic dominance; the authors thought to an *editing phase* in which the transparently dominated alternatives were eliminated but this generated non-transitivity problems. After to have extended his applicability from risk to uncertainty too and after to have built solid bridges with others valid models of *non-Expected Utility Theory*, the CPT needed only an axiomatic validation (i.e. a preference foundation), to be fully accepted also from the normative point of view, whereas the success from the descriptive point of view was incontestable. Several axiomatic basis for CPT were presented and, among them, the most well known are those of Wakker and Tversky (1993), Chateauneuf and Wakker (1999), Wakker and Zank (2002). In recent years CPT has obtained increasing space in applications in several fields: in business, finance, law, medicine, and political science (e.g., Benartzi and Thaler (1995); Barberis *et al.* (2001); Camerer (2000); Jolls *et al.* (1998); McNeil *et al.* (1982); Quattrone and Tversky (1988)) Despite the increasing interest in CPT-in the theory and in the practice-some critiques have been recently proposed: Levy and Levy (2002); Blavatskyy (2005); Birnbaum (2005); Baltussen *et al.* (2006); Birnbaum and Bahra (2007); Wu and Markle (2008). As well as the paradoxes of Allais (1953) and Ellsberg (1961) showed that for the EUT of Von Neumann and Morgenstern (1944) and for the Subjective Expected Utility Theory (SEUT) of Savage (1954) the weak point was the preference independence axiom, so, we believe that the relevant critic against CPT is the Gain-Loss Separability (GLS), i.e. the separate evaluation of losses and gains. More precisely, let $P = (x_1, p_1; ...x_n, p_n)$ be a *prospect* giving the outcome x_i with probability p_i, $i = 1, \ldots, n$. We indicate with P^+ the "positive part" of P, i.e. the prospect obtained from P by substituting all the losses with zero and we indicate with P^- the "negative part" of P, i.e. the prospect obtained from P substituting all the gains with zero. GLS means that the evaluation of P is obtained as sum of the value of P^+ and P^- :

$$V(P) = V(P^+) + V(P^-).$$

Wu and Markle (2008) refer to the following experiment: 81 participants gave their preferences as it is shown below (read $\mathcal{H} \succ \mathcal{L}$ "the prospect $\mathcal{H}$ is preferred to the prospect $\mathcal{L}$")

$$\mathcal{H} = \begin{pmatrix} 0.50\ \textit{chance} \\ \textit{at}\ \$4,200 \\ \\ 0.50\ \textit{chance} \\ \textit{at}\ \$-3,000 \end{pmatrix} \succ \begin{pmatrix} 0.75\ \textit{chance} \\ \textit{at}\ \$3,000 \\ \\ 0.25\ \textit{chance} \\ \textit{at}\ \$-4,500 \end{pmatrix} = \mathcal{L}$$

[52%] [48%]

$$\mathcal{H}^+ = \begin{pmatrix} 0.50\ \textit{chance} \\ \textit{at}\ \$4,200 \\ \\ 0.50\ \textit{chance} \\ \textit{at}\ \$0 \end{pmatrix} \prec \begin{pmatrix} 0.75\ \textit{chance} \\ \textit{at}\ \$3,000 \\ \\ 0.25\ \textit{chance} \\ \textit{at}\ \$0 \end{pmatrix} = \mathcal{L}^+$$

[15%] [85%]

$$\mathcal{H}^- = \begin{pmatrix} 0.50 \ chance \\ at \ \$0 \\ \\ 0.50 \ chance \\ at \ \$-3,000 \end{pmatrix} \prec \begin{pmatrix} 0.75 \ chance \\ at \ \$0 \\ \\ 0.25 \ chance \\ at \ \$-4,500 \end{pmatrix} = \mathcal{L}^-$$
$$[37\%] \qquad\qquad [63\%]$$

As can be seen, $\mathcal{H}$ is weakly preferred to $\mathcal{L}$, but when the two prospects are split in their respective positive and negative parts, a relevant majority prefers $\mathcal{L}^+$ to $\mathcal{H}^+$ and $\mathcal{L}^-$ to $\mathcal{H}^-$. GLS is violated and CPT cannot cover such a pattern of choice. In the sequel we will refer to this experiment as the "Wu-Markle paradox". In the CPT model the GLS implies the separation of the domain of the gains from the domain of the losses, with respect to a subjective *reference point*. This separation depends on a characteristic *S-shaped utility function*, steeper for losses than for gains, and on two different *weighting functions*, which distort in a different way probabilities relative to gains and losses. We aim to generalize CPT maintaining the utility function, but replacing the two weighting functions with a *bi-weighting function*. This is a function with two arguments, the first corresponding to the probability of a gain and the second to the probability of a loss of the same magnitude. We call this model the bipolar Cumulative Prospect Theory (bCPT). The bCPT will allow gains and losses within a mixed prospect to be evaluated conjointly. The motivation of this generalization are in the Wu-Markle paradox, where it is shown how people are more willing to accept the risk of a loss having the hope of a win and, on the converse, are more careful with respect to a possible gain having the risk of a loss. In other words, the evaluation of a possible loss seems mitigated if this risk comes together with a possible gain. For example, the evaluation of the loss of \$3,000 with a probability 0.5 in the prospect $H = (0\ , 0.5; \$ - 3,000,\ 0.5)$ seems different from the evaluation of the loss of the same amount with the same probability within the prospect $L = (\$4,200\ , 0.5;\ \$ - 3,000\ , 0.5)$, where the presence of the possible gain of \$4,200 has an evident mitigation role. On the contrary, the evaluation of a possible gain will seem us diminished in case of a risk of a loss. Therefore, the evaluation of the gain of \$4,200 with a probability 0.5 in the prospect $P = (\$4,200\ , 0.5;\ 0,\ 0.5)$ is different from the evaluation of the gain of the same amount with the same probability within the prospect $L = (\$4,200\ , 0.5;\ \$ - 3,000\ ,\ 0.5)$. This chapter is devoted to the description of bCPT in a risky context. In the next chapter we will face the problem of the preference foundation of the model. The bCPT can be axiomatized separately in the field of decision under risk and in the field of decision under uncertainty. The main tool to axiomatize the model in an uncertainty context is the *bipolar Choquet integral*. Following Schmeidler (1986) for the Choquet integral, we will present a fairly simple characterization of the *bipolar Choquet integral*.

2.2 Cumulative Prospect Theory

2.2.1 CPT fundamental concepts

The most important idea in CPT is the concept of gain-loss asymmetry. People perceive possible outcomes as either gains or losses with respect to a *reference point* rather than as absolute wealth levels. Therefore CPT replaces

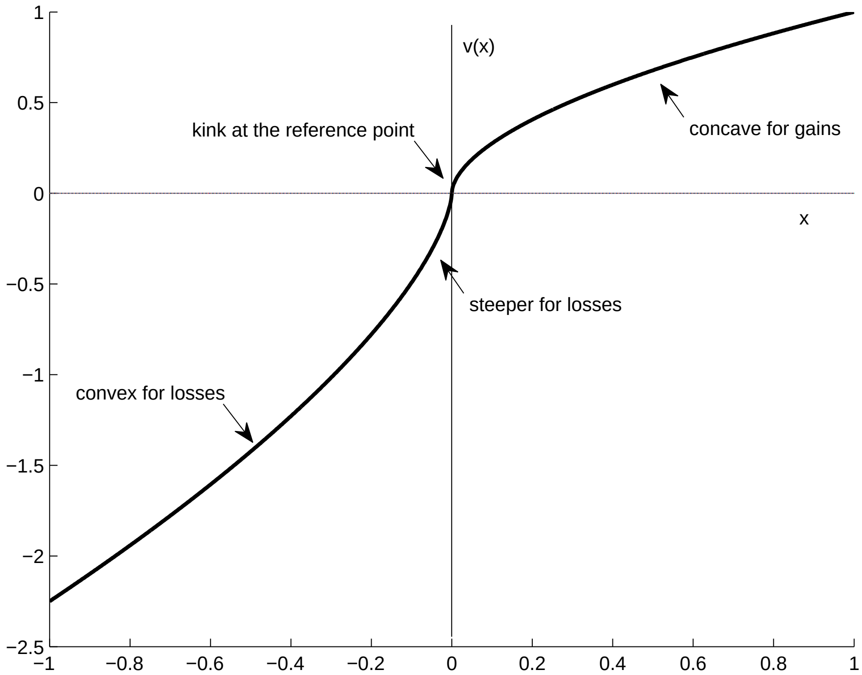

Figure 2.1: *CPT utility function*

the classical utility function with a characteristic S-shaped utility function (see Figure 2.1), null at the reference point, concave for gains and convex for losses, steeper for losses than for gains.

The other important idea in CPT is the notion of probability distortion, i.e. people tend to overweight very small probabilities and underweight average and large probabilities. Differently from PT, which assumes that people overweight small probability, independently from the relative outcomes, CPT adopts the fundamental idea of the Rank Dependent Utility Gilboa (1987); Quiggin (1982); Schmeidler (1986, 1989). This probability transformation is mathematically described by means of a *weighting function*

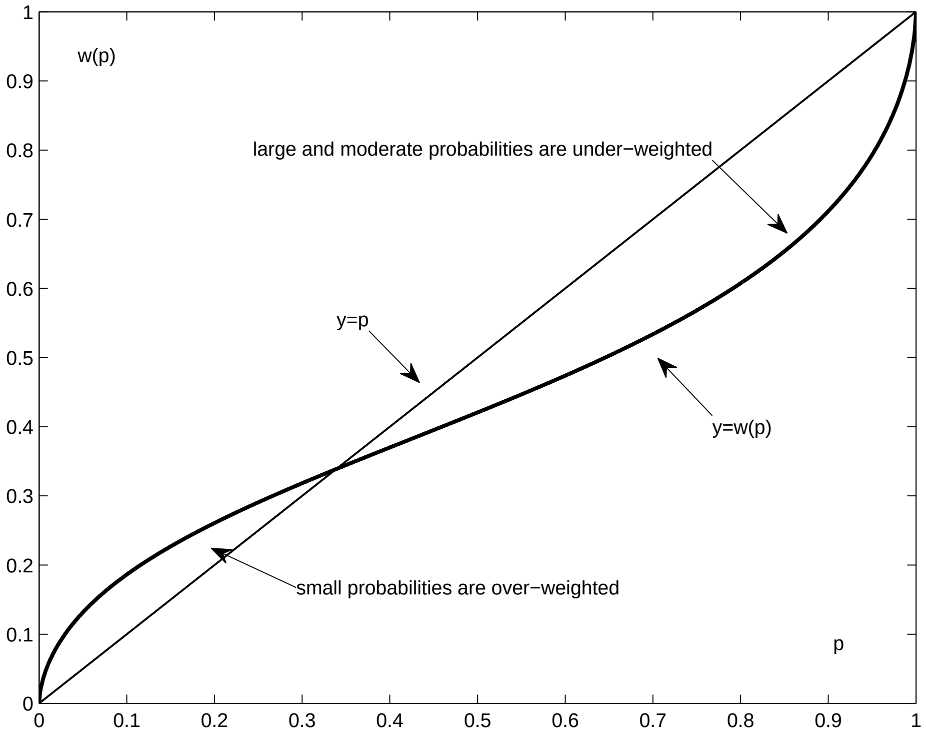

Figure 2.2: *CPT weighting function*

$w(p) : [0, 1] \rightarrow [0, 1]$, which typical inverse S-shape is shown in Figure 2.2.

2.2.2 The formal model

The outcomes set is the real line $\Re$ and zero has the role of reference point, so positive and negative outcomes are respectively gains and losses. A lottery $P = (x_1, p_1; \ldots ; x_n, p_n)$ with $x_i \in \Re,\ p_i \in [0; 1]\,,\ \sum p_i = 1$, is a finite probability distribution. We suppose that the lottery P is rank ordered, i.e. $x_1 \geq ... \geq x_k \geq 0 > x_{k+1} .. \geq x_n$, for some $k \in \{0, ...n\}$. Let us denote P^+ (P^-) the gain (loss) part of P, obtained by substituting all the losses (gains) in P with zero. A binary preference relation $\succsim$ over lotteries is given as primitive, where for all prospects P and Q, $P \succsim Q$ means the prospect P is at least as good as prospect Q. As usual $\succ$ and $\sim$ denote the asymmetric and symmetric part of $\succsim$. A real valued function $V(\cdot)$ over lotteries represents preferences if

$$P \succsim Q \qquad \text{iff} \qquad V(P) \geq V(Q).$$

CPT holds if the preference relation $\succsim$ can be represented by a CPT functional

$$V_{CPT}(P) = \sum_{j=1}^{n} \pi_j u(x_j) \tag{2.1}$$

where $u(\cdot) : \Re \rightarrow \Re$ is a continuous and strictly increasing S-shaped utility function, as described in Kahneman and Tversky (1979) and π_j are *decision weights*, defined as follows. A function $w : [0;1] \rightarrow [0;1]$ is *probability transformation* (or *weighting function*) if it is strictly increasing and satisfying the conditions $w(0) = 0$ and $w(1) = 1$.
If $j \leq k$, i.e. x_j is a gain, and w_+ is a (gain) probability transformation, then

$$\pi_j = w_+(p_1 + ... + p_j) - w_+(p_1 + ... + p_{j-1})$$

If $j > k$, i.e. x_j is a loss, and w_- is a (loss) probability transformation, then

$$\pi_j = w_-(p_j + ... + p_n) - w_-(p_{j+1} + ... + p_n)$$

Therefore, the (2.1) can be rewritten as

$$V_{CPT}(P) = V_{CPT}(P^+) + V_{CPT}(P^-) \tag{2.2}$$

where

$$V_{CPT}(P^{+}) = \sum_{j=1}^{k} \left[w_{+}(p_1 + \ldots + p_j) - w_{+}(p_1 + \ldots + p_{j-1}) \right] u(x_j) \qquad (2.3)$$

$$V_{CPT}(P^{-}) = \sum_{j=k+1}^{n} \left[w_{-}(p_j + \ldots + p_n) - w_{-}(p_{j+1} + \ldots + p_n) \right] u(x_j) \qquad (2.4)$$

Remark 2 *In the special case of a binary gamble* $P = (G, p_G; L, p_L)$ *containing a gain* G*, with probability* p_G *and a loss* L *with probability* p_L*, the CPT formula reduces to*

$$V_{CPT}(P) = u(G)w_{+}(p_G) + u(L)w_{-}(p_L)$$

Remark 3 *Using an integral representation the CPT functional assumes the following concise form*

$$V_{CPT}(P) = \int_0^{+\infty} w_{+}\left(\sum_{i:u(x_i)\geq t} p_i \right) dt \; - \int_0^{+\infty} w_{-}\left(\sum_{i:u(x_i)\leq -t} p_i \right) dt \qquad (2.5)$$

Despite the (2.5) emphasizes that in CPT gains and losses are evaluated separately, nothing avoid the above representation to be written by a unique integral

$$V_{CPT}(P) = \int_0^{+\infty} w_{+}\left(\sum_{i:u(x_i)\geq t} p_i \right) - w_{-}\left(\sum_{i:u(x_i)\leq -t} p_i \right) dt \qquad (2.6)$$

both the (2.5), (2.6) will be helpful in proving some results.

2.2.3 CPT parametrization

In this section we describe the "classical CPT" proposed by Tversky and Kahneman (1992). Kahneman and Tversky (KT) adopted two different weighting functions (for probability of gains and losses)

$$w_+(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{\frac{1}{\gamma}}} \quad , \quad w_-(p) = \frac{p^\delta}{[p^\delta + (1-p)^\delta]^{\frac{1}{\delta}}} \tag{2.7}$$

and a power utility function

$$u(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -\lambda|x|^\beta & \text{if } x < 0 \end{cases} \tag{2.8}$$

where typically the loss-coefficient is greater than the gain-coefficient, $\beta \geq \alpha$, while the *loss aversion* coefficient, λ, is typically assumed 2.25. Table 2.1 reports some estimations of parameters α, β, γ and δ presented in recent studies. In all the investigations after 1994, the coefficients α and β of the power utility functions are greater than the coefficients γ and δ of the relative weighting functions.

2.3 Bipolar Cumulative Prospect Theory

2.3.1 From CPT to bCPT

As we have seen in CPT the weighting function has the role to transform the (cumulated) probabilities. This transformation happens separately for the probabilities attached to gains and those attached to losses. As final

Table 2.1: Estimation of CPT parameters proposed in literature

	utility function		weighting functions	
Experimental study	α	β	γ	δ
Tversky and Kahneman (1992)	0.88	0.88	0.61	0.69
Camerer and Ho (1994)	0.37	-	0.56	-
Tversky and Fox (1995)	0.88	-	0.69	-
Wu and Gonzalez (1996)	0.52	-	0.71	-
Gonzalez and Wu (1999)	0.49	-	0.44	-
Abdellaoui (2000)	0.89	0.92	0.60	0.70
Bleichrodt and Pinto (2000)	0.77	-	0.67	0.55
Kilka and Weber (2001)	0.76-1.00	-	0.30-0.51	-
Abdellaoui *et al.* (2003)	0.91	-	0.76	-

result, see (2.5), the CPT functional can be seen as sum of the CPT functional relative to gains and that relative to losses. In bCPT the two weighting function of CPT are replaced with a two-variables bi-weighting function. This has in the first argument the (cumulated) probability of a gain and in the second argument the (cumulated) probability of a symmetric loss. The final result is a number within the closed interval [-1;1] which we cannot think as a distorted probability but as a distorted difference of the probability of gains and the probability of symmetric losses, generated in the process to evaluate conjointly gains and losses. In the sequel we propose some bi-weighting functions that are the natural extensions of the most note weighting functions proposed in the literature in the last years. These bi-weighting functions inherit the respective properties of the weighting functions from which they derive. One suggestion for future studies could be to determine which of these functions is the most suitable to generalize the CPT model. We choose-after presenting some generalizations-to use for the following part of the thesis the classical KT bi-weighting function.

2.3.2 The bi-weighting functions

As we have seen a weighting function is a strictly increasing function $w : [0;1] \to [0;1]$ satisfying the conditions $w(0) = 0$, $w(1) = 1$. In this section we shall generalize this concept to the bipolar context. In CPT the weighting function has the role to transform the (cumulated) probabilities; in our model we have a two-variables bi-weighting function which has in the first argument the (cumulated) probability of a gain with a utility greater or equal than a given level L and in the second argument the (cumulated) probability of a symmetric loss not smaller than $-L$. The final result is a number within the closed interval [-1;1] which we cannot think as a distorted probability but as a "*distorted difference of the probability of gains and the probability of symmetric losses*". Let us set

$$\mathcal{A} = \{(p,q) \in [0;1] \times [0;1] \text{ such that } p + q \leq 1\}$$

that is, in the $p-q$ plane, the triangle which vertexes are $O \equiv (0,0)$, $P \equiv (1,0)$ and $Q \equiv (0,1)$, see figure 2.3.

Definition 7 *We define bi-weighting function any function*

$$\omega(p,q) : \mathcal{A} \to [-1;1]$$

satisfying the following conditions:

- $\omega(p,q)$ *is increasing in* p *and decreasing in* q *(bi-monotonicity)*
- $\omega(1,0) = 1$, $\omega(0,1) = -1$ *and* $\omega(0,0) = 0$.

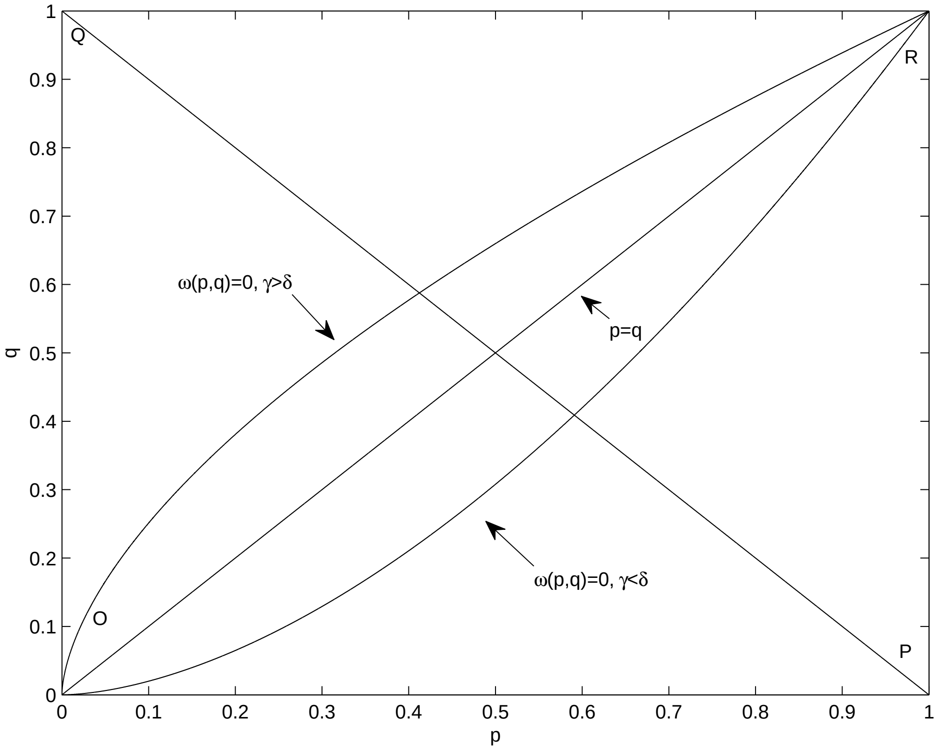

Figure 2.3: *the bi-weighting function domain*

In table 1 we propose some generalizations of well known weighting functions. They coincide with the original gain weighting function, ω_{+}, if $q = 0$ and with the opposite loss weighting function, $-\omega_{-}$, if $p = 0$. The following propositions hold.

Proposition 1 *The Kahneman-Tversky bi-weighting function with the parameters setting* $1/2 < \gamma, \delta < 1$, *is increasing in* p *and decreasing in* q.

Proposition 2 *The Latimore, Baker and Witte bi-weighting function with the parameters setting* $\alpha > 1/2$ *and* $0 < \gamma, \delta \leq 1$, *is increasing in* p *and decreasing in* q.

Proposed by	Original weighting function $w(p)$	Bipolarized form $w(p,q)$
Tversky and Kahneman (1992)	$\frac{p^\gamma}{[p^\gamma+(1-p)^\gamma]^{\frac{1}{\gamma}}}$	$\frac{p^\gamma-q^\delta}{[p^\gamma+(1-p)^\gamma]^{\frac{1}{\gamma}}+[q^\delta+(1-q)^\delta]^{\frac{1}{\delta}}-1}$
Lattimore, Baker and Witte (1992)	$\frac{\alpha p^\gamma}{\alpha p^\gamma+(1-p)^\gamma}$	$\frac{\alpha(p^\gamma-q^\delta)}{\alpha p^\gamma+(1-p)^\gamma+\alpha q^\delta+(1-q)^\delta}$
Prelec (1998)	$e^{-\beta(-lnp)^\alpha}$	$\frac{p^\gamma-q^\delta}{\lvert p^\gamma-q^\delta\rvert}e^{-\beta(-ln\lvert p^\gamma-q^\delta\rvert)^\alpha}$

Table 2.2: *original and bi-polarized weighting function*

Proposition 3 *The Prelec bi-weighting function with the parameters setting* $\beta \cong 1,\ \gamma,\delta > 0$ *and* $0 < \alpha < 1$ *is increasing in* p *and decreasing in* q*.*

The proofs of these propositions, together with a brief discussion about any of these generalization, are presented in appendix at the end of the chapter.

Remark 4 *What we called bi-monotonicity of the bi-weighting functions, i.e. the fact that they are increasing in the first argument (probability of gain in the model) and decreasing in the second argument (probability of loss in the model) will be crucial in proving that the bCPT functional satisfies the stochastic dominance principle.*

Remark 5 *The bi-weighting function we propose are typically inverse S-shaped in the space (see Figure 2.4)*

A typical feature of the weighting function described in Tversky and Kahneman (1992) is the inverse S-shape in the plane. Let us consider the bi-polarized form of the KT weighting function, preserving the original parameters estimation $\gamma = .61$ and $\delta = .69$

$$\omega(p,q) = \frac{p^{0.61} - q^{0.69}}{\left[p^{0.61} + (1-p)^{0.61}\right]^{\frac{1}{0.61}} + \left[q^{0.69} + (1-q)^{0.69}\right]^{\frac{1}{0.69}} - 1} \tag{2.9}$$

In Figure 2.4 we have plotted the (2.9). As can be seen the typical inverse S-Shape is generalized from the plane to the space. Clearly we are interested to the part of this plot such that $p + q = 1$.

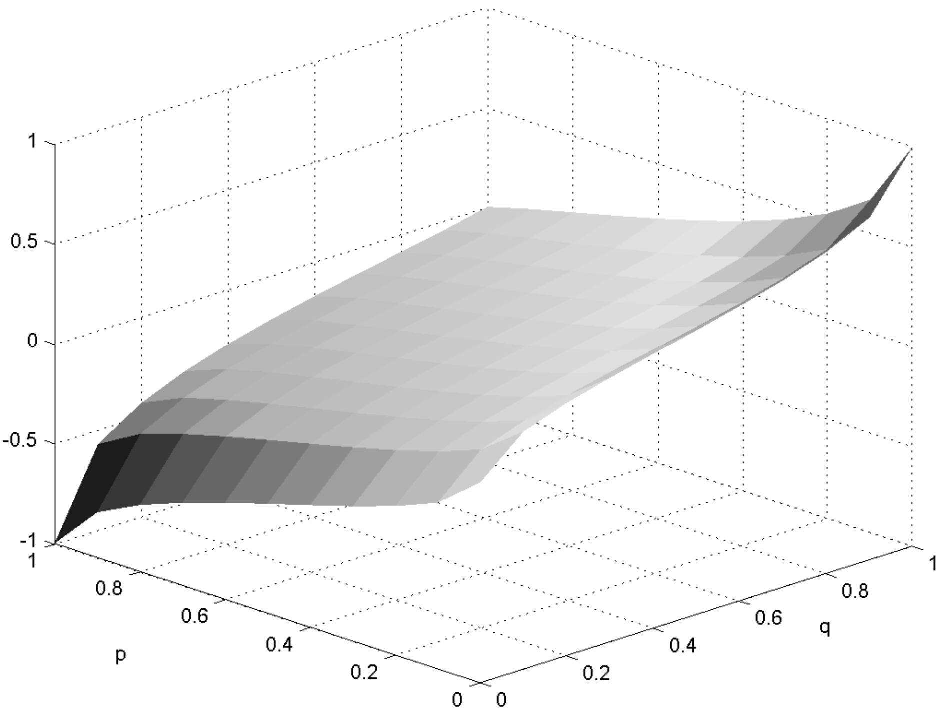

Figure 2.4: *bi-CPT weighting function*

2.3.3 The bipolar Cumulative Prospect Theory (bCPT)

As for CPT the outcomes set is the real line $\Re$ and zero has the role of the reference point, so positive and negative outcomes are respectively gains and losses. A risky prospect (or lottery) $P = (x_1, p_1; \ldots; x_n, p_n)$ is a finite probability distribution giving the outcome $x_j \in \Re$ with probability $p_j \in [0;1]$, $i = 1, 2, \ldots, n$. The outcomes are mutually exclusive and betting on P one of them is obtained, that is $\sum p_j = 1$,. A preference relation $\gtrsim$ over the prospects set is given as primitive. Let $u(.) : \Re \to \Re$ be an utility function (non necessarily S-shaped). If $I = \{1, 2, \ldots n\}$ for any $k = 1, 2, \ldots, n$, we define

$$I_k^+ = \{j \in I \text{ such that } u(x_j) \geq |x_k|\}$$

$$I_k^- = \{j \in I \text{ such that } u(x_j) \leq -|x_k|\}.$$

Let $(\cdot) : I \to I$ be an index permutation such that

$$|u(x_{(1)})| \leq |u(x_{(2)})| \leq \cdots \leq |u(x_{(n)})|.$$

Finally, let $\omega(p,q) : [0;1] \times [0;1] \to [-1;1]$ be a bi-weighting function, i.e. an increasing in p and decreasing in q, continuous function, satisfying the coherence conditions: $\omega(0,0) = 0$, $\omega(1,0) = 1$ and $\omega(0,1) = -1$. We define the bCPT value of P as

$$V_{bCPT}(P) = \sum_{i=1}^{n} \left(|u(x_{(i)})| - |u(x_{(i-1)})|\right) \omega \left(\sum_{j \in I_{(i)}^+} p_j, \sum_{k \in I_{(i)}^-} p_j \right) \tag{2.10}$$

where $u\left(x_{(0)}\right) = 0$ and the sums in ω arguments are replaced with zero if $I^+_{(i)}$ or $I^-_{(i)}$ are empty. Another way to formalize the bCPT functional is by means of the integral representation

$$V_{bCPT}(P) = \int_0^\infty \omega\left(\sum_{i:u(x_i)\geq t} p_i, \sum_{i:u(x_i)\leq -t} p_i\right) dt \tag{2.11}$$

which can be helpful to prove some properties.

2.3.4 bCPT and Stochastic Dominance

Let us remark that the bi-monotonicity of the weighting function, i.e. the fact that $\omega(p,q)$ is increasing in p and decreasing in q, ensures the bCPT model satisfies *Stochastic Dominance Principle*. This means that if prospect P stochastically dominates prospect Q then $V_{bCPT}(P) \geq V_{bCPT}(Q)$. The following theorem establishes this result.

Theorem 6 *Let us suppose that prospects are evaluated with the bipolar CPT. Then stochastic dominance is satisfied if and only if the bi-weighting function $\omega(p,q)$ is increasing in p and decreasing in q*

Proof. Trough the proof we shall consider the following formulation of the bipolar CPT functional of a lottery $P = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$:

$$V_{bCPT}(P) = \int_0^\infty \omega\left(\sum_{i:u(x_i)\geq t} p_i, \sum_{i:u(x_i)\leq -t} q_i\right) dt. \tag{2.12}$$

Let us suppose the bi-weighting function $\omega(p,q)$ is increasing in p and decreasing in q. Let us consider two lotteries $P = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$ and $Q = (y_1, q_1; y_2, q_2; \ldots; y_m, q_m)$ such that P stochastically dominates Q. This means that for all $t \in \mathfrak{R}$

$$\sum_{i:x_i \geq t} p_i \geq \sum_{i:y_i \geq t} q_i \tag{2.13}$$

or equivalently,

$$\sum_{i:x_i \leq t} p_i \leq \sum_{i:y_i \leq t} q_i \tag{2.14}$$

By the stochastic dominance of P over Q, we have that for all $t \in \mathfrak{R}^+$

$$\sum_{i:u(x_i) \geq t} p_i \geq \sum_{i:u(y_i) \geq t} q_i \tag{2.15}$$

and

$$\sum_{i:u(x_i) \leq -t} p_i \leq \sum_{i:u(y_i) \leq t} q_i \tag{2.16}$$

From 2.15 and 2.16, considering the monotonicity of $\omega(\cdot,\cdot)$, we have that for all $t \in \mathfrak{R}^+$

$$\omega\left(\sum_{i:u(x_i) \geq t} p_i, \sum_{i:u(x_i) \leq -t} p_i\right) \geq \omega\left(\sum_{i:u(x_i) \geq t} q_i, \sum_{i:u(x_i) \leq -t} q_i\right) \tag{2.17}$$

and by monotonicity of the integral we conclude that $V_{bCPT}(P) \leq V_{bCPT}(Q)$. Now, let us suppose the bi-weighting function $\omega(\cdot,\cdot)$ is not [increasing in p and decreasing in q], i.e. that there exist $(p,q), (\widetilde{p}, \widetilde{q}) \in [0,1]^2$ such that

$$\begin{cases} p \geq \widetilde{p} \\ q \leq \widetilde{q} \\ (p - \widetilde{p})^2 + (q - \widetilde{q})^2 > 0 \\ \omega(p,q) < \omega(\widetilde{p}, \widetilde{q}) \end{cases}$$

Let us consider $x > 0$ and $y < 0$ such that $u(x) = -u(y)$ and the two lotteries $R = (x, p; y, q)$ and $S = (x, \widetilde{p}; y, \widetilde{q})$. Even if R stochastically dominates S, it results

$$V_{bCPT}(R) = \omega(p, q) \cdot u(x) < \omega(\widetilde{p}, \widetilde{q}) \cdot u(x) = V_{bCPT}(S).$$

Q.E.D.

2.3.5 The relationship between CPT and bCPT

Now we wish to discuss the fundamental link between the CPT model and its generalization, the bCPT model. As we have seen, in CPT gains and losses are evaluated separately, whereas in bCPT they are evaluated conjointly. This gives rise to different approaches and then formulations. Let be given a lottery $P = (x_1, p_1; ...; x_n, p_n)$; an utility function $u(\cdot) : \mathfrak{R} \to \mathfrak{R}$; two weighting functions $\pi_- : [0,1] \to [0,1]$ and $\pi_+ : [0,1] \to [0,1]$; and a bi-weighting function $\omega(p, q) : \mathcal{A} \to [-1, 1]$. Using an integral representation we get

$$V_{CPT}(P) = \int_0^{+\infty} \pi_+ \left(\sum_{i:u(x_i) \geq t} p_i \right) dt - \int_0^{+\infty} \pi_- \left(\sum_{i:u(x_i) \leq -t} p_i \right) dt \tag{2.18}$$

$$V_{bCPT}(P) = \int_0^{+\infty} \omega \left(\sum_{i:u(x_i) \geq t} p_i, \sum_{i:u(x_i) \leq -t} p_i \right) dt \tag{2.19}$$

The separate evaluation of gains and losses in CPT is generalized to a conjoint evaluation of gains and losses in bCPT.

In order to link the above integrals we must establish the relation between a bi-weighting function and a weighting function. It is straightforward to note that given a bi-weighting function

$$\omega(p,q) : \mathcal{A} \to [-1;1]$$

we can define two weighting functions by setting for all $p,\ q \in [0,1]$

$$\pi_+(p) = \omega(p,0) : [0,1] \to [0,1]$$

$$\pi_-(q) = -\omega(0,q) : [0,1] \to [0,1]$$

and, on the converse, given two weighting functions

$$\pi_+(p) =: [0,1] \to [0,1]$$

$$\pi_-(q) =: [0,1] \to [0,1]$$

we can obtain a bi-weighting function by setting for all $p,\ q \in [0,1]$

$$\omega(p,q) = \pi_+(p) - \pi_-(q) : \mathcal{A} \to [-1;1]$$

In this last case we say that the bi-weighting function is *separable*, that means it can be represented as difference between two weighting functions.

In the next two propositions we will formalizes the relationship between the two models. The first proposition states that CPT and bCPT can be considered coincident for non-mixed prospects, i.e. for prospects non containing simultaneously gains and losses. This fact is of great importance, since CPT has been widely tested (with success) in situations involving only gains or only losses, as it will be pointed out at the end of the proof. The second proposition states that the CPT model can be considered a special case of the bCPT model provided that we use a *separable* bi-weighting function. This assertion too will be better discussed at the end of the proof.

Proposition 4 *For non mixed prospects (containing only gains or losses) the bCPT model coincides with the CPT model.*

Proof. In proving the proposition we clarify its meaning. Let us suppose the prospects are evaluated with the bCPT model and let us indicate with $u(\cdot) : \mathfrak{R} \to \mathfrak{R}$ the utility function and with $\omega(p, q) : \mathcal{A} \to [-1, 1]$ the bi-weighting function. Using the above considerations we can obtain two weighting function by setting $\pi_+(p) = \omega(p, 0)$ and $\pi_-(q) = -\omega(0, q)$ for all $p,\ q \in [0, 1]$. Now, let $P = (x_1, p_1; ...; x_n, p_n)$ be a prospect assigning the non-negative outcome $x_j \in \mathfrak{R}^+$ with probability p_j. Since P contains only gains (more precisely it does not contain losses), putting $\omega(p, 0) = \pi_+(p)$, we get:

$$V_{bCPT}(P) = \int_0^{+\infty} \omega\left(\sum_{i:u(x_i)\geq t} p_i\ ,\ \sum_{i:u(x_i)\leq -t} p_i \right) dt =$$

$$= \int_0^{+\infty} \omega\left(\sum_{i:u(x_i)\geq t} p_i\ ,\ 0 \right) dt\ =\ \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i \right) dt =$$

$$= \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i \right) dt\ -\ \int_0^{+\infty} \pi_-(\ 0\ )\, dt\ =\ V_{CPT}(P)$$

In the same manner, let $P = (x_1, p_1; ...; x_n, p_n)$ be a prospect assigning the non-positive outcome $x_j \in \mathfrak{R}^-$ with probability p_j. Since P contains only losses (more precisely it does not contain gains), putting $\omega(0,q) = -\pi_-(q)$, we get:

$$V_{bCPT}(P) = \int_0^{+\infty} \omega\left(\sum_{i:u(x_i)\geq t} p_i \ , \sum_{i:u(x_i)\leq -t} p_i \right) dt =$$

$$= \int_0^{+\infty} \omega\left(0 \ , \sum_{i:u(x_i)\leq -t} p_i \right) dt \ = \ - \int_0^{+\infty} \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i \right) dt =$$

$$= \int_0^{+\infty} \pi_+ (\ 0\ )\, dt \ - \ \int_0^{+\infty} \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i \right) dt \ = \ V_{CPT}(P)$$

The above reasoning can be replaced as follows. Let us suppose the prospects are evaluated with the CPT model and let us indicate with $u(\cdot) : \mathfrak{R} \to \mathfrak{R}$ the utility function and with $\pi_+(p) : [0,1] \to [0,1]$ and $\pi_-(q) : [0,1] \to [0,1]$ the two weighting functions (for gains and losses respectively). We can obtain a bi-weighting function by setting $\omega(p,q) = \pi_+(p) - \pi_-(q)$ for all $p, \ q \in [0,1]$ and replacing the steps in the above proof we get $V_{CPT}(P) = V_{bCPT}(P)$. We conclude that for non-mixed prospects the two model coincide, provided that we use weighting functions and bi-weighting function linked as it has been shown.

Q.E.D.

Wu and Markle (2008) assert: "*Prospect theory distinguishes itself from the classical theory of decision under risk, expected utility theory, in taking change in wealth rather than absolute wealth to be the relevant carrier of value (Kahneman and Tversky (1979); Tversky and Kahneman (1992)). This distinction has been applied with enormous success to applications in* *business, finance, law, medicine, and political science (e.g., Barberis* et al. *(2001); Camerer (2000); Jolls* et al. *(1998); McNeil* et al. *(1982); Quattrone and Tversky (1988)). Indeed, most important real world decisions are mixed gambles, involving some possibility of gain and some possibility of loss (MacCrimmon and Wehrung (1990); March and Shapira (1987)). This article investigates how individuals choose among mixed gambles ...”*

In the majority of works cited by Wu and Markle (2008), CPT has been tested with success for non mixed prospects. The fact that, in this context, bCPT coincides with CPT is, then, of great importance. Fist, because all the investigations regarding CPT can be seen as investigation regarding bCPT and secondarily since our model preserves all the descriptive powerful of CPT in a non-mixed context. On the converse, Wu and Markle (2008) expressed doubts about the gain loss separability: *“This article investigates how individuals choose among mixed gambles by examining a fundamental assumption of prospect theory, gain-loss separability. Simply stated, gain-loss separability requires that preferences for gains be independent of preferences for losses and, more strongly, that the valuation of a mixed gamble be the sum of the valuations of the gain and loss portions of that gamble. A failure of gain-loss separability has practical as well as theoretical implications. First,empirical findings gleaned from the numerous studies of single domain gambles will not necessarily generalize to the domain of mixed gambles. In addition, estimates of the probability weighting function to the choice data support our psychological interpretation that individuals are less sensitive to probability differences when making choices among mixed gambles than when faced with gambles involving all gains or all losses.”*

The authors supported their thesis with a large amount of data, showing systematic violations of gain-loss separability. In this case we wish to point out that if in a non mixed context CPT and bCPT can be considered the same model, in a situations involving both gains and losses they are different models. As we will soon see bCPT allows violation of gain-loss separability and is able to cover the majority of data in Wu and Markle (2008).

Proposition 5 *If the prospects are evaluated with the bCPT model with a separable bi-weighting function, than the representation coincides with that obtained with the CPT model. On the converse, if the prospects are evaluated with the CPT model, than the representation coincides with that obtained with the bCPT model with a separable bi-weighting function.*

Proof. Let us suppose the prospects are evaluated with the bCPT model, with a separable bi-weighting function

$$\omega(p,q) = \pi_+(p) - \pi_-(q) : \mathcal{A} \to [-1;1]$$

being π_+ and π_- two weighting function and let us denote by $u(\cdot) : \mathfrak{R} \to \mathfrak{R}$ the utility function. We get immediately:

$$V_{bCPT}(P) = \int_0^{+\infty} \omega\left(\sum_{i:u(x_i)\geq t} p_i \ , \sum_{i:u(x_i)\leq -t} p_i\right) dt =$$

$$= \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i\right) - \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i\right) dt =$$

$$= \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i\right) dt - \int_0^{+\infty} \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i\right) dt = V_{CPT}(P)$$

On the converse let us suppose the prospects are evaluated with the CPT model and let us denote by $u(\cdot) : \mathfrak{R} \to \mathfrak{R}$ the utility function and with $\pi_+(p) : [0,1] \to [0,1]$ and $\pi_-(q) : [0,1] \to [0,1]$ the two weighting functions (for gains and losses respectively). We can obtain a separable bi-weighting function by setting $\omega(p,q) = \pi_+(p) - \pi_-(q)$ for all $p,\ q \in [0,1]$ and replacing the steps in the above proof we get

$$\begin{aligned} V_{CPT}(P) &= \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i\right) dt - \int_0^{+\infty} \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i\right) dt = \\ &= \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i\right) - \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i\right) dt = \\ &= \int_0^{+\infty} \omega\left(\sum_{i:u(x_i)\geq t} p_i\ ,\ \sum_{i:u(x_i)\leq -t} p_i\right) dt = V_{bCPT}(P) \end{aligned}$$

Q.E.D.

Proposition 5 establishes the only situation in which CPT and bCPT are the same model (also for mixed prospects). More precisely, it asserts that the CPT can be regarded as a special case of bCPT, in the sense that there exists a (separable) bi-weighting function $\omega(p,q) = \pi_+(p) - \pi_-(q)$ such that $V_{bCPT}(P) = V_{CPT}(P)$ for all prospects P. This fact has two important consequences. First, it is relevant in order to provide a preference foundation for the model, since bCPT will need a less restrictive set of axioms with respect to CPT. Second, it seems to contradict Wu and Markle (2008) *"A failure of gain-loss separability has practical as well as theoretical implications … Finally, models that relax gain-loss separability will necessarily be more* *complex than models that assume separability."*

If in propositions 5 and 4 we have just analyzed the cases where CPT and bCPT coincide, in the next sections we will see the explanation, by using bCPT, of a recent paradox presented against CPT.

2.3.6 Explanation of the Wu-Markle paradox

Let us reconsider the Wu-Markle paradox described in the introduction. Wu and Markle (2008) refer the following experiment: 81 participants gave their preferences as it is shown below

$$\mathcal{H} = \begin{pmatrix} 0.50 \ chance \\ at \ \$4,200 \\ \\ 0.50 \ chance \\ at \ \$-3,000 \end{pmatrix} \succ \begin{pmatrix} 0.75 \ chance \\ at \ \$3,000 \\ \\ 0.25 \ chance \\ at \ \$-4,500 \end{pmatrix} = \mathcal{L}$$

[52%] [48%]

$$\mathcal{H}^+ = \begin{pmatrix} 0.50 \ chance \\ at \ \$4,200 \\ \\ 0.50 \ chance \\ at \ \$0 \end{pmatrix} \prec \begin{pmatrix} 0.75 \ chance \\ at \ \$3,000 \\ \\ 0.25 \ chance \\ at \ \$0 \end{pmatrix} = \mathcal{L}^+$$

[15%] [85%]

$$\mathcal{H}^- = \begin{pmatrix} 0.50 \ chance \\ at \ \$0 \\ \\ 0.50 \ chance \\ at \ \$-3,000 \end{pmatrix} \prec \begin{pmatrix} 0.75 \ chance \\ at \ \$0 \\ \\ 0.25 \ chance \\ at \ \$-4,500 \end{pmatrix} = \mathcal{L}^-$$

[37%] [63%]

As can be seen $\mathcal{H}$ is weakly preferred to $\mathcal{L}$, but when the two prospects are split in their respective positive and negative parts, a relevant majority prefers $\mathcal{L}^+$ to $\mathcal{H}^+$ and $\mathcal{L}^-$ to $\mathcal{H}^-$. GLS is violated and CPT cannot cover such a pattern of choice.

Wu and Markle (2008) suggested to use the the same model, CPT, with a different parametrization for mixed prospects and for prospects involving only gains or only losses. We report their general conclusions:
"In the last 50 years, a large body of empirical research has investigated how decision makers choose among risky gambles. Most of these findings can be accommodated by prospect theory. An S-shaped utility function and inverse S-shaped probability weighting function can model the reflection effect, the four-fold pattern of risk preferences, the common-ratio and common-consequence effects, as well as the generalization of these findings from risk to uncertainty. However, the majority of the existing empirical evidence has involved single-domain gambles. The emphasis on these gambles is sensible - they are easy for research participants to understand and can be studied in hypothetical situations as well as played out for real payoffs. The study of single-domain gambles is justified if the understanding gleaned from these investigations extends to the domain of mixed gambles. ***Our study indicates that mixed gamble behavior is described well by an S-shaped utility function and an inverse S-shaped probability weighting function. However, gain-loss separability fails, and hence different parameter values are needed for mixed gambles than single-domain gambles.*** *As a result, findings inferred from studies of single domain gambles may not extend automatically to mixed gambles. Our violations of gain-loss separa-*

bility appear to be systematic. Although a comprehensive study of gain-loss separability is beyond the scope of this paper, we encourage extensions of our tests to mixed gambles with different structures. Thus, even though future research will surely qualify the account of mixed gambles developed here, we nevertheless see our paper as moving us a step closer toward a fuller understanding of this important and understudied choice domain. In addition, we have proposed cognitive and affective explanations for the violations of gain-loss separability but have not provided direct evidence for either explanation. The role that these psychological accounts and others play in the general evaluation of mixed gambles awaits further investigation."

Despite the conclusions of Wu and Markle we are able to cover their paradox using bCPT, without changing the parameters in the passage from non mixed prospects to mixed prospects. If we use the bipolar CPT with the bi-polarized KT weighting functions (with the original parameters):

$$\omega(p,q) = \frac{p^{0.61} - q^{0.69}}{\left[p^{0.61} + (1-p)^{0.61}\right]^{\frac{1}{0.61}} + \left[q^{0.69} + (1-q)^{0.69}\right]^{\frac{1}{0.69}} - 1}$$

and the classical KT power utility function

$$u(x) = \begin{cases} x^{\alpha} & \text{if } x \geq 0 \\ -\lambda(-x)^{\alpha} & \text{if } x < 0 \end{cases}$$

we obtain

$$V_{bCPT}(\mathcal{H}) = -443.24 \quad > \quad V_{bCPT}(\mathcal{L}) = -453.76$$
$$V_{bCPT}(\mathcal{H}^{+}) = 649.19 \quad < \quad V_{bCPT}(\mathcal{L}^{+}) = 652.26$$
$$V_{bCPT}(\mathcal{H}^{-}) = -1,172.45 \quad < \quad V_{bCPT}(\mathcal{L}^{-}) = -1,083.04$$

These results agree with the preference relation $\gtrsim$ and gain loss separability is naturally covered. In our knowledge do not exist many models that are able to explain such a pattern of choice.

2.4 Appendix

In this appendix we prove propositions 1, 2 and 3. We remember that all the bi-weighting $\omega(p,q)$ functions we propose in this study are defined in

$$\mathcal{A} = \{(p,q) \in [0,1] \times [0,1] \quad \textit{such that} \quad p + q \leq 1\}$$

that is, in the $p-q$ plane, the triangle which vertexes are $O \equiv (0,0)$, $P \equiv (1,0)$ and $Q \equiv (0,1)$ (see 2.3). Moreover they coincide with the original gain weighting function, $\omega_{+}(p)$, if $q = 0$ and with the opposite loss weighting function, $-\omega_{-}(q)$, if $p = 0$. Since any bi-weighting generalization has as special case the original form, it inherits all its limitations. For instance, like the KT weighting function is not in general monotonic (see Rieger and Wang (2006); Ingersoll (2008)) so the bi-polarized form is not, in general, bi-monotonic, i.e. increasing in p and decreasing in q. We remark that it has this characteristic for all the relevant values of its parameters, where relevant means established from the previous literature. On the other hand, the Prelec bi-polarized weighting function is bi-monotonic (like the original) without limitations for the parameters. Another aspect to take into account in future behavioral investigation in field experiments is the form of the curve $\omega(p,q) = 0$. For all the three bi-weighting functions we are proposing this curve has equation $p^\gamma - q^\delta = 0$. It is represented by the arc $\widehat{OR}$ in figure 2.3; by choosing $\gamma > \delta$, $\gamma < \delta$ or $\gamma = \delta$, we have, respectively, a convex, a concave or a linear curvature.

2.4.1 The Kahneman-Tversky bi-weighting function

The first and most famous weighting function is that proposed in Tversky and Kahneman (1992)

$$\pi(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{\frac{1}{\gamma}}}$$

where the parameter gamma can be chosen differently for gains and losses and it is well known that the authors estimated $\gamma = 0.61$ and $\delta = 0.69$. For this weighting function we propose the following bipolar form

$$\omega(p,q) = \frac{p^\gamma - q^\delta}{[p^\gamma + (1-p)^\gamma]^{\frac{1}{\gamma}} + [q^\delta + (1-q)^\delta]^{\frac{1}{\delta}} - 1} \tag{2.20}$$

As the original KT weighting function is non monotonic for γ too much near to zero, see Rieger and Wang (2006); Ingersoll (2008), so it is the case of the bi-weighting function (2.20) when γ and δ are near zero. Proposition 1 establishes the parameter limitations preserving the bi-monotonicity of the bi-weighting function (2.20).

Proof of proposition 1.

For $x \in [0,1]$ and $\delta \in [0,1]$ it results $f(x) = [x^\delta + (1-x)^\delta]^{\frac{1}{\delta}} \geq 1$ since this function is continuous in the closed interval $[0,1]$, with $f(0) = f(1) = 1$, while $f'(x)$ is positive in $]0,1/2[$ and negative in $]1/2,1[$. In fact

$$f'(x) = \left[x^\delta + (1-x)^\delta\right]^{\frac{1}{\delta}-1}\left[x^{\delta-1} - (1-x)^{\delta-1}\right] \geq 0$$

$$\Leftrightarrow \left[x^{\delta-1} - (1-x)^{\delta-1}\right] \geq 0 \Leftrightarrow 1 \geq \left(\frac{x}{1-x}\right)^{1-\delta} \Leftrightarrow x \leq \frac{1}{2}$$

It follows that in (2.20) the denominator is positive and the sign depends on $p^\gamma - q^\delta$. If we start from the zero curve $\omega(p,q) = 0 \Leftrightarrow p^\gamma - q^\delta = 0$, that is the $\widehat{OB}$ curve in 2.5 , it is clear that an increase in p will bring us in the domain in which the function (2.20) is positive (OAB "triangle") while an increase in q will bring us in the domain in which the function is negative (OBC "triangle") and then, in this case, the function (2.20) is increasing in p and decreasing in q. Now it is sufficient to prove that $\omega(p,q)$ is increasing in p and decreasing in q within the two triangles, i.e. where $\omega(p,q) > 0$ (< 0) and $p,q > 0$. If $\omega(p,q) > 0$, and then if $p^\gamma - q^\delta > 0$ and since the function $Ln(x)$ is strictly increasing, it is sufficient to prove that $Ln(\omega(p,q))$ is increasing in p and decreasing in q. By differentiating w. r. t. the first variable:

$$\frac{\partial Ln\left[\omega(p,q)\right]}{\partial p} = \frac{\gamma p^{\gamma-1}}{p^\gamma - q^\delta} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot$$

$$\cdot \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1} \quad (2.21)$$

If $1/2 \le p < 1 \rightarrow \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \le 0$ and the (2.21) is positive. Suppose $0 < p < 1/2$, than the first summand in the (2.21) is positive and the second is negative. We have the following decreasing sequence:

$$\frac{\partial Ln\left[\omega(p,q)\right]}{\partial p} = \frac{\gamma p^{\gamma-1}}{p^{\gamma} - q^{\delta}} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot$$

$$\cdot \frac{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{\frac{1}{\gamma}} + \left[q^{\delta} + (1-q)^{\delta}\right]^{\frac{1}{\delta}} - 1} \ge^{1}$$

$$\ge \frac{\gamma p^{\gamma-1}}{p^{\gamma}} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot \frac{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{\frac{1}{\gamma}} + \left[q^{\delta} + (1-q)^{\delta}\right]^{\frac{1}{\delta}} - 1} \ge^{2}$$

$$\ge \frac{\gamma p^{\gamma-1}}{p^{\gamma}} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot \frac{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{\frac{1}{\gamma}}} =$$

$$= \frac{\gamma p^{\gamma-1}}{p^{\gamma}} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot \frac{1}{p^{\gamma} + (1-p)^{\gamma}} \ge^{3}$$

$$\ge \frac{\gamma\left(\frac{1}{p}\right)^{1-\gamma}}{p^{\gamma}} - \frac{\left(\frac{1}{p}\right)^{1-\gamma}}{p^{\gamma} + (1-p)^{\gamma}} = \left(\frac{1}{p}\right)^{1-\gamma} \cdot \left[\frac{\gamma}{p^{\gamma}} - \frac{1}{p^{\gamma} + (1-p)^{\gamma}}\right]$$

Then in order to prove that the (2.21) is non negative it is sufficient to show that the quantity in the last square bracket is non negative

$$\frac{\gamma}{p^{\gamma}} - \frac{1}{p^{\gamma} + (1-p)^{\gamma}} = \frac{\gamma\left[p^{\gamma} + (1-p)^{\gamma}\right] - p^{\gamma}}{p^{\gamma}\left[p^{\gamma} + (1-p)^{\gamma}\right]} \ge 0 \Leftrightarrow \gamma\left[p^{\gamma} + (1-p)^{\gamma}\right] - p^{\gamma} \ge 0$$

$$\Leftrightarrow \gamma(1-p)^{\gamma} \ge (\gamma)p^{\gamma} \Leftrightarrow \left(\frac{1-p}{p}\right)^{\gamma} \ge \frac{1-\gamma}{\gamma} \Leftrightarrow \frac{1-p}{p} \ge \left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{\gamma}}$$

Remembering that we are under the limitation $0 < p < 1/2$ the first term is greater than one and then the last is true if

$$\left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{\gamma}} \leq 1 \;\Leftrightarrow\; \gamma \geq \frac{1}{2}$$

and this is ensured by the hypothesis of proposition 1.

At this time we have proved that if $\omega(p,q) > 0$ then the function $\omega(p,q)$ is increasing in p. A dual demonstration will prove that if $\omega(p,q) < 0$ then the function is decreasing in q, i.e. the function $-\omega(p,q)$ is increasing in q. For this it is sufficient to exchange p with q and γ with δ and to repeat the previous passages. Now, in the case $\omega(p,q) > 0$ we turn out our attention to the first derivative with respect to q

$$\frac{\partial Ln\left[\omega(p,q)\right]}{\partial q} = \frac{-\delta q^{\delta-1}}{p^{\gamma}-q^{\delta}} - \left[\left(\frac{1}{q}\right)^{1-\delta} - \left(\frac{1}{1-q}\right)^{1-\delta}\right] \cdot \\ \cdot \frac{\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}-1}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}} + \left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}} - 1} \tag{2.22}$$

If $\left[\left(\frac{1}{q}\right)^{1-\delta} - \left(\frac{1}{1-q}\right)^{1-\delta}\right] \geq 0 \;\Leftrightarrow\; q \leq 1/2$ then the (2.22) is negative. Supposing $q > 1/2$ the first summand in the (2.22) is negative and the second is positive. Note that if $\gamma \geq \delta$ the curve which equation is $p^{\gamma} - q^{\delta} = 0$ coincides with the graph of the function $q = p^{\frac{\gamma}{\delta}}$ that is convex, like in 2.5, and within the domain

$$A^{+} = \left\{(p,q) \in [0;1] \times [0;1] \text{ such that } p+q \geq 1 \text{ and } p^{\gamma} - q^{\delta}\right\}$$

it is impossible that $q > 1/2$ and so we have finished the proof. On the other hand if $\gamma < \delta$ the graph of the function $q = p^{\frac{\gamma}{\delta}}$ is concave and within the domain A^+ there are points such that $q > 1/2$. For these reasons, from here we will suppose $q > 1/2$ and $\gamma < \delta$ and we will refer to 2.6.

From a sequence of increases it results:

$$\frac{\partial Ln\left[\omega(p,q)\right]}{\partial q} \leq {}^{4} \leq \frac{-\delta q^{\delta-1}}{p^{\gamma}-q^{\delta}} - \left[\left(\frac{1}{q}\right)^{1-\delta} - \left(\frac{1}{1-q}\right)^{1-\delta}\right] =$$

$$= q^{\delta-1}\left[\frac{-\delta}{p^{\gamma}-q^{\delta}} + \left(\frac{q}{1-q}\right)^{1-\delta} - 1\right]$$

Then it is sufficient to prove that

$$\frac{-\delta}{p^{\gamma}-q^{\delta}} + \left(\frac{q}{1-q}\right)^{1-\delta} - 1 \;\leq\; 0 \;\Leftrightarrow\; \left(\frac{q}{1-q}\right)^{1-\delta} \;\leq\; 1 + \frac{-\delta}{p^{\gamma}-q^{\delta}}$$

and this will follow from:

$$\frac{q}{1-q} \;\leq\; 1 + \frac{-\delta}{p^{\gamma}-q^{\delta}}$$

since

$$q > \frac{1}{2} \;\Rightarrow\; \frac{q}{1-q} > 1 \;\Rightarrow\; \left(\frac{q}{1-q}\right)^{1-\delta} \;\Rightarrow\; \frac{q}{1-q}$$

Summarizing, for our scope we must prove that

$$\frac{q}{1-q} \;\leq\; 1 + \frac{-\delta}{p^{\gamma}-q^{\delta}} \tag{2.23}$$

Under the restrictions we are working with, it is possible to elicit some limitations of the variables p, q, γ and δ. We have supposed $p^{\gamma} - q^{\delta} > 0$, $q > 1/2$ and $\delta > \gamma$, that in the 2.6 delimit the area ABC. Since the curvature of

$p^\gamma - q^\delta = 0$ is more accentuate when larger is the difference between γ and δ, a limit is, for us, the curve $p^{0.5} - q^1 = 0$, i.e. $q = \sqrt{p}$, which delimits the area ADE containing the area ABC. This consideration allows us to elicit some sure limitations for p and q: the "highest" point is the intersection between $q = \sqrt{p}$ and $p + q = 1$, that is $D(0.38; 0.62)$; the most "left-placed" point is the intersection between $q = \sqrt{p}$ and $q = 0.5$, that is $E(0.25; 0.5)$; we elicit $0.25 < p < 0.5$ and $0.5 < q < 0.62$. Consider the function $p^\gamma - q^\delta$, by differentiating, we can prove that it is increasing in p and δ and decreasing in q and γ, and then, using the elicited parameter limitations we have

$$p^\gamma - q^\delta \le \left(\frac{1}{2}\right)^{0.5} - \left(\frac{1}{2}\right)^1$$

which in turn implies

$$1 + \frac{-\delta}{p^\gamma - q^\delta} \ \ge\ 1 + \frac{\delta}{\left(\frac{1}{2}\right)^{0.5} - \left(\frac{1}{2}\right)^1} \tag{2.24}$$

Finally, the quantity $q/(1-q)$ is increasing in q and then by using the sup limitation of q it follows that

$$\frac{q}{1-q} \ \le\ \frac{0.62}{1-0.62} \tag{2.25}$$

Using (2.24) and (2.25) the (2.23) is true if it is true the:

$$\frac{0.62}{1-0.62} \ \le\ 1 + \frac{\delta}{\left(\frac{1}{2}\right)^{0.5} - \left(\frac{1}{2}\right)^1}$$

which gives $\delta > 0.131$ and this is perfectly within our basic limitations.

Similarly, by exchanging p with q and γ with δ it follows that $\omega(p,q)$ is increasing in p when $\omega(p,q) < 0$.

Q.E.D.

2.4.2 The Latimore, Baker and Witte bi-weighting function

Another widely used weighting function is

$$\pi(p) = \frac{\alpha p^\gamma}{\alpha p^\gamma + (1-p)^\gamma} \tag{2.26}$$

with $\gamma, \alpha > 0$. It was introduced by Lattimore *et al.* (1992); Goldstein and Einhorn (1987) and is known as *linear in log odd form*, since Gonzalez and Wu (1999) proved this property.

We propose the bipolar form of this weighting form:

$$\omega(p,q) = \frac{\alpha(p^\gamma - q^\delta)}{\alpha p^\gamma + (1-p)^\gamma + \alpha q^\delta + (1-q)^\delta} \tag{2.27}$$

with $\alpha > 1/2$ and $0 < \gamma, \delta \leq 1$ These parameter limitations allow us to preserve the space inverse S-shaped form of the function and to prove its bi-monotonicity. We are not worried about these restrictions, since they include many of the parameter estimations given for the form (2.26), as can be seen in table 2.3, (from Bleichrodt and Pinto (2000))

Proof of proposition 2.

For $x \in [0,1]$, $\alpha > 1/2$ and $\gamma \in]0,1]$ it results $f(x) = \alpha x^\gamma + (1-x)^\gamma \geq min\{1, \alpha\} > 1/2$. Since this function is continuous in the closed interval

authors	α	γ
Tversky and Fox (1995)	0.77	0.79
Wu and Gonzalez (1996)	0.84	0.68
Gonzalez and Wu (1999)	0.77	0.44
Abdellaoui (2000) (gains)	0.65	0.60
Abdellaoui (2000) (losses)	0.84	0.65
Bleichrodt and Pinto (2000)	0.816	0.550

Table 2.3: recent estimations of parameters for the (2.26)

$[0,1]$, with $f(0)=1$, $f(1)=\alpha$ and the second derivative is non-positive from zero to one:

$$f''(x)=\gamma(\gamma-1)\alpha x^{\delta-2}+\gamma(\gamma-1)(1-x)^{\delta-2}\le 0$$

It follows that in the (2.27) the denominator is positive under the limitation $\alpha>1/2$. Within its domain the first derivative of the (2.27) with respect to p is :

$$\frac{\partial\omega(p,q)}{\partial p}=\alpha\gamma\frac{(1-p)^{\gamma-1}\left(p^{\gamma-1}-q^{\delta}\right)+p^{\gamma-1}\left[2\alpha q^{\delta}+(1-q)^{\delta}-1\right]}{\left[\alpha p^{\gamma}+(1-p)^{\gamma}+\alpha q^{\delta}+(1-q)^{\delta}-1\right]^2} \tag{2.28}$$

Having chosen $\gamma\le 1$ the term $p^{\gamma-1}\ge 1$ for all $p\in]0,1]$ and since $q^{\delta}\le 1$ then $p^{\gamma-1}-q^{\delta}\ge 0$. On the other hand $(2\alpha q^{\delta}+1-q)^{\delta}-1\ge 0$ since for $x\in[0,1]$, $\alpha>1/2$ and $0<\delta\le 1$ the function $f(x)=2\alpha x^{\delta}+(1-x)^{\delta}\ge min\{1,2\alpha\}\ge 1$ since it is continuous in the closed interval $[0,1]$, with $f(0)=1$, $f(1)=2\alpha$ and the second derivative is non-positive from zero to one:

$$f''(x)=\gamma(\gamma-1)2\alpha x^{\delta-2}+\gamma(\gamma-1)(1-x)^{\delta-2}\le 0$$

Then (2.28) is non-negative and the (2.27) is increasing in p.

The first derivative with respect to q is

$$\frac{\partial\omega(p,q)}{\partial q} = \alpha\delta\frac{(1-q)^{\delta-1}\left(p^{\gamma}-q^{\delta-1}\right)-q^{\delta-1}\left[2\alpha p^{\gamma}+(1-p)^{\gamma}-1\right]}{\left[\alpha p^{\gamma}+(1-p)^{\gamma}+\alpha q^{\delta}+(1-q)^{\delta}-1\right]^{2}} \tag{2.29}$$

By the same argumentations, it is easy to see that it is non-positive and then the (2.27) is decreasing in q.

Q.E.D.

2.4.3 The Prelec bi-weighting function

One of the most famous alternative to the classical weighting function of Tversky and Kahneman (1992) is the *compound-invariant* form of Prelec (1998), which has two variants, with two parameters:

$$\pi(p) = e^{-\beta(-Lnp)^{\alpha}} \tag{2.30}$$

and with one parameter

$$\pi(p) = e^{-(-Lnp)^{\alpha}} \tag{2.31}$$

where $\beta \approx 1$ is variable for gains and for losses and $0 < \alpha < 1$. The Prelec weighting function is undefined for $p = 0$ but it is extended by continuity to the value of zero. We propose the following bi-weighting form:

$$\omega(p,q) = \frac{p^{\gamma}-q^{\delta}}{|p^{\gamma}-q^{\delta}|}e^{-\beta(-ln|p^{\gamma}-q^{\delta}|)^{\alpha}} \tag{2.32}$$

with $\beta \approx 1$; $\gamma, \delta > 0$ and $0 < \alpha < 1$. The term $\frac{p^\gamma - q^\delta}{|p^\gamma - q^\delta|}$ means ± 1, respectively if we are within the OBA or OBC "triangle" of figure 2.5. The (2.32) is undefined if $p^\gamma - q^\delta = 0$, in this case we extend the function by continuity setting $\omega(p,q) = 0$, following the original procedure. For the sake of simplicity we choose $\beta = 1$, moreover the two parameters γ and δ have the obvious motivation that we do not wish that $\omega(p,p) = 0$ necessarily. Note that $|p^\gamma - q^\delta| \in [0,1]$ and then the logarithm is non positive.

Proof of proposition 3.

If we start from the zero curve $\omega(p,q) = 0 \Leftrightarrow p^\gamma - q^\delta = 0$ that is the $\widehat{OB}$ curve in 2.5, it is clear that an increasing in p will bring them in the domain in which the function is positive (OAB "triangle") while an increasing in q will bring them in the domain in which the function is negative (OBC "triangle") and then, in this case, the function (2.32) is increasing in p and decreasing in q. Now it is sufficient to prove that $\omega(p,q)$ is increasing in p and decreasing in q within the two triangle, i.e. where $\omega(p,q) > 0$ or $\omega(p,q) < 0$ and $p, q > 0$. If $w(p,q) > 0$ and then if $p^\gamma - q^\delta > 0$ the (2.32) becomes: $\omega(p,q) = e^{-[-Ln(p^\gamma - q^\delta)]^\alpha}$ and by differentiating w. r. t. the two variables:

$$\frac{\partial \omega(p,q)}{\partial p} = e^{-[-Ln(p^\gamma - q^\delta)]^\alpha} \alpha \left[-Ln\left(p^\gamma - q^\delta\right)\right]^{\alpha-1} \frac{\gamma p^{\gamma-1}}{p^\gamma - q^\delta} > 0$$

$$\frac{\partial \omega(p,q)}{\partial p} = e^{-[-Ln(p^\gamma - q^\delta)]^\alpha} \alpha \left[-Ln\left(p^\gamma - q^\delta\right)\right]^{\alpha-1} \frac{-\delta q^{\delta-1}}{p^\gamma - q^\delta} < 0$$

This proves the property within the triangle OBA, where $\omega(p,q) > 0$. Similarly if $p^\gamma - q^\delta < 0$ the (2.32) becomes: $\omega(p,q) = -e^{-[-Ln(-p^\gamma + q^\delta)]^\alpha}$ and by differentiating w. r. t. the two variables:

$$\frac{\partial\omega(p,q)}{\partial p} = -e^{-\left[-Ln\left(-p^{\gamma}+q^{\delta}\right)\right]^{\alpha}}\alpha\left[-Ln\left(p^{\gamma}-q^{\delta}\right)\right]^{\alpha-1}\frac{-\gamma p^{\gamma-1}}{-p^{\gamma}+q^{\delta}} > 0$$

$$\frac{\partial\omega(p,q)}{\partial p} = -e^{-\left[-Ln\left(-p^{\gamma}+q^{\delta}\right)\right]^{\alpha}}\alpha\left[-Ln\left(p^{\gamma}-q^{\delta}\right)\right]^{\alpha-1}\frac{\delta q^{\delta-1}}{-p^{\gamma}+q^{\delta}} < 0$$

We conclude that the Prelec bi-weighting function has the requested property to be increasing in its first argument and decreasing in the second, for all the parameter values.

Q.E.D.

Notes

[1]since

$$\frac{\gamma p^{\gamma-1}}{p^{\gamma}-q^{\delta}} > \frac{\gamma p^{\gamma-1}}{p^{\gamma}}$$

[2]since from

$$\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}}-1\geq 0 \to \frac{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}}+\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}}-1} \leq \frac{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}}} \to$$

$$-\left[\left(\frac{1}{p}\right)^{1-\gamma}-\left(\frac{1}{1-p}\right)^{1-\gamma}\right]\frac{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}}+\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}}-1} \geq -\left[\left(\frac{1}{p}\right)^{1-\gamma}-\left(\frac{1}{1-p}\right)^{1-\gamma}\right]\frac{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}-1}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}}}$$

[3]since

$$-\left(\frac{1}{p}\right)^{1-\gamma} \leq -\left[\left(\frac{1}{p}\right)^{1-\gamma}-\left(\frac{1}{1-p}\right)^{1-\gamma}\right] \leq 0$$

[4] since from

$$1/2 < \gamma,\delta \leq 1 \;\to\; \left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}}-1\geq 0 \text{ and } \left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}}\geq 1 \to$$

$$\frac{\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}-1}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{\frac{1}{\gamma}}+\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}}-1} \leq \frac{\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}-1}}{\left[q^{\delta}+(1-q)^{\delta}\right]^{\frac{1}{\delta}}} = \frac{1}{q^{\delta}+(1-q)^{\delta}} \leq 1$$

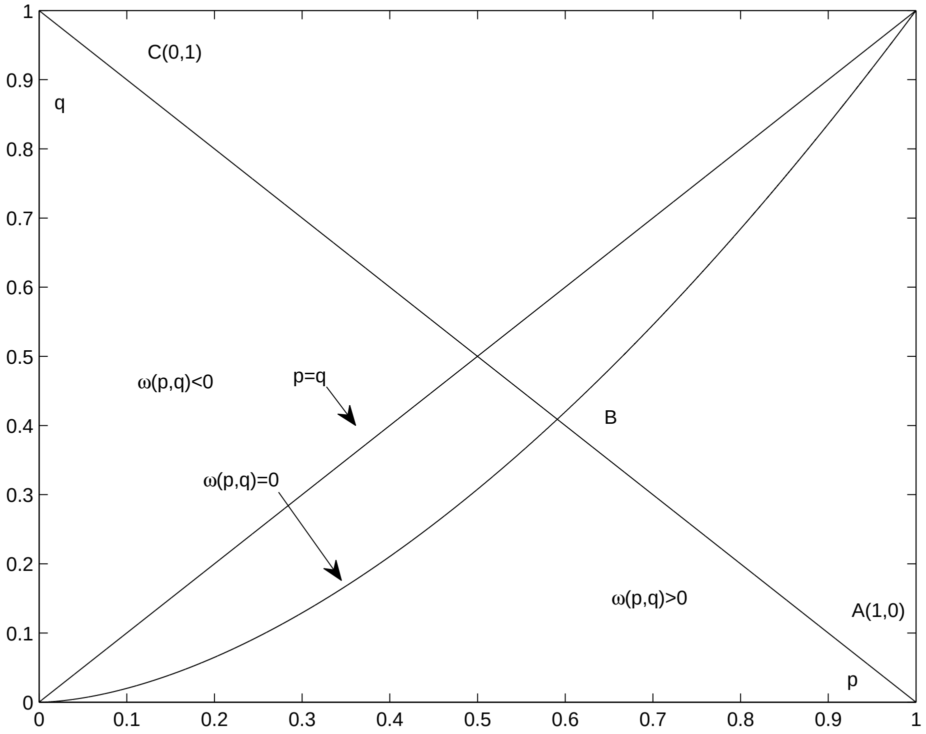

Figure 2.5: the KT bi-weighting function domain; in the case $\gamma > \delta$, the curve $q = p^{\gamma/\delta}$ is convex.

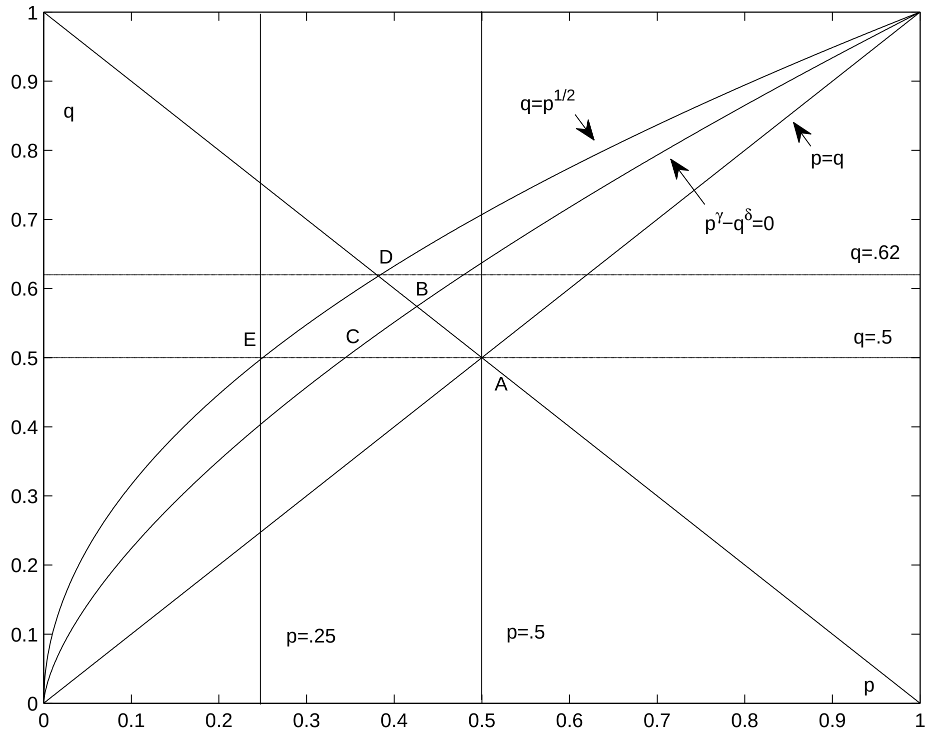

Figure 2.6: if $\gamma < \delta$ the curve $\widehat{OB} : q = p^{\gamma/\delta}$ is concave and its most accentuate curvature is that of $\widehat{OD} : q = \sqrt{p}$. The point $A(.5, .5)$ is the intersection between the lines $p = q$ and $p + q = 1$; the point B is the intersection between $p^\gamma - q^\delta = 0$ and $p + q = 1$; the point C is the intersection between $p^\gamma - q^\delta = 0$ and $q = .5$; the point $D(.38, .62)$ is the intersection between $q = \sqrt{p}$ and $p + q = 1$; the point $E(.25, .5)$ is the intersection between $q = \sqrt{p}$ and $q = .5$.

2.5 Concluding remarks

Any author, testing a theory, tries to bring the respondents in the boundary zones of their mental decision process, in the context of that theory. Generally, the use of moderate probability should avoid the *framing effect* and the *certainty effect* that occur with extreme probabilities. In CPT, if we think to the inverse S-shaped probability weighting function, in the field of moderate probabilities the model is very sensible. In fact as well as the small probabilities are over-estimated, so the moderate probabilities are strongly under-estimated. In this sense Levy and Levy (2002) opened the way, finding the first data involving moderate probabilities which caused problems (in prediction) to CPT. Despite just an year later Wakker (2003) demonstrated how the theory could accommodate for the data of Levy and Levy (2002), properly setting the parameters, successively different authors Baltussen *et al.* (2006); Birnbaum and Bahra (2007); Wu and Markle (2008) found the perfect mixture of gains, losses and associated probabilities to create unsolvable problems to CPT. The most relevant of this is the violation of *Gain Loss Separability* (GLS). GLS means that people in choosing between mixed prospects, evaluate separately gains from losses and then obtain a general evaluation by summing the two results. In this chapter we have extended the CPT model to the bCPT. In bCPT gains and losses within a mixed prospect are evaluated conjointly and not separately as in CPT. The most important paper denouncing the violation of GLS is that of Wu and Markle (2008). In their study it seems to appear a phenomenon that we should like to call *Gain-Loss-Hedging* (GLH). If we look trough, from up to down, the Table 1 of page 1326 in Wu and Markle (2008), with the preferences elicited from the reported percentages, it seems like if the Gain Loss Hedging first appears, denounced by means of the reversed preferences, and then disappears or, lose intensity so to be not enough strong to reverse the preferences. In the next chapter we will see in detail as the bCPT model seems to naturally capture the essence of the phenomenon. By way of example, we have just seen how bCPT is able to cover what we called the "Wu-Markle" paradox. We opened this chapter remembering how great, in the last years, has been the successful of the CPT model of Tversky and Kahneman (1992) to propose itself like the most valid alternative to the classical EUT. So the last thing we want is to renounce to these successes. In this view, our model is though so to coincide with the original in two distinct hypothesis. First if all the gambles involved in the choosing process are not mixed, i.e. they do not contain at the same time gains and losses. Second, using a *separable bi-weighting function* the bCPT model collapses to the CPT model. The use of a separable bi-weighting function corresponds to the axiomatic condition that people evaluate separately gains and losses, contained in the same mixed gamble.

Chapter 3

Explanation of some recent paradoxes against CPT

3.1 Recent literature against CPT

As discussed in the previous chapter this study aims to generalize CPT, in the most natural way, allowing gains and losses within a mixed prospect to be evaluated conjointly rather than separately. In this chapter we will see how our generalization is able to account for some paradoxes found in the recent literature. The major critique regards the most distinctive aspect of the model, the separate valuations of gains and losses (GLS). We have just argued how and why we retain the gain-loss separation at a perceptive level, in this preserving the characteristic S-shaped utility function of CPT, while we present a generalized way to evaluate the risky prospects, where gains and losses are estimated conjointly. In the following we shall focus our attention on two recent papers: Wu and Markle (2008) and Birnbaum and

Bahra (2007). Both of them report violations, by part of CPT, of the GLS. We will see how the bCPT is able to capture, at least partially, these errata predictions.

3.2 Wu and Markle (2008)

In table 1 of page 1326 the authors show several reversals between preferences for mixed gambles and their negative and positive parts. A mixed gamble H is preferred to a mixed gamble L, but the gain and the loss portions of L are preferred to the gain and the loss portions of H. This pattern of choice is inconsistent with CPT, since for any prospect P and for any parameters values setting: $V_{CPT}(P) = V_{CPT}(P^{+}) + V_{CPT}(P^{-})$. We prefer to say that the preferences are reversed in the other sense, i.e. there is an inversion when a person prefers the pure gamble L^{+} to the pure gamble H^{+} and prefers the pure gamble L^{-} to the pure gamble H^{-} but she prefers the mixed gamble H to the mixed gamble L. This, since the choices between the pure gambles involve less parameters and then, we think, are more "genuine" respect to the choice between the mixed gambles, where it seems that has to be considered a sort of Gain-Loss-Hedging (GLH). In our opinion the core of the Wu-Markle paper is to have found the more precise mixture (or combination) between gains, losses and their probabilities in the same prospects to denounce this GLH by means of the reversed preferences in the passage from the pure gambles to their combination in mixed gambles. In 3.1 we reproduce the Table 1 of page 1326 in Wu and Markle (2008) with the preferences elicited from the reported percentages found by the authors. In many cases

(tests 6,7, 10-18) the respondents preferred (in percentage) H to L while splitting the prospects into their respective positive and negative part the preferences were reversed, violating GLS.
To test our model we have used the bCPT functional

$$V_{bCPT}(P) = \int_0^\infty \omega\left(\sum_{i:u(x_i)\geq t} p_i, \sum_{i:u(x_i)\leq -t} p_i\right) dt \tag{3.1}$$

the KT bi-weighting function

$$\omega(p,q) = \frac{p^\gamma - q^\delta}{[p^\gamma + (1-p)^\gamma]^{\frac{1}{\gamma}} + [q^\delta + (1-q)^\delta]^{\frac{1}{\delta}} - 1} \tag{3.2}$$

with parameters $\gamma = 0.9$ and $\delta = 0.89$ and the classical KT power utility function

$$u(x) = \begin{cases} x_+^\alpha & \text{if } x \geq 0 \\ -\lambda(-x)_-^\alpha & \text{if } x < 0 \end{cases} \tag{3.3}$$

with parameters $\lambda = 1.77$, $\alpha_+ = 0.68$, and $\alpha_- = 0.79$.

As can be seen in table 3.1, our data are in the same directions of the preferences in all the pure positive choices except that in tests 13,23 and 25, in all the pure negative choices except in tests 9, 12-15, 17 and 19 and in all the mixed choices except in tests 3, 5 and 20. But, what we think is very interesting, is that the model is able to accommodate for the reversed preferences, totally in tests 6, 7, 10, 11, 16 and 18 and partially in test 12, 14, 15, and 17. The KT bi-weighting function seems able to predict the final choice, i.e. between the mixed gambles, and at the same time to naturally capture (totally or partially) what we have called the GLH, when this phenomenon appears. Future researches should be focused on the best fitting data in the set of the bi-weighting functions or on the discovery of new bi-weighting functions, that are non necessarily the extension of the well known weighting functions. Wu and Markle (2008), in their conclusions, suggest that "*the observed choice patterns are consistent with a process in which individuals are less sensitive to probability differences when choosing among mixed gambles than when choosing among either gain or loss gambles".*

So to accommodate for the violation of GLS it must be used different parameters in the same probability weighting function, for mixed and pure gambles. But this process could generate, we think, non-transitivity problems if we should choose in a set containing mixed as well as pure gambles. In Wu and Markle (2008), the chooses involve only mixed gambles or pure (co-signed) gambles, but if we should choose between a mixed and a pure gamble, what kind of parameters we should use? Those designed for mixed gambles or those designed for pure gambles? Another question is that, to admission of the same authors, the tests are thought to favorite the violation of GLS: *"the special configuration in table 1 shows how the highest and the lowest outcomes in H are better than the highest and lowest outcomes in L, so to contribute to suggest a dominance of H over L."*

For all these reasons we think is relevant to have found a function which, for the same parameters value, is able to capture, in many cases, the violation of GLS when it appears. At the end, we want to point out that, in order to find the best fitting set of parameters, we have forced not only those of the bi-weighting function (γ, δ) , but also those of the utility function (α_+, α_- and λ); but if the same authors admit that the test has been build to hard check the CPT model in its peculiarities, it is admissible to use the parameters in all their elasticity, since we are in a boundary zone in the mind of the Decision Makers in their choice process.

Table 3.1: application of bCPT to the data of Wu and Markle (2008)

Test	H gamble				L gamble				choice %			preferences			bCPT		
	g	p	l	1-p	g′	p′	l′	1-p′	H	H+	H-						
1	150	0,3	-25	0,7	75	0,8	-60	0,2	22	10	17	G+	G-	G	G+	G-	G
2	1800	0,05	-200	0,95	600	0,3	-250	0,7	21	17	15	G+	G-	G	G+	G-	G
3	1000	0,25	-500	0,75	600	0,5	-700	0,5	28	12	20	G+	G-	G	G+	G-	H
4	200	0,3	-25	0,7	75	0,8	-100	0,2	33	18	22	G+	G-	G	G+	G-	G
5	1200	0,25	-500	0,75	600	0,5	-800	0,5	43	21	25	G+	G-	G	G+	G-	H
6	750	0,4	-1000	0,6	500	0,6	-1500	0,4	51	26	25	G+	G-	HG	G+	G-	H
7	4200	0,5	-3000	0,5	3000	0,75	-6000	0,25	52	15	37	G+	G-	HG	G+	G-	H
8	4500	0,5	-1500	0,5	3000	0,75	-3000	0,25	48	17	47	G+	G-	GH	G+	G-	H
9	4500	0,5	-3000	0,5	3000	0,75	-6000	0,25	58	17	55	G+	H-	H	G+	G-	H
10	1000	0,3	-200	0,7	400	0,7	-500	0,3	51	48	28	G+	G-	HG	G+	G-	H
11	4800	0,5	-1500	0,5	3000	0,75	-3000	0,25	54	33	44	G+	G-	H	G+	G-	H
12	3000	0,01	-490	0,99	2000	0,02	-500	0,98	59	42	36	G+	G-	H	G+	H-	H
13	2200	0,4	-600	0,6	850	0,75	-1700	0,25	52	38	42	G+	G-	HG	H+	H-	H
14	2000	0,2	-1000	0,8	1700	0,25	-1100	0,75	58	34	48	G+	G-	H	G+	H-	H
15	1500	0,25	-500	0,75	600	0,5	-900	0,5	51	51	33	GH+	G-	HG	H+	H-	H
16	5000	0,5	-3000	0,5	3000	0,75	-6000	0,25	65	43	43	G+	G-	H	G+	G-	H
17	1500	0,4	-1000	0,6	600	0,8	-3500	0,2	59	48	41	G+	G-	H	G+	H-	H
18	2025	0,5	-875	0,5	1800	0,6	-1000	0,4	72	52	42	G+	G-	H	G+	G-	H
19	600	0,25	-100	0,75	125	0,75	-500	0,25	58	55	44	H+	G-	H	H+	H-	H
20	5000	0,1	-900	0,9	1400	0,3	-1700	0,7	40	47	53	G+	HG-	G	G+	G-	H
21	700	0,25	-100	0,75	125	0,75	-600	0,25	71	59	48	H+	H-	H	H+	G-	H
22	700	0,5	-150	0,5	350	0,75	-400	0,25	63	58	48	H+	GH-	H	H+	H-	H
23	1200	0,3	-200	0,7	400	0,7	-800	0,3	70	59	50	H+	H-	H	G+	H-	H
24	5000	0,5	-2500	0,5	2500	0,75	-6000	0,25	79	54	54	H+	H-	H	H+	H-	H
25	800	0,4	-1000	0,6	500	0,6	-1600	0,4	58	64	51	H+	H-	H	G+	H-	H
26	5000	0,5	-3000	0,5	2500	0,75	-6500	0,25	71	61	59	H+	H-	H	H+	H-	H
27	700	0,25	-100	0,75	100	0,75	-800	0,25	73	58	64	H+	H-	H	H+	H-	H
28	1500	0,3	-200	0,7	400	0,7	-1000	0,3	75	59	63	H+	H-	H	H+	G-	H
29	1600	0,25	-500	0,75	600	0,5	-1100	0,5	73	60	69	H+	H-	H	H+	H-	H
30	2000	0,4	-800	0,6	600	0,8	-3500	0,2	65	66	63	H+	H-	H	H+	H-	H
31	2000	0,25	-400	0,75	600	0,5	-1100	0,5	80	63	69	H+	H-	H	H+	H-	H
32	1500	0,4	-700	0,6	300	0,8	-3500	0,2	78	64	68	H+	H-	H	H+	H-	H
33	900	0,4	-1000	0,6	500	0,6	-1800	0,4	70	74	61	H+	H-	H	H+	H-	H
34	1000	0,4	-1000	0,6	500	0,6	-2000	0,4	78	71	70	H+	H-	H	H+	H-	H

$\alpha_+ = 0.68$, $\alpha_- = 0.79$, $\delta = 0.89$ $\lambda = 1.77$, $\gamma = 0.9$

3.3 Birnbaum-Bahra

In Birnbaum and Bahra (2007) the authors reported systematic violations of two behavioral properties implied by CPT, one is the just discussed GLS and the other is the property known as coalescing:

"*coalescing is the assumption that if there are two probability-consequences branches in a gamble leading to the same consequence, they can be combined by adding their probabilities.*"

For example, the three-branch gamble $A = (\$100, 0.25; \$100, 0.25; \$0, 0.5)$ is equivalent to the two-branch gamble $A' = (\$100, 0.5; \$0, 0.5)$. Our model is not able to accommodate for violation of coalescing, but we want to point out that, in their paper, Birnbaum and Bahra tested violation of coalescing presenting to the participants the gambles in terms of a container holding exactly 100 marbles of different colors. So, according to coalescing, $B' = (25 \text{ red to win } \$100; 75 \text{ white to win } \$0)$ should be considered equivalent to $B = (25 \text{ red to win } \$100; 25 \text{ white to win } \$0; 50 \text{ white to win } \$0)$. We are not sure that to present the gambles in this form is the same that to present the gambles with the cleared (numerically) probabilities, since any person which faces up B will ask himself what is the reason that the first 25 white marbles were not summed to the second 50 white marbles, it is admissible that she will think if they differ in some way. In any case, she will have an additional information or doubt to process and this could generate errors. As focused from Wu and Markle (2008) the examples of Birnbaum and Bahra (2007) to underline the GLS violation (implied from CPT) are less simple than theirs, but our model is able to accommodate for these violations modifying only the parameter γ from the value of 0.9 used to accommodate the majority of data in Wu and Markle (2008) to the value of 0.74. Next we report the part of the table 5 at page 1022 in Birnbaum and Bahra (2007) that, in the words of the same authors, form a test for the GLS. As we have said each gamble "*is described in terms of a container holding exactly 100 marbles of different colors, from which one marble would be drawn at random, and the color of that marble would determine the prize.*".

In the brackets are shown the percentages of each choose.

$$\mathcal{F} = \begin{pmatrix} 25\ \textit{black} \\ \textit{to win}\ \$100 \\ \\ 25\ \textit{white} \\ \textit{to win}\ \$0 \\ \\ 50\ \textit{pink} \\ \textit{to lose}\ \$50 \end{pmatrix} \succ \begin{pmatrix} 50\ \textit{blue} \\ \textit{to win}\ \$50 \\ \\ 25\ \textit{white} \\ \textit{to lose}\ \$0 \\ \\ 25\ \textit{red} \\ \textit{to lose}\ \$100 \end{pmatrix} = \mathcal{G}$$

[76%] [24%]

$$\mathcal{F}^+ = \begin{pmatrix} 25\ \textit{black} \\ \textit{to win}\ \$100 \\ \\ 25\ \textit{white} \\ \textit{to win}\ \$0 \\ \\ 50\ \textit{white} \\ \textit{to win}\ \$0 \end{pmatrix} \prec \begin{pmatrix} 25\ \textit{blue} \\ \textit{to win}\ \$50 \\ \\ 25\ \textit{blue} \\ \textit{to win}\ \$50 \\ \\ 50\ \textit{white} \\ \textit{to win}\ \$0 \end{pmatrix} = \mathcal{G}^+$$

[29%] [71%]

$$\mathcal{F}^- = \begin{pmatrix} 50\ \textit{white} \\ \textit{to lose}\ \$0 \\ \\ 25\ \textit{white} \\ \textit{to lose}\ \$0 \\ \\ 25\ \textit{red} \\ \textit{to lose}\ \$100 \end{pmatrix} \prec \begin{pmatrix} 25\ \textit{black} \\ \textit{to win}\ \$100 \\ \\ 25\ \textit{white} \\ \textit{to win}\ \$0 \\ \\ 50\ \textit{pink} \\ \textit{to loss}\ \$50 \end{pmatrix} = \mathcal{G}^-$$

$$[35\%] \qquad\qquad [65\%]$$

As can be seen $\mathcal{F}$ is preferred to $\mathcal{G}$, but when the two prospects are split in their respective positive and negative parts (according to coalescing) a relevant majority prefers $\mathcal{G}^+$ to $\mathcal{F}^+$ and $\mathcal{G}^-$ to $\mathcal{F}^-$. This violation of the GLS is clearly inconsistent with CPT. In order to evaluate these prospects we substitute the respective probabilities to the colors, so do the authors, by dividing for 100 any number of color within the prospects. Using the bipolar CPT with the bi-polarized KT weighting functions with parameters $\gamma = 0.74$, $\delta = 0.89$

$$\omega(p,q) = \frac{p^{0.74} - q^{0.89}}{[p^{0.74} + (1-p)^{0.74}]^{\frac{1}{0.74}} + [q^{0.89} + (1-q)^{0.89}]^{\frac{1}{0.89}} - 1}$$

and the classical KT power utility function with parameters $\lambda = 1.77$, $\alpha = 0.68$ and $\beta = 0.79$

$$u(x) = \begin{cases} x^{0.68} & \text{if } x \geq 0 \\ -1.77(-x)^{0.79} & \text{if } x < 0 \end{cases}$$

we obtain

$$V_{bCPT}(\mathcal{F}) = -11.07 \quad \geq \quad V_{bCPT}(\mathcal{G}) = -11.11$$

$$V_{bCPT}(\mathcal{F}^+) = 6.67 \quad \leq \quad V_{bCPT}(\mathcal{G}^+) = 6.71$$

$$V_{bCPT}(\mathcal{F}^-) = -19.28 \quad \leq \quad V_{bCPT}(\mathcal{G}^-) = -18.25$$

These results agree with the preference relation $\succsim$ and Gain Loss Separability is naturally covered. Again as in Wu and Markle (2008) we remark that there are no many models that are able to cover such pattern of choice.

3.4 Concluding remarks

Both in Wu and Markle (2008); Birnbaum and Bahra (2007) there are systematic violations of Gain Loss Separability. These data seems to support the hypothesis that people process in different ways mixed prospects and non-mixed prospects. If the cited authors have suggested to use different parametrization of the same model, we have shown how it is possible to cover their "paradoxes" by using the same model, bCPT, without changing the parameters setting in the passage from mixed prospects to prospects containing all gains or all losses.

Chapter 4

The bipolar Choquet Integral

4.1 Introduction

In the previous chapters we have seen a generalization of the Cumulative Prospect Theory (CPT) in the field of *risk*, i.e. when probabilities are assigned. We called this model the bipolar Cumulative Prospect Theory (bCPT) and we showed how it is able to accommodate for recent paradoxes regarding the CPT and, particularly, its most distinctive aspect, the separate evaluation of gains and losses. In this chapter we will extend the bCPT model from the field of *risk* to that of *uncertainty*, where the Decision Maker (DM) faces *events* with non-assigned probabilities. In this context she must use her subjective beliefs regarding the plausibility of any event, assigning to it a *subjective probability.* CPT can be extended from the field to decision under risk to that of decision under uncertainty by means of the two concepts of *capacity* and *Choquet integral with respect to a capacity* (Choquet (1953); Schmeidler (1986)). Similarly we extend bCPT by means of the two generalized concepts of *bi-capacity* and *bipolar Choquet integral with respect to a bi-capacity* (Grabisch and Labreuche (2005a,b); Greco *et al.* (2002)). A second problem we will face in this chapter is that of axiomatize bCPT, i.e. to find a preference foundation for the model. bCPT can be axiomatized separately in the field of decision under risk and in the field of decision under uncertainty. The main tool to axiomatize the model in an uncertainty context is the *bipolar Choquet integral*. In the following we present a fairly simple characterization of the *bipolar Choquet integral* (following Schmeidler (1986) for the Choquet integral).

4.2 Extension of bCPT to uncertainty

4.2.1 Bi-capacity and bipolar Choquet integral

In order to extend bCPT from the field of risk to that of uncertainty we need to generalize the concept of capacity and Choquet integral with respect to a capacity.
Let $\mathcal{S}$ be a non-empty set of states of the world and Σ an algebra of subsets of $\mathcal{S}$ (the *Events*). Let $\mathcal{B}$ denote the set of bounded real-valued Σ−measurable functions on $\mathcal{S}$ and $\mathcal{B}_0$ the set of simple (i.e. finite valued) functions in $\mathcal{B}$.

Definition 8 *A function $\nu : \Sigma \to [0,1]$ is a (normalized) capacity on Σ if*

- $\nu(\varnothing) = 0,\ \nu(\mathcal{S}) = 1$
- *for all $A, B \in \Sigma$ such that $A \subseteq B \Rightarrow \quad \nu(A) \leq \nu(B)$*

Choquet (1953) defined an integration operation with respect to the non-necessarily additive set function ν.

Definition 9 *Given a nonnegative valued function $f \in \mathcal{B}$ and a capacity $\nu : \Sigma \to [0,1]$, the Choquet integral of f with respect to ν is*

$$\int_S f(s)\ d\nu =: \int_0^\infty \nu(\{s \in S : f(s) \geq t\})\,dt$$

Successively Schmeidler (1986) extended this definition to all of $\mathcal{B}$

Definition 10 *Given a bounded real valued function $f \in \mathcal{B}$ and a capacity $\nu : \Sigma \to [0,1]$, the Choquet integral of f with respect to ν is*

$$\int_S f(s)\ d\nu =: \int_{-\infty}^0 [\nu(\{s \in S : f(s) \geq t\}) - 1]\,dt + \int_0^\infty \nu(\{s \in S : f(s) \geq t\})\,dt$$

Let us consider the set of all the couples of disjoint events

$$\mathcal{Q} = \{(A,B) \in \Sigma \times \Sigma : A \cap B = \varnothing\}$$

Definition 11 *A function $\mu_b : \mathcal{Q} \to [-1,1]$ is a bi-capacity on Σ if*

- $\mu_b(\varnothing,\varnothing) = 0$, $\mu_b(S,\varnothing) = 1$ *and* $\mu_b(\varnothing,S) = -1$
- $\mu_b(A,B) \leq \mu_b(C,D)$ *for all* $(A,B),(C,D) \in \mathcal{Q}$ *such that* $A \subseteq C \wedge B \supseteq D$

Grabisch and Labreuche (2005a,b); Greco et al. *(2002)*

Definition 12 *The bipolar Choquet integral of a simple function $f \in \mathcal{B}_0$ with respect to a bi-capacity μ_b is given by:*

$$\int_S f(s)\ d\mu_b =: \int_0^\infty \mu_b(\{s \in S : f(s) > t\}, \{s \in S : f(s) < -t\})dt$$

Grabisch and Labreuche (2005a,b); Greco et al. *(2002)*

4.2.2 Two different approaches

Since we are working with *simple acts* $f \in \mathcal{B}_0$ it follows that the set $f(\mathcal{S}) = \{f(s) \mid s \in \mathcal{S}\}$ is finite, so, for the sake of simplicity, let us suppose that $\mathcal{S} = \{s_1, s_2, \cdots, s_n\}$. An uncertain *act* can then be expressed as a vector $f = (f(s_1), s_1; \cdots; f(s_n), s_n)$ where the outcome $f(s_i) \in \mathfrak{R}$ will be obtained if the state of world s_i will occur. As usual let us indicate with f_+ the positive part of f, i.e. $f_+(s) = f(s)$ if $f(s) \geq 0$ and $f_+(s) = 0$ if $f(s) < 0$; similarly f_- indicates the negative part of f obtained from f substituting all the gains with zero. The dual capacity of a capacity $\nu : \Sigma \to [0,1]$ is defined as $\widehat{\nu}(A) = 1 - \nu(A^c)$ for all $A \in \Sigma$.
In order to evaluate the acts let be given an utility function $u(\cdot) : \mathfrak{R} \to \mathfrak{R}$, two capacity (one for gains, one for losses) $\nu_+ : \mathcal{S} \to [0,1]$ and $\nu_- : \mathcal{S} \to [0,1]$ and and a bi-capacity $\mu_b : \mathcal{Q} \to [-1,1]$. By using CPT or bCPT the evaluation of an act $f = (f(s_1), s_1; \cdots; f(s_n), s_n)$ is obtained in different way by means of their respective functionals.
In CPT we sum the Choquet integral of $u(f_+)$ with respect to ν_+ with the Choquet integral of $u(f_-)$ with respect to the dual of ν_-, by getting *a separate*

evaluation of gains and losses

$$V_{CPT}(f) = \int_{\mathcal{S}} u\left[f_+(s)\right] d\nu_+ \;+\; \int_{\mathcal{S}} u\left[f_-(s)\right] d\widehat{\nu_-} \tag{4.1}$$

The (4.1) can be rewritten as

$$V_{CPT}(f) = \int_0^{\infty} \nu_+\left(\{s_j : u(f(s_j)) \geq t\}\right) dt - \int_{-\infty}^{0} \nu_-\left(\{s_i : u(f(x_i)) \leq t\}\right) dt \tag{4.2}$$

In bCPT we calculate the bipolar Choquet integral of $u(f)$ with respect to μ_b getting a *conjointly evaluation of gains and losses*

$$V_{bCPT}(\mathcal{P}) = \int_{\mathcal{S}} u\left[f(s)\right] d\mu_b = \int_0^{+\infty} \mu_b\left(\{s_i : u(x_i) > t\}, \{s_i : u(x_i) < -t\}\right) dt \tag{4.3}$$

In this paragraph it has been underlined how the main difference between CPT and bCPT is in their different approach to the evaluation of an act f. In CPT two different capacities, one for states of world corresponding to gains and one for those corresponding to losses, allow the positive and negative part of f to be evaluated separately and then summed. In bCPT a bi-capacity allows gains and losses within f to be evaluated conjointly. In the next section we will see the situations where the two model coincide and, more precisely, where CPT can be seen as a special case of bCPT. As in a risk-context this fact will occur for non mixed acts or by using a *separable bi-capacity.*

4.2.3 Link between CPT and bCPT

First we wish to discuss some properties and links with previous results. Let us identify $(A,B) \in \mathcal{Q}$ with the double-indicator function $(A,B)^* \in \mathcal{B}_0$ defined as follows:

$$(A,B)^*(s) = \begin{cases} 1 & \text{if } s \in A \\ -1 & \text{if } s \in B \\ 0 & \text{if } s \notin A \cup B \end{cases}$$

Since

$$\int_{\mathcal{S}} (A,B)^* \mu_b = \int_0^1 \mu_b(A,B)dt = \mu_b(A,B) \quad \Rightarrow$$

the functional $\int_{\mathcal{S}} \mu_b$, i.e. the bipolar Choquet integral, can be considered as an extension of the bi-capacity μ_b from $\mathcal{Q}$ to $\mathcal{B}_0$.
If $\mu_b : \mathcal{Q} \to [-1,1]$ is a bi-capacity, then we can define a capacity ν_+ as follows: for all $E \in \Sigma$

$$\nu_+(E) = \mu_b(E, \varnothing)$$

If $f \in \mathcal{B}_0$ is such that $f(s) \geq 0$ for all $s \in S$, then

$$\int_S f(s)\ d\mu_b = \int_0^\infty \mu_b(\{s \in S : f(s) > t\},\ \varnothing\ )\ dt =$$

$$= \int_0^\infty \nu_+(\{s \in S : f(s) > t\})\ dt = \int_S f(s)\ d\nu_+ \quad \Rightarrow$$

Remark 6 *For acts taking only non-negative values, the bipolar Choquet integral collapses to the classical Choquet integral.*

If $\mu_b : \mathcal{Q} \to [-1,1]$ is a bi-capacity, then we can define a capacity ν_- as follows: for all $E \in \Sigma$

$$\nu_- (E) = -\mu_b (\varnothing, E)$$

If $f \in \mathcal{B}_0$ is such that $f(s) \leq 0$ for all $s \in S$, then

$$\int_S f(s)\ d\mu_b = \int_0^\infty \mu_b(\ \varnothing,\ \{s \in S : f(s) < -t\})\ dt =$$

$$= -\int_0^\infty \nu_-(\{s \in S : f(s) < -t\})\ dt = \int_{\mathcal{S}} (f(s)\ )d\widehat{\nu_-} \quad \Rightarrow$$

Remark 7 *For acts taking only non-positive values, the bipolar Choquet integral collapses to the classical Choquet integral*

By using the remarks 6, 7 we can establish the following important relationship between CPT and bCPT (which sense has been clarified in the previous discussion)

Remark 8 *For non-mixed acts, i.e. for acts containing all non positive or non negative values, the bCPT model coincides with the CPT model*

On the other hand, let be given two capacity $\nu_+ : \mathcal{S} \to [0,1]$ and $\nu_- : \mathcal{S} \to [0,1]$, it is straightforward noting that we can define a *separable bi-capacity* by posing for all $(A, B) \in \mathcal{Q}$

$$\mu_b(A, B) = \nu_+(A) - \nu_-(B)$$

Now we can establish the second situation where CPT can be seen as a special case of bCPT. The following prepares

Remark 9 *The bipolar Choquet integral with respect to a separable bi-capacity is the sum of two Choquet integrals*

In fact, let be $f \in \mathcal{B}_0$ a simple function and $\mu_b(A, B) = \nu_+(A) - \nu_-(B)$ a separable bi-weighting function, we get

$$\int_S f(s)\ d\mu_b =: \int_0^\infty \mu_b(\{s \in S : f(s) > t\}, \{s \in S : f(s) < -t\})dt =$$

$$= \int_0^\infty [\nu_+(\{s \in S : f(s) > t\}) - \nu_-(\{s \in S : f(s) < -t\})\ dt] =$$

$$= \int_0^\infty \nu_+(\{s \in S : f(s) > t\})dt \ - \ \int_0^\infty \nu_-(\{s \in S : f(s) < -t\})dt =$$

$$\int_S f_+(s)\ d\nu_+ \ + \ \int_S f_-(s)\ d\widehat{\nu}_-$$

being f_+ (f_-) the non negative (positive) part of the act f and $\widehat{\nu}_-$ the dual capacity of ν. As direct consequence of the last remark we can state that

Remark 10 *The bCPT model with a separable bi-weighting function coincides with the CPT model.*

Another way to state the 10 is to assert that *the CPT model is the bCPT model with a separable bi-capacity.* In this case, CPT will appear as a special case of bCPT. In the remind of this chapter we will face the problem of the preference foundation of bCPT. As we have just seen the main tool to extend bCPT from the field of risk to that of uncertainty is the bipolar Choquet integral with respect to a bi-capacity. We will present a fairly simple characterization of the *bipolar Choquet integral* (following Schmeidler (1986) for the Choquet integral).

4.3 The characterization theorem

4.3.1 Properties and main theorem

In this section we first give the concept of *absolutely co-monotonic and co-signed acts* which are the special acts for which the bipolar Choquet integral is additive. In a second time we will list some properties of the functional. Finally we will state the main theorem, i.e. the characterization theorem. This meaning that we will give some properties which, if satisfied by an assigned functional, will characterize uniquely the bipolar Choquet integral. In this we replay what Schmeidler (1986) did for the Choquet integral.

Definition 13 *Two real valued functions $f, g : \mathcal{S} \to \Re$ are absolutely co-monotonic and cosigned (a.c.c.) if*

- *their absolute value are co-monotonic, i.e.*

$$(\; |f(s)| - |f(t)| \;) \cdot (\; |g(s)| - |g(t)| \;) \geq 0 \quad \forall s, t \in \mathcal{S}$$

- *are co-signed, i.e.*

$$f(s) \cdot g(s) \geq 0 \quad \forall s \in \mathcal{S}$$

Let us suppose that μ_b is a bi-capacity and let us indicate with $\mathcal{I} = \int_{\mathcal{S}} \mu_b$ the bipolar Choquet integral with respect to μ_b. The next proposition list the properties of $\mathcal{I}$. Given to the importance of this section, the proofs of these properties are presented in the main text.

Proposition 6 *The functional $\mathcal{I}$ satisfies the following properties*

- *(P1) Monotonicity.*

$$f(s) \geq g(s) \quad \forall s \in \mathcal{S} \quad \Rightarrow \quad \mathcal{I}(f) \geq \mathcal{I}(g)$$

- *(P2) Positive homogeneity. For all $a > 0$, and f, $a \cdot f \in \mathcal{B}_0$*

$$\mathcal{I}(a \cdot f) = a \cdot \mathcal{I}(f)$$

- *(P3) Bipolar-idem-potency. For all $\lambda > 0$*

$$\mathcal{I}(\lambda(\mathcal{S}, \varnothing)^*) = \lambda \qquad \mathcal{I}(\lambda(\varnothing, \mathcal{S})^*) = -\lambda$$

- *(P4) Additivity for acts a.c.c. If $f, g \in \mathcal{B}_0$ are a.c.c.* $\Rightarrow$

$$\mathcal{I}(f + g) = \mathcal{I}(f) + \mathcal{I}(g)$$

Proof. Supposing $f(s) \geq g(s)$ for all $s \in \mathcal{S}$, then $\{s : f(s) > t\} \supseteq \{s : g(s) > t\}$ and $\{s \in S : f(s) < -t\} \subseteq \{s \in S : g(s) < -t\}$ such that (P1) follows from monotonicity of bicapacity and integral.
For all $a > 0$ and for all $f \in B_0, af \in B_0$, taking $t = az$, by definition we get

$$I(af) = \int_0^\infty \mu_b(\{s \in S : f(s) > \frac{t}{a}\}, \{s \in S : f(s) < -\frac{t}{a}\})dt =$$

$$\int_0^\infty \mu_b(\{s \in S : f(s) > z\}, \{s \in S : f(s) < -z\}) \; adz = aI(f).$$

which is (P2).

For $\gamma > 0$, by homogeneity, $I\left(\gamma(S,\varnothing)^*\right) = \gamma I(S,\varnothing)^* = \gamma\mu_b(S,\varnothing) = \gamma$. If $\gamma < 0$, then $I\left(\gamma(S,\varnothing)^*\right) = -\gamma I(\varnothing,S)^* = -\gamma\mu_b(\varnothing,S) = \gamma$. Note also that $I\left(0(S,\varnothing)^*\right) = I\left((\varnothing,\varnothing)^*\right) = \mu_b(\varnothing,\varnothing) = 0$. $I(\lambda(\varnothing,S)^*) = -\lambda$ can be obtained analogously. Thus (P3) is proved.

Let $f, g \in B_0$ be two acts a.c.c., then following Schmeidler (1986)-remark 4 there exist

- a partition of S into k pairwise disjoint subsets of Σ, $(E_i)_{i=1}^k$, such that for each E_i there exist E_i^+ and E_i^- with $E_i^+ \cup E_i^- = E_i$ and $E_i^+ \cap E_i^- = \varnothing$
- two k-list of numbers $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$ and $0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_k$

such that

$$f = \sum_{i=1}^{k} \alpha_i (E_i^+, E_i^-)^* \qquad , \qquad g = \sum_{i=1}^{k} \beta_i (E_i^+, E_i^-)^*$$

It follows that

$$f + g = \sum_{i=1}^{k} (\alpha_i + \beta_i)(E_i^+, E_i^-)^*$$

By the definition of bipolar Choquet integral,

$$I(f+g) = I(f) + I(g)$$

this proves (P4).

Q.E.D.

The following characterization theorem extends the result with respect to Choquet integral in Schmeidler (1986) to bipolar Choquet integral

Theorem 7 *Let* $\mathcal{J} : \mathcal{B}_0 \to \Re$ *satisfy*

- $\mathcal{J}((S,\varnothing)^*) = 1$ *and* $\mathcal{J}((\varnothing,S)^*) = -1$
- *(P1) Monotonicity*
- *(P4) Additivity for acts a.c.c.*

then, by assuming $\mu_b(A,B) = \mathcal{J}[(A,B)^*] \quad \forall (A,B) \in \mathcal{Q}$,

$$\Downarrow$$

$$\mathcal{J}(f) = \mathcal{I}(f) = \int_S f(s)\ d\mu_b \quad \forall f \in \mathcal{B}_0$$

Remark 11 *The properties (P2), homogeneity of degree one and (P3), bipolar idem-potency, are not among the hypothesis of Theorem 7 since they are implied by additivity for absolutely co-monotonic and cosigned acts (P4) and monotonicity (P1)*

Proof. Let $f \in B_0$ be a simple function with image $f(S) = \{x_1, x_2, \dots, x_n\}$. Let $(\cdot) : N \to N$ be a permutation of indexes in $N = \{1, 2, \dots, n\}$ such that $|x_{(1)}| \le |x_{(2)}| \le \dots \le |x_{(n)}|$. f can be written as sum of double-indicator functions, i.e.

$$f = \sum_{i=1}^{n} \left(\left|x_{(i)}\right| - \left|x_{(i-1)}\right|\right) \left(A(f)_{(i)}, B(f)_{(i)}\right)^*$$

where $A(f)_{(i)} = \left\{s \in S :\ f(s) \ge \left|x_{(i)}\right|\right\}$, $B(f)_{(i)} = \left\{s \in S :\ f(s) \le -\left|x_{(i)}\right|\right\}$ and $\left|x_{(0)}\right| = 0$.

Observe that the simple functions $\left(A(f)_{(i)}, B(f)_{(i)}\right)^*$ for $i = 1, 2, \ldots, n$ are a.c.c., as well as the simple functions $\left(|x_{(i)}| - |x_{(i-1)}|\right)\left(A(f)_{(i)}, B(f)_{(i)}\right)^*$ for $i = 1, 2, \ldots, n$. On the basis of this observation, applying (P4), homogeneity and the definition of $\mu_b(A, B)$ we get the thesis as follows:

$$J(f) = J\left[\sum_{i=1}^{n}\left(|x_{(i)}| - |x_{(i-1)}|\right)\left(A(f)_{(i)}, B(f)_{(i)}\right)^*\right] =$$

$$= \sum_{i=1}^{n} J\left[\left(|x_{(i)}| - |x_{(i-1)}|\right)\left(A(f)_{(i)}, B(f)_{(i)}\right)^*\right] =$$

$$= \sum_{i=1}^{n}\left(|x_{(i)}| - |x_{(i-1)}|\right) J\left[\left(A(f)_{(i)}, B(f)_{(i)}\right)^*\right] =$$

$$= \sum_{i=1}^{n}\left(|x_{(i)}| - |x_{(i-1)}|\right)\mu_b\left(A(f)_{(i)}, B(f)_{(i)}\right) = \int_S f d\mu_b$$

Q.E.D.

We have characterized the bipolar Choquet integral for simple acts, i.e. for acts taking only a finite number of values. A natural question is if this functional can be extended from $\mathcal{B}_0$ to the set of bounded real valued functions $\mathcal{B}$. In this work we are not able to obtain this extension, but we will get some helpful suggestions in the next section encouraging future researches over this argument. Finally, in the last section of this chapter we will discuss some coherence conditions, regarding the concept of bi-capacity. This since we aim to build a model representing preferences of people and, in this view, we must avoid some contradictory situations which could arise using bi-capacities.

4.3.2 Continuity and extension to bounded functions

In order to extend the definition 12 to all of B, let us remember some well known results. The set $\mathcal{B}$ is a subspace of the linear space $\Re^{\mathcal{S}}$, with the *(sup)norm* $\|f\|_\infty = sup\{|f|\}$ and $\mathcal{B}_0$ is dense in $\mathcal{B}$. That is $\mathcal{B}$ is the norm closure of $\mathcal{B}_0$ with respect to the distance induced by the sup-norm. A functional $J : \mathcal{B}_0 \to \Re$ is said Holder continuous if there exist $L > 0$ and $\alpha \in (0,1)$ such that $|J(f) - J(g)| \leq L[d(f,g)]^\alpha$, for all $f, g \in \mathcal{B}_0$, where $d(f,g)$ is a metric on B_0. It is said Lipschitz continuous if is Holder continuous with $\alpha = 1$, i.e., if there exist a constant L such that $|J(f) - J(g)| \leq Ld(f,g)$, for all $f, g \in \mathcal{B}_0$. Lipschitzianity implies uniform continuity: for any $\epsilon > 0$ there exist $\delta > 0$ such that $|J(f) - J(g)| \leq \epsilon$ for all $f, g \in B_0$ with $d(f,g) < \delta$. The following results are well known.

Theorem 8 *Let X be a subset of some metric space S and $f : X \to \Re$ uniformly continue in X, then f can be uniquely extended to a uniform continue function to all of $\overline{X}$.*

A more general result is the following

Theorem 9 *Let $f : D \to Y$ be a Lipschitzian function, with D, Y metric spaces and Y complete (i.e. every Cauchy's sequence is convergent), let D be dense in X, then f has a unique extension to all of X, Lipschitzian with the same constant L.*

After these preliminaries we now, first remember that the Choquet integral is an uniform continuous functional and then we show how the bipolar Choquet integral does not preserve this property, even if we reduce to the set of simple and bounded functions.

Remark 12 *If $I_\nu : B_0 \to \mathfrak{R}$ is the Choquet functional, w.r.t. some capacity ν and since I_ν is additive for co-monotonic function (including constants) then I_ν is Lipschitz continuous.*

In fact, let $f, g \in B_0$ be two simple function with $I_\nu(f) > I_\nu(g)$ and $\delta = d(f,g)$, then $I_\nu(f) \leq I_\nu(g + \delta S^*) = I_\nu(g) + \delta$ and $|I_\nu(f) - I_\nu(g)| = I_\nu(f) - I_\nu(g) \leq \delta$.

Remark 13 *The bipolar Choquet functional is not uniform continuous, even if we reduce to simple acts.*

Example 1. Let Σ be the Algebra of finite union of intervals of $S =]0, +\infty[$ and let $\mathcal{Q} = \{(A,B) \in \Sigma \times \Sigma : A \cap B = \varnothing\}$. We define a bicapacity $\mu : \mathcal{Q} \to \{-1, 0, 1\}$ by posing $\mu(A,B) = -1, 0, 1$ if the sum of the lengths of the intervals forming A is respectively minor, equal or major of those forming B. For any even integer n let $f_n : S \to \mathfrak{R}$ be the simple function defined by

$$f_n(s) = \begin{cases} -k & \text{if } s \in]k-1, k] \quad k = 1, 3, \ldots, n-1 \\ k & \text{if } s \in]k-1, k] \quad k = 2, 4, \ldots, n \\ 0 & \text{if } s > n \end{cases}$$

It is easy to see that $I_\mu(f_n(s)) = n/2$ and $I_\mu(f_n(s) + \delta) = n/2 + n \cdot \delta$, for any $\delta \in (0,1)$. So we can build two functions as close as we want (δ) whose bipolar Choquet integrals differ by as much as we want.

The following example shows how the problem persists also if we reduce to simple acts.

Example 2. With the same setting of previous example, let $\mathcal{B}_0^M$ be the set of simple, real valued and Σ-measurable functions with value in $[-M, M]$. For any even integer n let $f_n : S \to \Re$ be the simple function defined by

$$f_n(s) = \begin{cases} -k\frac{M}{n} & \text{if } s \in](k-1)\frac{M}{n}, k\frac{M}{n}] \qquad k = 1, 3, \ldots, n-1 \\ \\ k\frac{M}{n} & \text{if } s \in]k - 1\frac{M}{n}, k\frac{M}{n}] \qquad k = 2, 4, \ldots, n \\ \\ 0 & \text{if } s > n \end{cases}$$

It is easy to see that $I_\mu(f_n(s)) = M/2$ and $I_\mu(f_n(s) + \delta) = M/2 + n \cdot \delta$, for any $\delta \in (0, 1)$. So we can build two functions as close as we want (δ) whose bipolar Choquet integrals differ by as much as we want.

The previous examples teach us that in order to obtain the uniform continuity, probably we must check for some hypothesis of continuity on the bi-capacity. Only after having obtained such a result we could think to extend the definition by means of the limit. Let $f \in B$ be a bounded act and let μ_b be a bicapacity on Σ, the bipolar Choquet integral of f will be given by

$$I(f) = \int_S f d\mu_b \equiv \lim_{n\to\infty} I(g_n)$$

where $\{g_n\}$ is any sequence of simple functions such that

$$\lim_{n\to\infty} g_n(s) = f(s) \; \forall s \in S$$

The question to find the conditions allowing this extension is an open problem for future research.

4.3.3 Coherence conditions.

The bipolar Choquet integral should represent preference under uncertainty. In this case it is reasonable to expect that there is some belief about plausibility of events $A \subseteq \mathcal{S}$ that should not depend on what is gained or lost in other events. In this context it is reasonable to imagine that the value given by a bi-capacity μ_b to $(A,B) \in Q$ is not decreasing with the plausibility of A and non-increasing with the plausibility of B. If this is true, then one has to expect that should not be possible to have $\mu_b(A,C) > \mu_b(B,C)$ and $\mu_b(A,D) < \mu_b(B,D)$. In fact, this would mean that act $(A,C)^*$ would be preferred to act $(B,C)^*$, revealing a greater credibility of A over B, and act $(A,D)^*$ would be preferred to act $(B,D)^*$, revealing a greater credibility of B over A. Similar situations arise when $\mu_b(C,A) > \mu_b(C,B)$ and $\mu_b(D,A) < \mu_b(D,B)$, or $\mu_(A,C) > \mu_b(B,C)$ and $\mu_(D,A) > \mu_b(D,B)$. Taking into account such situations, we shall analyze in detail the following coherence conditions:

(A1) $(A,C)^* \succ (B,C)^* \Rightarrow (A,D)^* \succ (B,D)^*$,

for all $(A,C),(B,C),(A,D),(B,D) \in Q$,

(A2) $(C,A)^* \succ (C,B)^* \Rightarrow (D,A)^* \succ (D,B)^*$,

for all $(C,A),(C,B),(D,A),(D,B) \in Q$,

(A3) for any $A,B \subseteq S$ *there exist one* $C \subseteq S \setminus (A \cup B)$ *such that*

$(A,C)^* \succ (B,C)^* \Leftrightarrow (C,A)^* \prec (C,B)^*$

(A4) $(A,C)^* \succ (B,C)^* \Leftrightarrow (C,A)^* \prec (C,B)^*$,

for all $(A,C),(B,C),(C,A),(C,B) \in Q$, *(A5)* $(A,C)^* \succ (B,C)^* \Leftrightarrow (D,A)^* \prec (D,B)^*$,

for all $(A,C),(B,C),(D,A),(D,B) \in Q$.

Theorem 10 *The following proposition hold*

1) *If (A1) holds, then there exists a capacity* ν_1 *on* S *and a function*

$$\omega_1 : \{(v,B) : v = \nu_1(A), (A,B) \in Q\} \to [-1,1],$$

such that $\mu(A,B) = \omega_1(\nu_1(A),B)$ *for all* $(A,B) \in Q$, *with function* ω_1 *increasing in the first argument and non increasing with respect to inclusion in the second argument;*

2) *If (A2) holds, then there exists a capacity* ν_2 *on* S *and a function*

$$\omega_2 : \{(A,v) : v = \nu_2(B), (A,B) \in Q\} \to [-1,1],$$

such that $\mu(A,B) = \omega_2(A,\nu_2(B))$ *for all* $(A,B) \in Q$, *with function* ω_2 *non decreasing with respect to inclusion in the first argument and decreasing in the second argument;*

3) *If (A1) and (A2) hold, then there exist two capacities* ν_1 *and* ν_2 *on* S *and a function*

$$\omega_3 : \{(u,v) : u = \nu_1(A), v = \nu_2(B), (A,B) \in Q\} \to [-1,1],$$

such that $\mu(A,B) = \omega_3(\nu_1(A),\nu_2(B))$ *for all* $(A,B) \in Q$, *with function* ω_3 *increasing in the first argument and decreasing in the second argument;*

4) If (A1), (A2) and (A3) hold, then there exists a capacity ν on S and a function

$$\omega : \{(u,v) : u = \nu(A), v = \nu(B), (A,B) \in Q\} \to [-1,1],$$

such that $\mu(A,B) = \omega(\nu(A), \nu(B))$ for all $(A,B) \in Q$, with function ω increasing in the first argument and decreasing in the second argument;

5) If (A1) and (A4) hold, then there exists a capacity ν on S and a function

$$\omega : \{(u,v) : u = \nu(A), v = \nu(B), (A,B) \in Q\} \to [-1,1],$$

such that $\mu(A,B) = \omega(\nu(A), \nu(B))$ for all $(A,B) \in Q$, with function ω increasing in the first argument and decreasing in the second argument;

6) If (A2) and (A4) hold, then there exists a capacity ν on S and a function

$$\omega : \{(u,v) : u = \nu(A), v = \nu(B), (A,B) \in Q\} \to [-1,1]$$

such that $\mu(A,B) = \omega(\nu(A), \nu(B))$ for all $(A,B) \in Q$, with function ω increasing in the first argument and decreasing in the second argument;

7) If (A5) holds, then there exists a capacity ν on S and a function

$$\omega : \{(u,v) : u = \nu(A), v = \nu(B), (A,B) \in Q\} \to [-1,1],$$

such that $\mu(A,B) = \omega(\nu(A), \nu(B))$ for all $(A,B) \in Q$, with function ω increasing in the first argument and decreasing in the second argument.

Proof.

1) Let us define $\nu_1(A) = \mu_b(A, \varnothing)$. For all $(A, C), (B, C), (A, D), (B, D) \in Q$, it is not possible to have $\mu_b(A, C) = \mu_b(B, C)$ and $\mu_b(A, D) > \mu_b(B, D)$, because, for (A1), $\mu_b(A, D) > \mu_b(B, D)$ would imply $\mu_b(A, C) > \mu_b(B, C)$, too. Thus,

$$\mu_b(A, C) = \mu_b(B, C) \Rightarrow \mu_b(A, D) = \mu_b(B, D),$$

for all $(A, C), (B, C), (A, D), (B, D) \in Q$. Consequently,

$$\mu_b(A, \varnothing) = \mu_b(B, \varnothing) \Rightarrow \mu_b(A, C) = \mu_b(B, C),$$

from which we get

$$\nu_1(A) = \nu_1(B) \Rightarrow \mu_b(A, C) = \mu_b(B, C),$$

for all $(A, C), (B, C) \in Q$. We can therefore define function ω_1 as follows: $\omega_1(\nu_1(A), B) = \mu_b(A, B)$, for all $(A, B) \in Q$.

For (A1) we have

$$\mu_b(A, \varnothing) > \mu_b(B, \varnothing) \Rightarrow \mu_b(A, C) > \mu_b(B, C)$$

for all $(A, C), (B, C) \in Q$, i.e.

$$\nu_1(A) > \nu_1(B) \Rightarrow \mu_b(A, C) > \mu_b(B, C),$$

and consequently

$$\nu_1(A) > \nu_1(B) \Rightarrow \omega_1(\nu_1(A), C) > \omega_1(\nu_1(B), C),$$

which means that ω_1 is increasing in the first argument. Monotonicity of bipolar capacity gives the monotonicity of function ω_1 with respect to the second argument.

2) It can be proved analogously to 1), by defining $\nu_2(A) = -\mu_b(\varnothing, A)$.

3) By 1) and 2).

4) Condition (A3) ensures that capacities ν_1 and ν_2 agree in the sense that for all $A, B \subseteq S$

$$\nu_1(A) > \nu_1(B) \Leftrightarrow \nu_2(A) > \nu_2(B). \quad (i)$$

Indeed, applying the definition of ν_1, (A1), (A3), (A2) and the definition of ν_2, we get

$$\nu_1(A) > \nu_1(B) \Rightarrow \mu_b(A, \varnothing) > \mu_b(B, \varnothing) \Rightarrow$$

$$\mu_b(A, C) > \mu_b(B, C) \Leftrightarrow \mu_b(C, A) < \mu_b(C, B)$$

$$\Rightarrow \mu_b(\varnothing, A) < \mu_b(\varnothing, B) \Rightarrow \nu_2(A) > \nu_2(B),$$

i.e.

$$\nu_1(A) > \nu_1(B) \Rightarrow \nu_2(A) > \nu_2(B). \quad (ii)$$

Analogously, applying the definition of ν_2, (A2), (A3), (A1) and the definition of ν_1, we get

$$\nu_2(A) > \nu_2(B) \Rightarrow \mu_b(\varnothing, B) > \mu_b(\varnothing, A) \Rightarrow$$

$$\mu_b(C,A) > \mu_b(C,B) \Leftrightarrow \mu_b(A,C) < \mu_b(B,C)$$

$$\Rightarrow \mu_b(A,\varnothing) < \mu_b(B,\varnothing) \Rightarrow \nu_1(A) > \nu_1(B),$$

i.e.

$$\nu_2(A) > \nu_2(B) \Rightarrow \nu_1(A) > \nu_1(B). \quad (iii)$$

By (i) and (ii) we get (iii). (iii) implies also that for all $A, B \subseteq S$

$$\nu_1(A) = \nu_1(B) \Leftrightarrow \nu_2(A) = \nu_2(B). \quad (iv)$$

By (i) and (iv), there exists an increasing function $g : \{v \in [0,1] : \exists A \subseteq S$ for which $\nu_2(A) = v\} \to [0,1]$ such that $\nu_2(A) = g(\nu_1(A))$. Thus we can define a function $\omega : \{(u,v) : u = \nu_1(A), v = \nu_1(B), (A,B) \in Q\} \to [-1,1]$ defined as follows: for all $A,B) \in Q$

$$\omega(\nu_1(A), \nu_1(B)) = \omega_3(\nu_1(A), g(\nu_1(B)))$$

where ω_3 is the function defined in point 3). For the monotonicity of function ω_3 and g, ω is increasing in the first argument and decreasing in the second argument.

5) Observe that (A3) is a particular case of (A4), which, given $A, B \subseteq S$ holds for any $C \subseteq S$ such that $(A,C), (B,C), (C,A), (C,B) \in Q$ (observe that if $(A,C), (B,C) \in Q$, then also $(C,A), (C,B) \in Q$). Using (A4), (A1) and again (A4) we get (A2) as follows: for all $(C,A), (C,B), (A,D), (B,D) \in Q$

$$\mu_b(C,A) > \mu_b(C,B) \Leftrightarrow \mu_b(A,C) < \mu_b(B,C) \Rightarrow$$

$$\Rightarrow \mu_b(A,D) < \mu_b(B,D) \Leftrightarrow \mu_b(D,A) > \mu_b(D,B)$$

With (A1), (A2) and (A3) we can apply 4) and obtain the result we looked for.

6) Analogously to 5), using (A4), (A2) and again (A4) we get (A1) as follows: for all $(C,A),(C,B),(A,D),(B,D) \in Q$

$$\mu_b(A,C) > \mu_b(B,C) \Leftrightarrow \mu_b(C,A) < \mu_b(C,B) \Rightarrow$$

$$\mu_b(D,A) < \mu_b(D,B) \Leftrightarrow \mu_b(A,D) > \mu_b(B,D)$$

With (A1), (A2) and (A3) we can apply 4) and obtain the result we looked for.

7) (A4) is a specific case of (A5), which, given $A,B \subseteq S$ holds for any $C,D \subseteq S$ such that $(A,C),(B,C),(D,A),(D,B) \in Q$. Using (A4), (A1) can obtained as follows:

$$\mu_b(A,C) > \mu_b(B,C) \Leftrightarrow \mu_b(D,A) < \mu_b(D,B) \Leftrightarrow \mu_b(A,D) > \mu_b(B,D).$$

With (A1) and (A4) we can apply 5) and we get the thesis.

Q.E.D.

4.4 Concluding remarks

In this chapter we have extended the bCPT model to the field of uncertainty where the probabilities attached to the *Events* are unknown. To this scope we have used the concept of *bipolar Choquet integral* with respect to a *bi-capacity*, due to Grabisch and Labreuche (2005a,b); Greco *et al.* (2002). Moreover we have presented a fairly simple characterization of this integral for *simple acts*.

Conclusion

We opened this thesis remembering how, in the last thirty years, an enormous amount of work has been done to develop alternatives to the classical Expected Utility Theory. One of the greatest strengths of recent work in decision theory has been the interplay between theoretical development and experimental investigation. The first wave of alternatives to Expected Utility Theory were developed in an attempt to explain observed regularities in behavior which contravened the received theory. Those new theories where then subjected to experimental tests, and in the light of the result of those tests, new theories were developed and old ones were revised. This healthy process continues. Theorists have also become more aware of the findings of experimental psychology, and have tried to construct theories which are compatible with those findings. Cumulative Prospect Theory of Tversky and Kahneman (1992) is considered nowadays one of the most valid alternative to Expected Utility Theory. It was, perhaps, not a case that the authors - Kahneman and Tversky - are two psychologists. They have been able to capture, in their theory, the "irrationality" of some human behavior in the process of choice. In very recent years (Wu and Markle (2008)) new experimental results have shown some limitations of Cumulative Prospect Theory, particularly in denouncing how people tend to evaluate conjointly gains and losses, showing a phenomenon which we called *gain-loss hedging*. Cumulative Prospect Theory predicts that gains and losses are processed differently and separately and is not able to cover situations in which gain-loss hedging appears. The model we present in this thesis, bipolar Cumulative Prospect Theory, is a "natural" generalization of Cumulative Prospect Theory, which is able to accommodate for the gain-loss hedging, allowing gains and losses to be evaluated conjointly. In the light of the previous discussion, regarding the evolution of the decision theory, we think that our work is a stimulating contribution to the research.

Bibliography

Abdellaoui, M. (2000). Parameter-free elicitation of utility and probability weighting functions. *Management Science*, **46**(11), 1497–1512.

Abdellaoui, M., Vossmann, F., and Weber, M. (2003). *Choice-based elicitation and decomposition of decision weights for gains and losses under uncertainty*. Centre for Economic Policy Research.

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école Américaine. *Econometrica: Journal of the Econometric Society*, **21**(4), 503–546.

Allais, M. and Hagen, O. (1979). *Expected utility hypotheses and the Allais paradox*. Reidel Dordrecht.

Anscombe, F. and Aumann, R. (1963). A definition of subjective probability. *Annals of Mathematical statistics*, **34**(1), 199–205.

Arrow, K. (1964). The role of securities in the optimal allocation of risk-bearing. *The Review of Economic Studies*, **31**(2), 91–96.

Baltussen, G., Post, T., and van Vliet, P. (2006). Violations of cumulative prospect theory in mixed gambles with moderate probabilities. *Management Science*, **52**(8), 1288.

Barberis, N., Huang, M., and Santos, T. (2001). Prospect Theory and Asset Prices. *Quarterly Journal of Economics*, **116**(1), 1–53.

Benartzi, S. and Thaler, R. (1995). Myopic loss aversion and the equity premium puzzle. *The Quarterly Journal of Economics*, **110**(1), 73–92.

Bernoulli, D. (1738). Specimen theoriae novae de mensura sortis. . *Commentarii Academiae scientiarum imperialis Petropolitanae*, **5**(2), 175–192.

Bernoulli, D. (1954). Exposition of a new theory on the measurement of risk. *Econometrica: Journal of the Econometric Society*, **22**(1), 23–36.

Birnbaum, M. (2005). Three new tests of independence that differentiate models of risky decision making. *Management Science*, **51**(9), 1346–1358.

Birnbaum, M. and Bahra, J. (2007). Gain-loss separability and coalescing in risky decision making. *Management Science*, **53**(6), 1016–1028.

Blavatskyy, P. (2005). Back to the St. Petersburg paradox? *Management Science*, **51**(4), 677–678.

Bleichrodt, H. and Pinto, J. (2000). A parameter-free elicitation of the probability weighting function in medical decision analysis. *Management Science*, **46**(11), 1485–1496.

Camerer, C. (1995). Individual decision making. *The handbook of experimental economics*, **3**, 587–704.

Camerer, C. (2000). Prospect theory in the wild: Evidence from the field. *Advances in Behavioral Economics*.

Camerer, C. and Ho, T. (1994). Violations of the betweenness axiom and nonlinearity in probability. *Journal of Risk and Uncertainty*, **8**(2), 167–196.

Chateauneuf, A. and Wakker, P. (1999). An axiomatization of cumulative prospect theory for decision under risk. *Journal of Risk and Uncertainty*, **18**(2), 137–145.

Chew, S. and MacCrimmon, K. (1979). *Alpha-nu choice theory: A generalization of expected utility theory*. The University of British Columbia.

Choquet, G. (1953). Theory of capacities. *Ann. Inst. Fourier*, **5**(131-295), 54.

Cramer, G. (1728). Letter to Nicolas Bernoulli. In: Bernoulli (1738). pages 211–213.

De Finetti, B. (1931). Sul significato soggettivo della probabilita. *Fundamenta mathematicae*, **17**, 298–329.

De Finetti, B. (1937). *La prevision: ses lois logiques, ses sources subjectives*. Institut Henri Poincaré.

Debreu, G. (1959). *Theory of value: An axiomatic analysis of economic equilibrium*. Yale Univ Pr.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, **75**(4), 643–669.

Fishburn, P. (1970). Utility theory for decision making.

Fishburn, P. (1978). On Handa's" New theory of cardinal utility" and the maximization of expected return. *The Journal of Political Economy*, **86**(2), 321–324.

Fishburn, P. (1983). Transitive measurable utility. *Journal of Economic Theory*, **31**(2), 293–317.

Gilboa, I. (1987). Expected utility with purely subjective non-additive probabilities. *Journal of Mathematical Economics*, **16**(1), 65–88.

Goldstein, W. and Einhorn, H. (1987). Expression theory and the preference reversal phenomena. *Psychological Review*, **94**(2), 236.

Gonzalez, R. and Wu, G. (1999). On the shape of the probability weighting function. *Cognitive Psychology*, **38**(1), 129–166.

Grabisch, M. and Labreuche, C. (2005a). Bi-capacities–I: definition, M "obius transform and interaction. *Fuzzy sets and systems*, **151**(2), 211–236.

Grabisch, M. and Labreuche, C. (2005b). Bi-capacities–II: the Choquet integral. *Fuzzy sets and systems*, **151**(2), 237–259.

Greco, S., Matarazzo, B., and Slowinski, R. (2002). Bipolar Sugeno and Choquet integrals. In *EUROFUSE Workshop on Informations Systems*, pages 191–196.

Hausman, D. (1992). *The inexact and separate science of economics*. Cambridge Univ Pr.

Herstein, I. and Milnor, J. (1953). An axiomatic approach to measurable utility. *Econometrica, Journal of the Econometric Society*, pages 291–297.

Holt, C. (1986). Preference reversals and the independence axiom. *The American Economic Review*, **76**(3), 508–515.

Ingersoll, J. (2008). Non-monotonicity of the Tversky-Kahneman probability-weighting function: a cautionary note. *European Financial Management*, **14**(3), 385–390.

Jolls, C., Sunstein, C., and Thaler, R. (1998). A behavioral approach to law and economics. *Stanford Law Review*, **50**(5), 1471–1550.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, **47**(2), 263–291.

Karni, E. (1985). *Decision making under uncertainty: the case of state-dependent preferences*. Harvard University Press Cambridge, MA.

Karni, E. and Safra, Z. (1987). " Preference reversal" and the observability of preferences by experimental methods. *Econometrica: Journal of the Econometric Society*, **55**(3), 675–685.

Karni, E. and Schmeidler, D. (1991). Atemporal dynamic consistency and expected utility theory* 1. *Journal of Economic Theory*, **54**(2), 401–408.

Keynes, J. (1921). *A treatise on probability*. Macmillan & Co., Ltd.

Kilka, M. and Weber, M. (2001). What determines the shape of the probability weighting function under uncertainty? *Management Science*, **47**(12), 1712–1726.

Knight, F. (1921). Risk, Uncertainty and Profit, 1921. *Boston and New York*.

Kraft, C., Pratt, J., and Seidenberg, A. (1959). Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, **30**(2), 408–419.

Lattimore, P., Baker, J., and Witte, A. (1992). The influence of probability on risky choice:: A parametric examination. *Journal of Economic Behavior & Organization*, **17**(3), 377–400.

Levy, M. and Levy, H. (2002). Prospect Theory: Much ado about nothing? *Management Science*, **48**(10), 1334–1349.

Lichtenstein, S. and Slovic, P. (1971). Reversals of preference between bids and choices in gambling decisions. *Journal of experimental psychology*, **89**(1), 46–55.

Lindman, H. (1971). Inconsistent preferences among gambles. *Journal of Experimental Psychology*, **89**(2), 390–397.

Loomes, G. and Sugden, R. (1982). Regret theory: An alternative theory of rational choice under uncertainty. *The Economic Journal*, **92**(368), 805–824.

Loomes, G. and Sugden, R. (1983). A rationale for preference reversal. *The American Economic Review*, **73**(3), 428–432.

Loomes, G. and Sugden, R. (1987). Some implications of a more general form of regret theory* 1. *Journal of Economic Theory*, **41**(2), 270–287.

Loomes, G., Starmer, C., and Sugden, R. (1989). Preference reversal: information-processing effect or rational non-transitive choice? *The Economic Journal*, **99**(395), 140–151.

MacCrimmon, K. and Wehrung, D. (1990). Characteristics of risk taking executives. *Management science*, **36**(4), 422–435.

Machina, M. (1982). " Expected Utility" Analysis without the Independence Axiom. *Econometrica: Journal of the Econometric Society*, **50**(2), 277–323.

March, J. and Shapira, Z. (1987). Managerial perspectives on risk and risk taking. *Management science*, pages 1404–1418.

Marschak, J. (1950). Rational behavior, uncertain prospects, and measurable utility. *Econometrica: Journal of the Econometric Society*, **18**(2), 111–141.

McNeil, B., Pauker, S., Sox Jr, H., and Tversky, A. (1982). On the elicitation of preferences for alternative therapies. *New England journal of medicine*, **306**(21), 1259–1262.

Menger, K. (1934). The role of uncertainty in economics. *Essays in Mathematical Economics in Honor of Oskar Morgenstern.*

Prelec, D. (1998). The probability weighting function. *Econometrica*, **66**(3), 497–527.

Quattrone, G. and Tversky, A. (1988). Contrasting rational and psychological analyses of political choice. *The American political science review*, **82**(3), 719–736.

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior & Organization*, **3**(4), 323–343.

Ramsey, F. (1926). Truth and probability. *Histoy of Economic Thought Chapters*, pages 156–198.

Rieger, M. and Wang, M. (2006). Cumulative prospect theory and the St. Petersburg paradox. *Economic Theory*, **28**(3), 665–679.

Samuelson, P. (1952). Probability, utility, and the independence axiom. *Econometrica: Journal of the Econometric Society*, **20**(4), 670–678.

Savage, L. (1954). Foundations of statistics.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society*, **97**(2), 255–261.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica: Journal of the Econometric Society*, **57**(3), 571–587.

Segal, U. (1988). Does the preference reversal phenomenon necessarily contradict the independence axiom? *The American Economic Review*, **78**(1), 233–236.

Shackle, G. (1949). A non-additive measure of uncertainty. *The Review of Economic Studies*, pages 70–74.

Shackle, G. (1952). *Expectation in economics*. University Press.

Slovic, P. *et al.* (1995). The construction of preference. *American Psychologist*, **50**(5), 364–371.

Starmer, C. (2000). Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk. *Journal of Economic Literature*, **38**, 332–382.

Starmer, C. and Sugden, R. (1989). Violations of the independence axion in common ratio problems: An experimental test of some competing hypotheses. *Annals of Operations Research*, **19**(1), 79–102.

Tammi, T. (1997). *Essays on the rationality of experimentation in economics: The case of preference reversal research*. Joensuun yliopisto.

Tversky, A. and Fox, C. (1995). Weighing risk and uncertainty. *Psychological review*, **102**(2), 269–283.

Tversky, A. and Kahneman, D. (1981). The framing of decisions and the psychology of choice. *Science*, **211**(4481), 453.

Tversky, A. and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, **5**(4), 297–323.

Tversky, A. and Thaler, R. (1990). Anomalies: preference reversals. *The Journal of Economic Perspectives*, **4**(2), 201–211.

Tversky, A., Slovic, P., and Kahneman, D. (1990). The causes of preference reversal. *The American Economic Review*, **80**(1), 204–217.

Von Neumann, J. and Morgenstern, O. (1944). *Theories of games and economic behavior*. Princeton University Press Princeton, NJ.

Wakker, P. (2003). The Data of Levy and Levy (2002)" Prospect Theory: Much Ado About Nothing?" Actually Support Prospect Theory. *Management Science*, **49**(7), 979–981.

Wakker, P. and Tversky, A. (1993). An axiomatization of cumulative prospect theory. *Journal of risk and uncertainty*, **7**(2), 147–175.

Wakker, P. and Zank, H. (2002). A simple preference foundation of cumulative prospect theory with power utility. *European Economic Review*, **46**(7), 1253–1271.

Wu, G. and Gonzalez, R. (1996). Curvature of the probability weighting function. *Management Science*, **42**(12), 1676–1690.

Wu, G. and Markle, A. (2008). An empirical test of gain-loss separability in prospect theory. *Management Science*, **54**(7), 1322–1335.

Yaari, M. (1987). The dual theory of choice under risk. *Econometrica: Journal of the Econometric Society*, pages 95–115.