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Superalgebras with superinvolution

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Contents

Introduction

Let A be an associative algebra over a field F of characteristic zero. If $\mathbb{Z}_2$ is the cyclic group of order 2, a $\mathbb{Z}_2$-grading on A is a decomposition of A, as a vector space, into the direct sum of subspaces $A = A_0 \oplus A_1$ such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$. The subspaces A_0 and A_1 are the homogeneous components of A and their elements are called homogeneous of degree zero (even elements) and of degree one (odd elements), respectively. The $\mathbb{Z}_2$-graded algebras are usually called superalgebras.

A superinvolution $*$ on the superalgebra $A = A_0 \oplus A_1$ is a graded linear map (a map preserving the grading) $* : A \to A$ such that $(a^*)^* = a$, for all $a \in A$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$, for any homogeneous elements $a, b \in A_0 \cup A_1$, of degrees $|a|$, $|b|$, respectively. Since $\mathrm{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i : a^* = a\}$ and $A_i^- = \{a \in A_i : a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively. From now on, we shall refer to a superalgebra with superinvolution simply as a $*$-algebra.

The superinvolutions play a prominent role in the setting of Lie and Jordan superalgebras. The skew elements of a $*$-algebra have a structure of Lie superalgebra under the graded bracket: $[a, b] = ab - (-1)^{|a||b|}ba$. Similarly, the symmetric elements form a Jordan superalgebra under the supersymmetrized product: $a \circ b = ab + (-1)^{|a||b|}ba$. Many of the classical simple Lie and Jordan superalgebras (see [31, 48]) arise in this way.

The purpose of this thesis is to present, in the setting of $*$-algebras, some of the most interesting and challenging problems of combinatorial PI-theory (the theory of polynomial identities), which have already been addressed in the field of associative algebras or of algebras with involution.

If A is an associative algebra over a field F of characteristic zero, an effective way of measuring the polynomial identities satisfied by A is provided by its sequence of codimensions $c_n(A), n = 1, 2, \ldots$. Such a sequence was introduced by Regev in [50] and, in characteristic zero, gives an actual quantitative measure of the identities satisfied by a given algebra. The most important feature of the sequence of codimensions, proved in [50], is that in case A is a PI-algebra, i.e., it satisfies a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded. Later in [32, 33], Kemer showed that, given any PI-algebra A, $c_n(A)$, $n = 1, 2, \ldots$, cannot have intermediate growth, i.e., either it is polynomially bounded or it grows exponentially. Much effort has been put into the study of algebras with polynomial codimensions growth ([16, 17, 18, 37, 38, 39]). In this setting a celebrated theorem of Kemer ([33]) characterizes them as follows. If G is the infinite dimensional Grassmann algebra over F and UT_2 is the algebra of 2×2 upper triangular matrices over F, then $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if G and UT_2 do not satisfy all the identities of A.

One of the aims of this thesis is to prove a similar result in the setting of $*$-algebras.

Another aim is inspired from the famous theorem of Amitsur and Levitzki. Recall that the standard polynomial of degree r in the non-commutative variables $x_1, \ldots, x_r$ is defined as $St_r(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} (\text{sgn}\sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}$. The result of Amitsur and Levitzki states that St_{2n} is an identity for the algebra of $n \times n$ matrices over a commutative ring. The original proof (see [3]) was highly combinatorial and based on the properties of the matrix units. Several different proofs were given afterwards (see for instance [49, 51, 55, 56]). If F is a field of characteristic different from 2, it is not hard to prove that St_{2n} is, up to a scalar, the only identity of minimal degree of $M_n(F)$, the algebra of $n \times n$ matrices over F.

In this thesis we prove similar results in the case of $*$-algebras. More precisely, we find standard identities of minimal degree in the setting of matrix $*$-algebras and we show, in this way, that the Amitsur-Levitzki theorem can be improved by considering only certain kinds of matrices. The idea draws inspiration from the results of Kostant and Rowen, concerning algebras with involution. It is well known (see, for instance [53, Theorem 3.1.62]) that, up to isomorphism, we have only the transpose and the symplectic involution on the matrix algebra $M_n(F)$. In [35], Kostant reproved the Amitsur-Levitzki theorem by showing that it is equivalent to a theorem in Lie cohomology. Moreover, he showed the power of his method by proving that $St_{2n-2}(K_1, \ldots, K_{2n-2}) = 0$, for all n even, where K_i, $i = 1, \ldots, 2n-2$, are $n \times n$ skew-symmetric matrices with respect to the transpose involution. In 1974, Rowen (see [52]) reproved Kostant's theorem through a graph-theoretical approach and was able to extend this result to the case n odd (see also [30]). Inspired from the results of Procesi and Razmyslov (in [46, 49] they showed that the Amitsur-Levitzki theorem formally follows from the Hamilton-Cayley polynomial), Rowen, in [54], presented a simple proof of Kostant's theorem and solved the analogous question for the symplectic involution s. In particular he showed that $St_{2n-2}(S_1, \ldots, S_{2n-2}) = 0$, where S_i, $i = 1, \ldots, 2n-2$, are $n \times n$ symmetric matrices with respect to s.

The first chapter of this thesis is introductory. We start with the basic definitions and properties of associative algebras with polynomial identities (Section 1). A polynomial $f(x_1, \ldots, x_n)$, in non-commuting variables, is a polynomial identity for the algebra A if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$. The set of all polynomial identities of A, $\text{Id}(A)$, is an ideal of the free associative algebra $F\langle X \rangle$, where $X = \{x_1, x_2, \ldots\}$ is a countable set. Moreover $\text{Id}(A)$ is a T-ideal of $F\langle X \rangle$, i.e., an ideal invariant under all the endomorphisms of $F\langle X\rangle$. We also introduce the notion of variety of algebras: if A is a PI-algebra, the variety $\mathcal{V}$ generated by A, $\mathrm{var}(A)$, is the class of all algebras B satisfying the same set of identities of A. In Section 2 we give a brief introduction to the classical representation theory of the symmetric group and of the general linear group via the theory of Young diagrams. Section 3 is devoted to the study of associative PI-algebras. We define the sequence of codimensions, $c_n(A)$, and cocharacters, $\chi_n(A)$, for an associative PI-algebra A. In the last part of this section we focus our attention to varieties of polynomial growth (their sequence of codimensions is polynomially bounded) and we prove several characterizations concerning them. Recall that the growth of a variety $\mathcal{V}$ is defined as the growth of the sequence of codimensions of any algebra A generating $\mathcal{V}$. The next two sections have a similar structure than the third one. We give analogous definitions and properties in the setting of algebras with involution and of superalgebras.

The sixth section is devoted to the main objects of the thesis, that is the superalgebras with superinvolution ($*$-algebras). In this setting, we introduce the analogous objects than those of the ordinary case. In particular we define the free $*$-algebra, the $*$-polynomial identities, the sequence of $*$-codimensions and cocharacters, and, moreover, we prove some basic results. Finally, we consider the particular case of matrix algebras with superinvolution (Section 7) and present the so-called proper polynomials (Section 8).

In the second chapter we give some characterizations concerning finite dimensional $*$-algebras with polynomial growth of the $*$-codimensions. In Section 1 we introduce some finite dimensional $*$-algebras generating $*$-varieties (varieties of $*$-algebras) of almost polynomial growth, i.e., $*$-varieties with exponential growth such that every proper subvariety has polynomial growth. Then we present a Wedderburn-Malcev theorem for finite dimensional $*$-algebras (Section 2). In the final section we prove several results characterizing $*$-varieties of polynomial growth. Between them, in the main theorem of the chapter, we give the analogous result than that Kemer has proved for associative algebra: in case A is a finite dimensional $*$-algebra, the sequence of $*$-codimensions $c_n^*(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if the $*$-variety generated by A does not contain three explicitly described $*$-algebras.

In the third chapter, we completely classify all subvarieties and all minimal subvarieties of the $*$-varieties of almost polynomial growth generated by a finite dimensional $*$-algebra. A $*$-variety $\mathcal{V}$ is minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$, for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n^*(\mathcal{U}) \approx q'n^t$, with $t < k$. We shall see that the three $*$-algebras which appear in the main result of Chapter 3 are the only finite dimensional $*$-algebras generating $*$-varieties of almost polynomial growth. In the last section we describe the $*$-algebras whose $*$-codimensions are bounded by a linear function.

Finally, the fourth chapter is devoted to the study of standard identities in matrix $*$-algebras.

If $n = k + h$, we define a particular $\mathbb{Z}_2$-grading on the matrix algebra $M_n(F)$ and, in this way, it becomes a superalgebra, denoted by $M_{k,h}(F)$. If the field F is algebraically closed of characteristic different from 2, then any non trivial $\mathbb{Z}_2$-grading on $M_n(F)$ is isomorphic to $M_{k,h}(F)$, for some k and h. Moreover, in [47], Racine proved that, up to isomorphism, it is possible to define on $M_{k,h}(F)$ only the orthosymplectic and the transpose superinvolution. In this chapter we find the minimal degree for which the standard polynomial vanishes when evaluated in homogeneous symmetric or skew-symmetric matrices of $M_{k,h}(F)$ in the case of both transpose or orthosymplectic superinvolution. In the final section, we make a systematic study of the identities of the algebra $M_{1,1}(F)$ endowed with the transpose superinvolution: we compute a generating set of the ideal of polynomial identities and we find the decomposition of the corresponding character into irreducibles.

Chapter 1

Preliminaries

In this chapter we present some preliminary concepts which allow us to treat the main arguments of the thesis.

In the first section we give some basic definitions and well-known results concerning the theory of polynomial identities for associative algebras. We start by introducing the free algebra $F\langle X\rangle$ and the T-ideal of the identities, $\mathrm{Id}(A)$, of an associative algebra A. Then we discuss the so-called multilinear polynomials and some results concerning them.

The second section is devoted to the description of the ordinary representation theory of the symmetric group S_n and of the general linear group. In this part we introduce the so-called highest weight vectors.

In third, forth and fifth section we focus our attention to PI-algebras (algebras satisfying a non-trivial polynomial identities), algebras with involution (antiautomorphism of order 2) and superalgebras ($\mathbb{Z}_2$-graded algebra), respectively. For each of these structures we introduce the corresponding sequence of codimensions ($\sharp$-codimensions, supercodimensions) and cocharacters, the notion of varieties of algebras (algebras with involution, superalgebras) and present results concerning varieties ($\sharp$-varieties, supervarieties) of polynomial growth.

In the sixth section we discuss the main objects of the thesis, that is the superalgebras with superinvolution. They are $\mathbb{Z}_2$-graded algebras endowed with a graded linear map of order 2, with a suitable property similar to that of an involution. In order to simplify the notation we shall refer to the superalgebras with superinvolution simply as $*$-algebra and we shall use the prefix "$*$-" to denote, in the setting of $*$-algebras, the analogous objects introduced in the case of associative algebra (for instance, free $*$-algebra, $*$-polynomial identities, $*$-codimensions sequence, and so on).

Finally, we consider the particular case of matrix algebras with superinvolution (Section 7) and present the so-called proper polynomials (Section 8).

1.1 Basic definitions

We start with the definition of free algebra. Let F be a field and X a countable set. The free associative algebra on X over F is the algebra $F\langle X\rangle$ of polynomials in the non-commuting indeterminates $x_1, x_2, \ldots \in X$. A basis of $F\langle X\rangle$ is given by all words in the alphabet X, adding the empty word 1. Such words are called monomials and the product of two monomials is defined by juxtaposition. The elements of $F\langle X\rangle$ are called polynomials and if $f \in F\langle X\rangle$, then we write $f = f(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n \in X$ are the only variables occurring in f.

We define $\deg u$, the degree of a monomial u, as the length of the word u. Also $\deg_{x_i} u$, the degree of u in the indeterminate x_i, is the number of the occurrences of x_i in u. Similarly, the degree $\deg f$ of a polynomial $f = f(x_1, \ldots, x_n)$ is the maximum degree of a monomial in f and $\deg_{x_i} f$, the degree of f in x_i, is the maximum degree of $\deg_{x_i} u$, for u monomial in f.

The algebra $F\langle X\rangle$ is defined, up to isomorphism, by the following universal property: given an associative F-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of algebras $F\langle X\rangle \to A$. The cardinality of X is called the rank of $F\langle X\rangle$.

Now we can define one of the main objects of the thesis.

Definition 1.1. *Let A be an associative F-algebra and $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$. We say that f is a polynomial identity for A, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$.*

We may also say that A satisfies $f \equiv 0$ or, sometimes, that f itself is an identity of A. Since the trivial polynomial $f = 0$ is an identity for any algebra A, we say that A is a PI-algebra if it satisfies a non-trivial polynomial identity.

Example 1.1. *If A is a commutative algebra, then A is a PI-algebra since it satisfies the identity $[x_1, x_2] \equiv 0$, where $[x_1, x_2] = x_1x_2 - x_2x_1$ denotes the Lie commutator of x_1 and x_2.*

Example 1.2. *If A is a nilpotent algebra, with $A^n = 0$, then A is a PI-algebra since it satisfies the identity $x_1 \cdots x_n \equiv 0$.*

Definition 1.2. *Given an algebra A, the two-sided ideal of polynomials identities of A is defined as*

$$Id(A) = \{f \in F\langle X\rangle : f \equiv 0 \text{ on } A\}.$$

Recalling that an ideal I of $F\langle X\rangle$ is a T-ideal if $\varphi(I) \subseteq I$, for all endomorphism φ of $F\langle X\rangle$, it is easy to check that $\mathrm{Id}(A)$ is a T-ideal of $F\langle X\rangle$. Moreover, given a T-ideal I, it is proved that $\mathrm{Id}(F\langle X\rangle/I) = I$. Then all T-ideals of $F\langle X\rangle$ are actually ideals of polynomial identities for a suitable algebra A.

Since many algebras may correspond to the same set of polynomial identities (or T-ideal) we need to introduce the notion of variety of algebras.

Definition 1.3. *Given a non-empty set $S \subseteq F\langle X\rangle$, the class of all algebras A such that $f \equiv 0$ on A, for all $f \in S$, is called the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S.*

A variety $\mathcal{V}$ is called non-trivial if $S \neq 0$ and $\mathcal{V}$ is proper if it is non-trivial and contains a non-zero algebra.

Example 1.3. *The class of all commutative algebras forms a proper variety with $S = \{[x_1, x_2]\}$.*

Notice that if $\mathcal{V}$ is the variety determined by the set S and $\langle S\rangle_T$ is the T-ideal of $F\langle X\rangle$ generated by S, then $\mathcal{V}(S) = \mathcal{V}(\langle S\rangle_T)$ and $\langle S\rangle_T = \bigcap_{A\in\mathcal{V}} \mathrm{Id}(A)$. We write $\langle S\rangle_T = \mathrm{Id}(\mathcal{V})$.

In the following definition we introduce the concept of relatively free algebra.

Definition 1.4. *Let $\mathcal{V}$ be a variety, $A \in \mathcal{V}$ an algebra and $Y \subseteq A$ a subset of A. We say that A is relatively free on Y (with respect to $\mathcal{V}$), if for any algebra $B \in \mathcal{V}$ and for every function $\alpha : Y \to B$, there exists a unique homomorphism $\beta : A \to B$ extending α.*

When $\mathcal{V}$ is the variety of all algebras, this is just the definition of a free algebra on Y. The cardinality of Y is called the rank of A. Relatively free algebras are easily described in terms of free algebras (see, for instance, [26, Theorem 1.2.4]).

Theorem 1.1.1. *Let X be a non-empty set, $F\langle X\rangle$ a free algebra on X and $\mathcal{V}$ a variety with corresponding ideal $Id(\mathcal{V}) \subseteq F\langle X\rangle$. Then $F\langle X\rangle/Id(\mathcal{V})$ is a relatively free algebra on the set $\bar{X} = \{x + Id(\mathcal{V}) \mid x \in X\}$. Moreover, any two relatively free algebras with respect to $\mathcal{V}$ of the same rank are isomorphic.*

The following theorem (see for instance [26, Theorem 1.2.5]) explains the connection between T-ideals and varieties of algebras.

Theorem 1.1.2. *There is a one-to-one correspondence between T-ideals of $F\langle X\rangle$ and varieties of algebras. More precisely, a variety $\mathcal{V}$ corresponds to the T-ideal of identities $Id(\mathcal{V})$ and a T-ideal I corresponds to the variety of algebras satisfying all the identities of I.*

If $\mathcal{V}$ is a variety and A is an algebra such that $\mathrm{Id}(A) = \mathrm{Id}(\mathcal{V})$, then we say that $\mathcal{V}$ is the variety generated by A and we write $\mathcal{V} = \mathrm{var}(A)$. Also, we shall refer to $F\langle X\rangle/\mathrm{Id}(\mathcal{V})$ as the relatively free algebra of the variety $\mathcal{V}$ of rank $|X|$.

If the ground field F is infinite, then the study of polynomial identities of an algebra A over F can be reduced to the study of the so-called homogeneous or multilinear polynomials. This reduction is very useful because these kinds of polynomials are easier to deal with.

Let $F_n = F\langle x_1, \ldots, x_n\rangle$ be the free algebra of rank $n \geq 1$ over F. This algebra can be naturally decomposed as

$$F_n = F_n^{(1)} \oplus F_n^{(2)} \oplus \cdots$$

where, for every $k \geq 1$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. The $F_n^{(i)}$s are called the homogeneous components of F_n. This decomposition can be further refined as follows: for every $k \geq 1$, write

$$F_n^{(k)} = \bigoplus_{i_1+\cdots+i_n=k} F_n^{(i_1,\ldots,i_n)}$$

where $F_n^{(i_1,\ldots,i_n)}$ is the subspace spanned by all monomials of degree i_1 in x_1, …, i_n in x_n.

Definition 1.5. *A polynomial $f \in F_n^{(k)}$, for some $k \geq 1$, is called homogeneous of degree k. If $f \in F_n^{(i_1,\ldots,i_n)}$ it will be called multihomogeneous of multidegree $(i_1, \ldots, i_n)$. We also say that a polynomial f is homogeneous in the variable x_i if x_i appears with the same degree in every monomial of f.*

Among multihomogeneous polynomials a special role is played by the multilinear ones.

Definition 1.6. *A polynomial $f \in F\langle X\rangle$ is called linear in the variable x_i if x_i occurs with degree* 1 *in every monomial of f. Moreover, f is called multilinear if f is linear in each of its variables (multihomogeneous of multidegree $(1, \ldots, 1)$).*

It is always possible to reduce an arbitrary polynomial to a multilinear one. This so-called process of multilinearization can be found, for instance, in [26, Chapter 1].

We now introduce the notion of alternating polynomials and state a proposition concerning them.

Definition 1.7. *Let $f(x_1, \ldots, x_n, t_1, \ldots, t_m)$ be a polynomial linear in each of the variables $x_1, \ldots, x_n$. We say that f is alternating in the variables $x_1, \ldots, x_n$ if, for any $1 \leq i < j \leq n$, the polynomial f becomes zero when we substitute x_i instead of x_j.*

Example 1.4. *Let S_r denotes the symmetric group on $\{1, \ldots, r\}$. The standard polynomial of degree r*

$$St_r(x_1, \ldots, x_r) = \sum_{\sigma\in S_r} (sgn\sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}$$

is alternating in each of its variables.

Proposition 1.1. *Let $f(x_1, \ldots, x_n, t_1, \ldots, t_m)$ be a polynomial alternating in the variables $x_1, \ldots, x_n$ and let A be an F-algebra. If $a_1, \ldots, a_n \in A$ are linearly dependent over F, then $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$, for all $b_1, \ldots, b_m \in A$.*

We conclude this section with the following result, emphasizing the importance of multilinear polynomials. For a proof of it see [26, Corollary 1.3.9] and previous results.

Theorem 1.1.3. *If char$F = 0$, every T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains.*

1.2 The representation theory

1.2.1 Finite dimensional representations

In this section we survey the information on finite dimensional representations. We refer to [26, Chapter 2] for the results of this section. For a start, let V be a vector space over a field F and let $GL(V)$ be the group of invertible endomorphisms of V. We recall the following.

Definition 1.8. *A representation of a group G on V is a homomorphism of groups $\rho : G \to GL(V)$.*

Let us denote by $End(V)$ the algebra of F-endomorphisms of V. If FG is the group algebra of G over F and ρ is a representation of G on V, it is clear that ρ induces a homomorphism of F-algebras $\rho' : FG \to End(V)$ such that $\rho'(1_{FG}) = 1$.

Throughout we shall be dealing only with the case of finite dimensional representation. In this case, $n = \dim_F V$ is called the dimension or the degree of the representation ρ. Now, a representation of a group G uniquely determines a finite dimensional FG-module (or G-module) in the following way. If $\rho : G \to GL(V)$ is a representation of G, V becomes a (left) G-module by defining $gv = \rho(g)(v)$, for all $g \in G$ and $v \in V$. It is also clear that if L is a G-module which is finite dimensional as a vector space over F, then $\rho : G \to GL(L)$, such that $\rho(g)(l) = gl$, for $g \in G$ and $l \in L$, defines a representation of G on L.

Definition 1.9. *If $\rho : G \to GL(V)$ and $\rho' : G \to GL(W)$ are two representations of a group G, we say that ρ and ρ' are equivalent, and we write $\rho \sim \rho'$, if V and W are isomorphic as G-modules.*

Definition 1.10. *A representation $\rho : G \to GL(V)$ is irreducible if V is an irreducible G-module and ρ is completely reducible if V is the direct sum of its irreducible submodules.*

The basic tool for studying the representations of a finite group in case char$F = 0$, is the Maschke's theorem. Recall that an algebra A is simple if $A^2 \neq 0$ and it contains no non-trivial two-sided ideals and A is semisimple if $J(A) = 0$, where $J(A)$ is the Jacobson radical of A.

Theorem 1.2.1. *Let G be a finite group and let char$F = 0$ or char$F = p > 0$ and $p \nmid |G|$. Then the group algebra FG is semisimple.*

We now present two famous results of Wedderburn and Wedderburn-Artin concerning simple and semisimple artinian rings. Recall that a ring R is said artinian if it satisfies the descending chain condition for ideals (i.e., if every strictly descending sequence of ideals eventually terminates).

Theorem 1.2.2. *Let R be a ring. Then*

1. *R is simple artinian if and only if $R \cong M_k(D)$, the ring of $k \times k$ matrices over a division ring D, $k \geq 1$.*

2. *R is semisimple artinian if and only if $R = I_1 \oplus \cdots \oplus I_n$, where $I_1, \ldots, I_n$ are simple artinian rings and they are all minimal two-sided ideals of R.*

As a consequence, it follows that, under the hypothesis of Maschke's theorem,

$$FG \cong M_{n_1}(D^{(1)}) \oplus \cdots \oplus M_{n_k}(D^{(k)}),$$

where $D^{(1)}, \ldots, D^{(k)}$ are finite dimensional division algebras over F. In light of these results one can classify all the irreducible representations of G: L is an irreducible G-module if and only if L is an irreducible $M_{n_i}(D^{(i)})$-module, for some i. On the other hand, $M_{n_i}(D^{(i)})$ has, up to isomorphism, only one irreducible module, isomorphic to $\sum_{j=1}^{n_i} D^{(i)} e_{ij}$, where the e_{ij}s denote the usual matrix units.

From the above it can also be deduced that every G-module V is completely reducible. Hence if $\dim_F V < \infty$, V is the direct sum of a finite number of irreducible G-modules. We record this fact in the following.

Corollary 1.1. *Let G be a finite group and let $charF = 0$ or $charF = p > 0$ and $p \nmid |G|$. Then every representation of G is completely reducible and the number of inequivalent irreducible representations of G equals the number of simple components in the Wedderburn decomposition of the group algebra FG.*

Recall that an element $e \in FG$ is an idempotent if $e^2 = e$. It is well known that, since FG has finite dimensional and it is semisimple, every one-sided ideal of FG is generated by an idempotent. Moreover every two-sided ideal of FG is generated by a central idempotent. We say that an idempotent is minimal (resp. central) if it generates a minimal one-sided (resp. two-sided) ideal. We record this in the following.

Proposition 1.2. *If L is an irreducible representation of G, then $L \cong J_i$, a minimal left ideal of $M_{n_i}(D^{(i)})$, for some $i = 1, \ldots k$. Hence there exists a minimal idempotent $e \in FG$ such that $L \cong FGe$.*

If F is a splitting field for the group G, e.g., F is algebraically closed, the following properties hold.

Proposition 1.3. *Let F be a splitting field for the group G. Then the number of non-equivalent irreducible representations of G equals the number of conjugacy classes of G.*

Since by Corollary 1.1 this number equals the number of simple components of FG, it follows that, when F is a splitting field for G, it equals the dimension of the center of FG over F.

A basic tool in representation theory is provided by the theory of characters. From now on, we assume that F is a splitting field for G of characteristic zero and let $\mathrm{tr} : End(V) \to F$ be the trace function on $End(V)$.

Definition 1.11. *Let $\rho : G \to GL(V)$ be a representation of G. Then the map $\chi_\rho : G \to F$ such that $\chi_\rho(g) = tr(\rho(g))$ is called the character of the representation ρ. Moreover, $\dim_F V = \deg \chi_\rho$ is called the degree of the character χ_ρ.*

We say that the character χ_ρ is irreducible if ρ is irreducible. Moreover it is easy to prove that χ_ρ is constant on the conjugacy classes of G, i.e., χ_ρ is a class function of G. Notice that $\chi_\rho(1) = \deg \chi_\rho$.

1.2.2 S_n-representations

We now describe the ordinary representation theory of the symmetric group S_n, $n \geq 1$. We refer to [26, Chapter 2] for the results of this section.

Since $\mathbb{Q}$, the field of rational numbers, is a splitting field for S_n, for any field F of characteristic zero, the group algebra FS_n has a decomposition into semisimple components which are algebras of matrices over the field F itself. Moreover, by Proposition 1.3, the number of irreducible non-equivalent representations equals the number of conjugacy classes of S_n. Recall the following.

Definition 1.12. *Let $n \geq 1$ be an integer. A partition λ of n is a finite sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geq \cdots \geq \lambda_r$ and $\sum_{i=1}^{r} \lambda_i = n$. In this case we write $\lambda \vdash n$.*

It is well known that the conjugacy classes of S_n are indexed by the partition of n. If $\sigma \in S_n$, we decompose σ into the product of disjoint cycles, including 1-cycles, and this decomposition is unique if we require that $\sigma = \pi_1 \cdots \pi_r$, with $\pi_1, \ldots, \pi_r$ cycles of length $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$, respectively. Then the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ uniquely determines the conjugacy class of σ. Since all the irreducible characters of S_n are indexed by the partitions of n, let us denote by χ_λ the irreducible S_n-character corresponding to $\lambda \vdash n$.

Proposition 1.4. *Let F be a field of characteristic zero and $n \geq 1$. There is a one-to-one correspondence between irreducible S_n-characters and partitions of n. Let $\{\chi_\lambda \mid \lambda \vdash n\}$ be a complete set of irreducible characters of S_n and let $d_\lambda = \chi_\lambda(1)$ be the degree of χ_λ, $\lambda \vdash n$. Then*

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F),$$

where $I_\lambda = e_\lambda F S_n$ *and* $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ *is, up to a scalar, the unit element of* I_λ.

Definition 1.13. *Let* $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. *The Young diagram associated to* λ *is the finite subset of* $\mathbb{Z} \times \mathbb{Z}$ *defined as* $D_\lambda = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i = 1, \dots, r,\ j = 1, \dots, \lambda_i\}$.

The array of the boxes denoting D_λ is such that the first coordinate i (the row index) increases from top to bottom and the second coordinate j (the column index) increases from left to right.

For a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ we shall denote by $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ the conjugate partition of λ in which $\lambda'_1, \dots, \lambda'_s$ are the lengths of the columns of D_λ. Hence $D_{\lambda'}$ is obtained from D_λ by flipping D_λ along its main diagonal.

Definition 1.14. *Let* $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. *A Young tableau* T_λ *of the diagram* D_λ *is a filling of the boxes of* D_λ *with the integers* $1, 2, \dots, n$. *We shall also say that* T_λ *is a tableau of shape* λ.

Of course there are $n!$ distinct tableaux. Among these a prominent role is played by the so-called standard tableaux.

Definition 1.15. *A tableau* T_λ *of shape* λ *is standard if the integers in each row and in each column of* T_λ *increase from left to right and from top to bottom, respectively.*

There is a strict connection between standard tableaux and degrees of the irreducible S_n-characters.

Theorem 1.2.3. *Given a partition* $\lambda \vdash n$, *the number of standard tableaux of shape* λ *equals* d_λ, *the degree of* χ_λ, *the irreducible character corresponding to* λ.

Next we give a formula to compute the degree d_λ of the irreducible character χ_λ: the hook formula. First we need some further terminology.

Given a diagram D_λ, $\lambda \vdash n$, we identify a box of D_λ with the corresponding point (i, j).

Definition 1.16. *For any box* $(i, j) \in D_\lambda$, *we define the hook number of* (i, j) *as* $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, *where* λ' *is the conjugate partition of* λ.

Note that h_{ij} counts the number of boxes in the "hook" with edge in (i, j), i.e., the boxes to the right and below (i, j).

Proposition 1.5. *The number of standard tableaux of shape* $\lambda \vdash n$ *is* $d_\lambda = \frac{n!}{\prod_{i,j} h_{ij}}$, *where the product runs over all boxes of* D_λ.

We now describe a complete set of minimal left ideals of FS_n. Given a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, we denote by $T_\lambda = D_\lambda(a_{ij})$ the tableau T_λ of shape λ in which a_{ij} is the integer in the box (i, j). Then we can give the following definitions.

Definition 1.17. *The row-stabilizer of* T_λ *is*

$$R_{T_\lambda} = S_{\lambda_1}(a_{11}, a_{12}, \dots, a_{1\lambda_1}) \times \cdots \times S_{\lambda_r}(a_{r1}, a_{r2}, \dots, a_{r\lambda_r})$$

where $S_{\lambda_i}(a_{i1}, a_{i2}, \dots, a_{i\lambda_i})$ *denotes the symmetric group acting on the integers* $a_{i1}, a_{i2}, \dots, a_{i\lambda_i}$.

Definition 1.18. *The column-stabilizer of* T_λ *is*

$$C_{T_\lambda} = S_{\lambda'_1}(a_{11}, a_{21}, \dots, a_{\lambda'_1 1}) \times \cdots \times S_{\lambda'_r}(a_{1\lambda_1}, a_{2\lambda_1}, \dots, a_{\lambda'_s \lambda_1})$$

where $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ *is the conjugate partition of* λ.

Hence R_{T_λ} and C_{T_λ} are the subgroups of S_n consisting of all permutations stabilizing the rows and the columns of T_λ, respectively.

Definition 1.19. *For a given tableau* T_λ *we define* $e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (sgn\tau)\sigma\tau.$

It can be shown that $e_{T_\lambda}^2 = a e_{T_\lambda}$, where $a = \frac{n!}{d_\lambda}$ is a non-zero integer, i.e., e_{T_λ} is an essential idempotent of FS_n.

Given a partition $\lambda \vdash n$, the symmetric group S_n acts on the set of Young tableaux of shape λ as follows. If $\sigma \in S_n$ and $T_\lambda = D_\lambda(a_{ij})$, then $\sigma T_\lambda = D_\lambda(\sigma(a_{ij}))$. This action has the property that $R_{\sigma T_\lambda} = \sigma R_{T_\lambda} \sigma^{-1}$ and $C_{\sigma T_\lambda} = \sigma C_{T_\lambda} \sigma^{-1}$. It follows that $e_{\sigma T_\lambda} = \sigma e_{T_\lambda} \sigma^{-1}$. We record the most important fact about e_{T_λ} in the following.

Proposition 1.6. *For every Young tableau* T_λ *of shape* $\lambda \vdash n$*, the element* e_{T_λ} *is a minimal essential idempotent of* FS_n *and* $FS_n e_{T_\lambda}$ *is a minimal left ideal of* FS_n*, with character* χ_λ*. Moreover, if* T_λ *and* T'_λ *are Young tableaux of the same shape, then* e_{T_λ} *and* $e_{T'_\lambda}$ *are conjugated in* FS_n *through some* $\sigma \in S_n$.

The above proposition says that, for any two tableaux T_λ and T'_λ of the same shape $\lambda \vdash n$, $FS_n e_{T_\lambda} \cong FS_n e_{T'_\lambda}$, as S_n-modules.

We now regard the group S_n embedded in S_{n+1} as the subgroup of all permutations fixing the integer $n+1$. The next theorem gives a decomposition into irreducibles of any S_n-module induced up to S_{n+1}. Let us denote by L_λ an irreducible S_n-module corresponding to the partition $\lambda \vdash n$. We have the following.

Theorem 1.2.4. *Let the group* S_n *be embedded into* S_{n+1} *as the subgroup fixing the integer* $n+1$. *Then*

1. *If $\lambda \vdash n$, then $L_\lambda \uparrow S_{n+1} \cong \sum_{\mu\in\lambda^+} L_\mu$, where λ^+ is the set of all partitions of $n+1$ whose diagram is obtained from D_λ by adding one box.*

2. *If $\mu \vdash n+1$, then $L_\mu \downarrow S_n \cong \sum_{\lambda\in\mu^-} L_\lambda$, where μ^- is the set of all partitions of n whose diagram is obtained from D_μ by deleting one box.*

We now embed the group $S_n \times S_m$ into S_{n+m} by letting S_m act on $\{n+1, \ldots, n+m\}$. Recall that if N is an S_n-module and N' is an S_m-module, then $N \otimes N'$ has a natural structure of $(S_n \times S_m)$-module.

Definition 1.20. *If N is an S_n-module and N' is an S_m-module, then the outer tensor product of N and N' is defined as*

$$N \widehat{\otimes} N' \cong \left(N \otimes N'\right) \uparrow S_{n+m}.$$

Recall that (m) denotes a one-row partition $\mu \vdash m$, i.e., $\mu_1 = m$. In the following theorem we present the so-called Young rule.

Theorem 1.2.5. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and $m \geq 1$. Then*

$$N_\lambda \widehat{\otimes} N_{(m)} \cong \sum M_\mu$$

where the sum runs over all partitions μ of $n+m$ such that we have $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_{n+m} \geq \lambda_{n+m}$.

Definition 1.21. *An unordered partition of n is a finite sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_t)$ such that $\sum_{i=1}^{t} \alpha_i = n$. In this case we write $\alpha \models n$.*

Definition 1.22. *Let $\lambda \vdash n$ and $\alpha \models n$. A (generalized) Young tableau of shape λ and content α is a filling of the diagram D_λ by positive integers in such a way that the integer i occurs exactly α_i times.*

Definition 1.23. *A Young tableau is semistandard if the numbers are non-decreasing along the rows and strictly increasing down the columns.*

We now consider the obvious partial order on the set of partitions. Let $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$ and $\mu = (\mu_1, \ldots, \mu_q) \vdash m$, then $\lambda \geq \mu$ if and only if $p \geq q$ and $\lambda_i \geq \mu_i$, for all $i = 1, \ldots, p$. In the language of Young diagrams $\lambda \geq \mu$ means that D_μ is a subdiagram of D_λ.

If $\lambda \geq \mu$, we define the skew-partition $\lambda \setminus \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$. The corresponding diagram $D_{\lambda\setminus\mu}$ is the set of boxes of D_λ which do not belong to D_μ.

Definition 1.24. *A skew-tableau $T_{\lambda\setminus\mu}$ is a filling of the boxes of the skew-diagram $D_{\lambda\setminus\mu}$ with distinct natural numbers. If repetitions occur, then we have the notion of generalized skew-tableau. We also have the natural notion of standard and semistandard skew-tableau.*

Definition 1.25. *Let $\alpha = (\alpha_1, \ldots, \alpha_t) \models n$. We say that α is a lattice permutation if for each j the number of i's which occur among $\alpha_1, \ldots, \alpha_j$ is greater than or equal to the number of $(i+1)$'s for each i.*

We can now formulate the Littlewood-Richardson rule.

Theorem 1.2.6. *Let $\lambda \vdash n$ and $\mu \vdash m$. Then*

$$N_\lambda \widehat{\otimes} N_\mu \cong \sum_{\nu \vdash n+m} k^{\mu}_{\nu \setminus \lambda} \, N_\nu$$

where $k^{\mu}_{\nu\setminus\lambda}$ is the number of semistandard tableau of shape $\nu \setminus \lambda$ and content μ which yield lattice permutations when we read their entries from right to left and downwards.

1.2.3 Representations of the general linear group

In this section we survey the information on representation theory of the general linear group over an algebraically closed field F of characteristic zero. We restrict our attention to the case when $GL_m = GL_m(F)$ acts on the free associative algebra of rank m and consider the so-called polynomial representation of GL_m, which have many properties similar to those of the representations of finite groups. We refer to [11, Chapter 12] for the results of this section.

Definition 1.26. *The representation $\phi : GL_m \to GL_s$ of the general linear group GL_m is called polynomial if the entries of the $s \times s$ matrix $\phi(a_{ij})$ are polynomial functions of the entries of the $m \times m$ matrix a_{ij}, for all $a_{ij} \in GL_m$. When all the entries of $\phi(a_{ij})$ are homogeneous polynomials of degree k, then ϕ is a homogeneous representation of degree k.*

Let $F_m\langle X\rangle = F\langle x_1, \ldots, x_m\rangle$ denote the free associative algebra in m variables and let $U = \operatorname{span}_F \{x_1, \ldots, x_m\}$. The action of the group $GL_m \cong GL(U)$ on $F_m\langle X\rangle$ can be obtained extending diagonally the natural left action of GL_m on the space U by defining:

$$g(x_{i_1}, \ldots, x_{i_k}) = g(x_{i_1}) \cdots g(x_{i_k}), \quad g \in GL_m, \; x_{i_1}, \ldots, x_{i_k} \in F_m\langle X\rangle.$$

Actually, $F_m\langle X\rangle$ is a polynomial GL_m-module, i.e., the corresponding representation is polynomial.

Let F_m^n be the space of homogeneous polynomials of degree n in the variables $x_1, \ldots, x_m$. Then F_m^n is a (homogeneous polynomial) submodule of $F_m\langle X\rangle$. We observe that

$$F_m^n = \bigoplus_{i_1+\cdots+i_m=n} F_m^{(i_1,\ldots,i_m)}$$

where $F_m^{(i_1,\ldots,i_m)}$ is the multihomogeneous subspace spanned by all monomials of degree i_1 in $x_1, \ldots,$ i_m in x_m.

The following theorem states a result similar to Maschke's theorem about the complete reducibility of GL_m-modules, valid for the polynomial representations of GL_m.

Theorem 1.2.7. *Every polynomial GL_m-module is a direct sum of irreducible homogeneous polynomial submodules.*

The irreducible homogeneous polynomial GL_m-modules are described by partition of n in no more than m parts and Young diagrams.

Theorem 1.2.8. *Let $P_m(n)$ denote the set of all partitions of n with at most m parts (i.e. whose diagrams have height at most m).*

1. *The pairwise non isomorphic irreducible homogeneous polynomial GL_m-modules of degree $n \geq 1$ are in one-to-one correspondence with the partitions of $P_m(n)$. We denote by W^λ an irreducible GL_m-module related to λ.*

2. *Let $\lambda \in P_m(n)$. Then the GL_m-module W^λ is isomorphic to a submodule of F_m^n. Moreover the GL_m-module F_m^n has a decomposition:*
$$F_m^n \cong \sum_{\lambda \in P_m(n)} d_\lambda W^\lambda,$$
where d_λ is the dimension of the irreducible S_n-module corresponding to the partition λ.

3. *As a subspace of F_m^n, the vector space W^λ is multihomogeneous, i.e.*
$$W^\lambda = \bigoplus_{i_1+\cdots+i_m=n} W^{\lambda,(i_1,\ldots,i_m)},$$
where $W^{\lambda,(i_1,\ldots,i_m)} = W^\lambda \cap F_m^{(i_1,\ldots,i_m)}$.

We want to show that if $W^\lambda \subseteq F_m^n$, then W^λ is cyclic and generated by a multihomogeneous polynomial of multidegree $\lambda_1, \ldots, \lambda_k$, with $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_m^n$.

We observe first that the symmetric group S_n acts from the right on F_m^n by permuting the places in which the variables occur, i.e., for all $x_{i_1}, \ldots, x_{i_n} \in F_m^n$ and for all $\sigma \in S_n$, we have $x_{i_1} \cdots x_{i_n} \sigma^{-1} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n)}}$.

Let now $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_m^n$. We denote by s_λ the following polynomial of F_m^n:
$$s_\lambda = s_\lambda(x_1, \ldots, x_{\lambda_1'}) = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}\left(x_1, \ldots, x_{h_i(\lambda)}\right),$$
where $h_i(\lambda)$ is the height of the i-th column of the diagram of λ and $St_r(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} (\text{sgn}\sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}$ is the standard polynomial of degree r. Note that by definition s_λ is multihomogeneous of multidegree $(\lambda_1, \ldots, \lambda_k)$.

Theorem 1.2.9. *The following conditions hold:*

1. *The element s_λ generates an irreducible GL_m-submodule W of F_m^n isomorphic to W^λ.*

2. *Every submodule $W^\lambda \subseteq F_m^n$ is generated by a non-zero polynomial called highest weight vector of W^λ, of the type*
$$f_\lambda = s_\lambda \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \quad \alpha_\sigma \in F. \tag{1.1}$$
The highest weight vector is unique up to a multiplicative constant and it is contained in the one-dimensional vector space $W^{\lambda,(i_1,\dots,i_m)}$.

3. *Let $\sum_{\sigma \in S_n} \alpha_\sigma \sigma \in FS_n$. If $s_\lambda \sum_{\sigma \in S_n} \alpha_\sigma \sigma \neq 0$, then it generates an irreducible submodule $W \cong W^\lambda$, $W \subseteq F_m^n$.*

By Theorem 1.2.8 the multiplicity of W^λ in F_m^n is equal to the dimension d_λ of the irreducible S_n-module corresponding to the partition λ. Hence every $W \cong W^\lambda \subset F_m^n$ is a submodule of the direct sum of d_λ isomorphic copies of W^λ and the problem is how to find the highest weight vectors of those d_λ modules.

We fix $\lambda = (\lambda_1, \dots, \lambda_k) \in P_m^n$ and let T_λ be a Young tableau. We denote by f_{T_λ} the highest weight vector obtained from (1.1) by considering the only permutation $\sigma \in S_n$ such that the first column of T_λ is filled from top to bottom with the integers $\sigma(1), \dots, \sigma(h_1(\lambda))$, in this order, the second column is filled with $\sigma(h_1(\lambda)+1), \dots, \sigma(h_1(\lambda)+h_2(\lambda))$ and so on.

Proposition 1.7. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \in P_m^n$ and let $W^\lambda \subseteq F_m^n$. The highest weight vector f_λ of W^λ can be expressed uniquely as a linear combination of the polynomials f_{T_λ} with T_λ standard tableau.*

1.3 PI-algebras

In this section we focus our attention on associative algebras satisfying a non-trivial polynomial identity (PI-algebras). One of the most interesting and challenging problems in combinatorial PI-theory is that of finding numerical invariants allowing to classify the T-ideals of $F\langle X \rangle$. A very useful numerical invariant that can be attached to a T-ideal is given by the sequence of codimensions. It was introduced by Regev in [50] and measures the rate of growth of the multilinear polynomials lying in a given T-ideal.

In order to define this object, let A be a PI-algebra over a field F of characteristic zero and $\mathrm{Id}(A)$ its T-ideal of identities. We introduce

$$P_n = \mathrm{span}\left\{ x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n \right\}$$

the vector space of multilinear polynomials in the variables $x_1, \ldots, x_n$ in the free algebra $F\langle X\rangle$. According to Theorem 1.1.3, since char$F = 0$, then Id(A) is determined by its multilinear polynomials and so it is generated, in the free associative algebra $F\langle X\rangle$, by the subspace

$$(P_1 \cap \mathrm{Id}(A)) \oplus (P_2 \cap \mathrm{Id}(A)) \oplus \cdots \oplus (P_n \cap \mathrm{Id}(A)) \oplus \cdots .$$

It is clear that if A satisfies all the identities of some PI-algebra B, then $P_n \cap \mathrm{Id}(A) \supseteq P_n \cap \mathrm{Id}(B)$ and so $\dim(P_n \cap \mathrm{Id}(A)) \geq \dim(P_n \cap \mathrm{Id}(B))$, for all $n = 1, 2, \ldots$. Therefore the dimensions of the spaces $P_n \cap \mathrm{Id}(A)$ give us, in some sense, the growth of the identities of the algebra A.

Definition 1.27. *The non-negative integer*

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap Id(A)}$$

is called the n-th codimension of the algebra A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the codimensions sequence of A.

The most important feature of the sequence of codimensions was proved by Regev in [50]. We recall such result in the following.

Theorem 1.3.1 (Regev, 1972). *If A is a PI-algebra then $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded.*

We now introduce an action of the symmetric group S_n on P_n. From now on, we assume that char$F = 0$. We start with the following lemma concerning arbitrary irreducible S_n-module.

Lemma 1.1. *Let L be an irreducible left S_n-module with character $\chi(L) = \chi_\lambda$, $\lambda \vdash n$. Then L can be generated, as an S_n-module, by an element of the form $e_{T_\lambda} f$, for some $f \in L$ and some Young tableau T_λ. Moreover, for any Young tableau T'_λ of shape λ, there exists $f' \in L$ such that $L = FS_n e_{T'_\lambda} f'$.*

The previous lemma says that, given a partition $\lambda \vdash n$ and a Young tableau T_λ of shape λ, any irreducible S_n-module L such that $\chi(L) = \chi_\lambda$, can be generated by an element of the form $e_{T_\lambda} f$, for some $f \in L$. By the definition of R_{T_λ}, for any $\sigma \in R_{T_\lambda}$, we have that $\sigma e_{T_\lambda} f = e_{T_\lambda} f$, i.e., $e_{T_\lambda} f$ is stable under the R_{T_λ}-action. The number of R_{T_λ}-stable elements in an arbitrary S_n-module L is closely related to the number of irreducible S_n-submodules of L having character χ_λ.

Lemma 1.2. *Let T_λ be a Young tableau corresponding to $\lambda \vdash n$ and let L be an S_n-module such that $L = L_1 \oplus \cdots \oplus L_m$, where $L_1, \ldots, L_m$ are irreducible S_n-submodules with character χ_λ. Then m is equal to the number of linearly independent elements $g \in L$ such that $\sigma g = g$, for all $\sigma \in R_{T_\lambda}$.*

Let now $\varphi : FS_n \to P_n$ be the map defined by $\varphi\left(\sum_{\sigma\in S_n}\alpha_\sigma\sigma\right) = \sum_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$. It is clear that φ is a linear isomorphism. This isomorphism turns P_n into an S_n-bimodule (i.e., an abelian group that is both a left and a right S_n-module and in which the left and right multiplications are compatible). If σ, $\tau \in S_n$, then

$$\sigma\left(x_{\tau(1)}\cdots x_{\tau(n)}\right) = x_{\sigma\tau(1)}\cdots x_{\sigma\tau(n)} = \left(x_{\sigma(1)}\cdots x_{\sigma(n)}\right)\tau.$$

The interpretation of the left S_n-action on a polynomial $f(x_1,\ldots,x_n) \in P_n$, for $\sigma \in S_n$, is

$$\sigma f(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}),$$

that is, of permuting the variables according to σ. Since T-ideals are invariant under permutations of the variables, we obtain that the subspace $P_n \cap \mathrm{Id}(A)$ is invariant under this action, that is $P_n \cap \mathrm{Id}(A)$ is a left S_n-submodule of P_n. Hence

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

has an induced structure of left S_n-module.

Definition 1.28. *For $n \geq 1$, the S_n-character of $P_n(A) = P_n/(P_n \cap Id(A))$, denoted by $\chi_n(A)$, is called n-th cocharacter of A.*

If A is an algebra over a field of characteristic zero, we can decompose the n-th cocharacter into irreducibles as follows:

$$\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

Our next goal is to present several results concerning algebras of polynomial growth.

Definition 1.29. *Let A be a PI-algebra. A has polynomial growth if its sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded, i.e., $c_n(A) \leq an^b$, for some constants a and b.*

If $\mathcal{V}$ is a variety of algebras, then the growth of $\mathcal{V}$ is defined as the growth of the sequence of codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}(A)$. Hence we have the following.

Definition 1.30. *A variety $\mathcal{V}$ has polynomial growth if its sequence of codimensions $c_n(\mathcal{V})$, $n = 1, 2, \ldots$, is polynomially bounded. We say that $\mathcal{V}$ has almost polynomial growth if $c_n(\mathcal{V})$, $n = 1, 2, \ldots$, is not polynomially bounded but any proper subvariety of $\mathcal{V}$ has polynomial growth.*

Actually, the condition $c_n(\mathcal{V})$ not polynomially bounded simply means that $\mathcal{V}$ has exponential growth. In fact, the following result, due to Kemer, holds.

Theorem 1.3.2 ([32, 33]). *Let A be a PI-algebra. Then $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded or it grows exponentially.*

In order to present another celebrated theorem of Kemer, let us define two suitable algebras generating varieties of almost polynomial growth. Let G be the infinite dimensional Grassmann algebra over F, i.e., the algebra generated by a countable set of elements $\{e_1, e_2, \ldots\}$ satisfying the condition $e_i e_j = -e_j e_i$ and let UT_2 be the algebra of 2×2 upper triangular matrices over F.

Such algebras, generating varieties of almost polynomial growth, were extensively studied in [36] and [44]. In the following theorem we collect the results of [36, 44], concerning the T-ideals of these two algebras. Recall that $\langle f_1, \ldots, f_n \rangle_T$ denotes the T-ideal generated by the polynomials $f_1, \ldots, f_n \in F\langle X \rangle$.

Theorem 1.3.3.
1. *$Id(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle_T$.*
2. *$Id(G) = \langle [[x_1, x_2], x_3] \rangle_T$.*

Now we are ready to state the celebrated result of Kemer.

Theorem 1.3.4 (**Kemer, 1979**). *A variety of algebras $\mathcal{V}$ has polynomial growth if and only if G, $UT_2 \notin \mathcal{V}$.*

Corollary 1.2. *The varieties $var(G)$ and $var(UT_2)$ are the only varieties of almost polynomial growth.*

The next theorem characterizes the algebras of polynomial growth in terms of their cocharacter sequence.

Theorem 1.3.5 ([32]). *Let A be a PI-algebra. Then $c_n(A)$ is polynomially bounded if and only if there exists a constant q such that, for all $n \geq 1$,*

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 \leq q}} m_\lambda \chi_\lambda.$$

As a consequence, it is possible to prove the following.

Theorem 1.3.6. *If $\mathcal{V}$ is a variety of algebras with polynomial growth then $\mathcal{V} = var(A)$, for some finite dimensional algebra A.*

Finally, we present three results of Giambruno and Zaicev characterizing finite dimensional algebras generating varieties of polynomial growth.

Proposition 1.8 ([24]). *Let $\mathcal{V}$ be a variety of algebras over an algebraically closed field F of characteristic zero. Then $\mathcal{V}$ has polynomial growth if and only if $\mathcal{V} = var(A)$ for some finite dimensional algebra A such that*

1. *$A = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ a vector space direct sum of F-algebras where for $i = 1, \ldots, m$, $A_i = B_i + J_i$, $B_i \cong F$, J_i is a nilpotent ideal of A_i and A_0, $J_1, \ldots, J_m$ are nilpotent right ideals of A;*

2. *for all $i, k \in \{1, \ldots, m\}$, $i \neq k$, $A_iA_k = 0$ and $B_iA_0 = 0$.*

Moreover, if J is the Jacobson radical of A, then $Id(A) = Id(A_1 \oplus A_0) \cap \cdots \cap Id(A_m \oplus A_0) \cap Id(J)$.

Theorem 1.3.7 ([24]). *A variety of algebras $\mathcal{V}$ has polynomial growth if and only if $\mathcal{V} = var(A_1 \oplus \cdots \oplus A_m)$, where $A_1, \ldots, A_m$ are finite dimensional algebras over F and $\dim_F A_i/J(A_i) \leq 1$, for all $i = 1, \ldots, m$.*

If in Theorem 1.3.5 we take A to be a finite dimensional algebra (as we may by Theorem 1.3.6), we can find an upper bound for the number of boxes below the first row of any diagram in the n-th cocharacter of A (see [24, 32]).

Theorem 1.3.8. *Let A be a finite dimensional algebra. Then A has polynomial growth if and only if*

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda \chi_\lambda,$$

where q is such that $J(A)^q = 0$.

We now collect these results in the following.

Theorem 1.3.9. *For a variety of algebras $\mathcal{V}$, the following conditions are equivalent:*

1) $c_n(\mathcal{V})$ is polynomially bounded.

2) $G, UT_2 \notin \mathcal{V}$.

3) $\mathcal{V} = var(A)$, where $A = A_1 \oplus \cdots \oplus A_m$, with $A_1, \ldots, A_m$ finite dimensional algebras over F such that $\dim A_i/J(A_i) \leq 1$, for all $i = 1, \ldots, m$.

4) For all $n \geq 1$, there exists a constant q such that $\chi_n(\mathcal{V}) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 \leq q}} m_\lambda \chi_\lambda$. If $\mathcal{V} = var(A)$ with A finite dimensional algebra, then q is such that $J(A)^q = 0$.

1.4 Algebras with involution

Let A be an associative algebra over a field F. From now until the end of this section the field F is of characteristic zero.

Definition 1.31. *An involution on A is a linear map $\sharp : A \to A$ of order two ($(a^\sharp)^\sharp = a$, for all $a \in A$) such that, for all $a, b \in A$,*

$$(ab)^\sharp = b^\sharp a^\sharp.$$

Let A be an algebra with involution $\sharp$. We write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^\sharp = a\}$ and $A^- \{a \in A \mid a^\sharp = -a\}$ denote the sets of symmetric and skew elements of A, respectively. Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $F\langle X, \sharp\rangle = F\langle x_1, x_1^\sharp, x_2, x_2^\sharp, \ldots\rangle$ be the free associative algebra with involution on X over F. It is useful to regard to $F\langle X, \sharp\rangle$ as generated by symmetric and skew variables: if for $i = 1, 2, \ldots,$ we let $x_i^+ = x_i + x_i^\sharp$ and $x_i^- = x_i - x_i^\sharp$, then

$$F\langle X, \sharp\rangle = F\langle x_1^+, x_1^-, x_2^+, x_2^-, \ldots\rangle.$$

Definition 1.32. *A $\sharp$-polynomial $f(x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_m^-) \in F\langle X, \sharp\rangle$ is a $\sharp$-identity of the algebra with involution A, and we write $f \equiv 0$, if, for all $s_1, \ldots, s_n \in A^+$, $k_1, \ldots, k_m \in A^-$, we have $f(s_1, \ldots, s_n, k_1, \ldots, k_m) = 0$.*

We denote by $\mathrm{Id}^\sharp(A) = \{f \in F\langle X, \sharp\rangle \mid f \equiv 0 \text{ on } A\}$ the $T^\sharp$-ideal of $\sharp$-identities of A, i.e., $\mathrm{Id}^\sharp(A)$ is an ideal of $F\langle X, \sharp\rangle$ invariant under all endomorphisms of the free algebra commuting with the involution $\sharp$. It is well known that in characteristic zero, every $\sharp$-identity is equivalent to a system of multilinear $\sharp$-identities. Let

$$P_n^\sharp = \mathrm{span}_F\left\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \left\{x_i^+, x_i^-\right\},\ i = 1, \ldots, n\right\}$$

be the vector space of multilinear polynomials of degree n in the variables $x_1^+, x_1^-, \ldots, x_n^+, x_n^-$. Hence for every $i = 1, \ldots, n$ either x_i^+ or x_i^- appears in every monomial of $P_n^\sharp$ at degree 1 (but not both). The study of $\mathrm{Id}^\sharp(A)$ is equivalent to the study of $P_n^\sharp \cap \mathrm{Id}^\sharp(A)$, for all $n \geq 1$ and we can give the following definition.

Definition 1.33. *The non-negative integer*

$$c_n^\sharp(A) = \dim_F \frac{P_n^\sharp}{P_n^\sharp \cap Id^\sharp(A)},\ n \geq 1,$$

is called the n-th $\sharp$-codimension of A.

Given a non-empty set $S \subseteq F\langle X, \sharp\rangle$, the class of all algebras with involution A such that $f \equiv 0$ on A, for all $f \in S$, is called the $\sharp$-variety $\mathcal{V} = \mathcal{V}(S)$ determined by S. Moreover, if A is an algebra with involution, we write $\mathrm{var}^\sharp(A)$ to denote the $\sharp$-variety generated by A, i.e., the class of all algebras with involution B such that $f \equiv 0$ on B, for all $f \in \mathrm{Id}^\sharp(A)$.

As in the case of associative algebras we have the following.

Definition 1.34. *Let A be an algebra with involution. A has polynomial growth if its sequence of $\sharp$-codimensions $c_n^\sharp(A)$, $n = 1, 2, \ldots$, is polynomially bounded.*

At the same way we define the growth of a $\sharp$-variety $\mathcal{V}$ as the growth of the sequence of $\sharp$-codimensions of any algebra with involution A generating $\mathcal{V}$. Then a $\sharp$-variety $\mathcal{V}$ has polynomial growth if $c_n^\sharp(\mathcal{V})$, $n = 1, 2, \ldots$, is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^\sharp(\mathcal{V})$ is not polynomially bounded but any proper subvariety of $\mathcal{V}$ has polynomial growth.

In order to present an analogous result than that Kemer proved for associative algebras, let us consider the following suitable algebras with involution generating $\sharp$-varieties of almost polynomial growth.

For a start, let UT_n be the algebra of $n \times n$ upper triangular matrices over F. One can define on UT_n the so-called "reflection" involution *ref* in the following way: if $a \in UT_n$, $a^{ref} = ba^tb^{-1}$, where a^t denotes the usual transpose and b is the permutation matrix

$$b = e_{1,n} + e_{2,n-1} + \cdots + e_{n-1,2} + e_{n,1},$$

where the e_{ij}s denote the usual matrix units. Clearly a^{ref} is the matrix obtained from a by reflecting a along its secondary diagonal: if $a = (a_{ij})$ then $a^{ref} = (b_{ij})$ where $b_{ij} = a_{n+1-j\ n+1-i}$.

Let now $M = \left\{ \begin{pmatrix} u & r & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & v \\ 0 & 0 & 0 & u \end{pmatrix} \mid u, r, s, v \in F \right\}$ be a subalgebra of UT_4 endowed with the reflection involution and let $F \oplus F$ be the two-dimensional commutative algebra endowed with the exchange involution *ex* defined, for all $(a, b) \in F \oplus F$, by $(a, b)^{ex} = (b, a)$.

In the following theorem we collect the results of [19, 45] concerning this two algebras with involution. Recall that $\langle f_1, \ldots, f_n\rangle_{T^\sharp}$ denotes the $T^\sharp$-ideal generated by the polynomials $f_1, \ldots, f_n \in F\langle X, \sharp\rangle$.

Theorem 1.4.1. *The algebras $F \oplus F$ with exchange involution and M with reflection involution generate $\sharp$-varieties of almost polynomial growth. Moreover*

1. *$Id^\sharp(F \oplus F) = \langle [x_1^+, x_2^+], [x^+, x^-], [x_1^-, x_2^-]\rangle_{T^\sharp}$.*

2. $Id^\sharp(M) = \langle x_1^- x_2^- \rangle_{T^\sharp}$.

We remark that, in [45], Mishchenko and Valenti proved that, in the finite dimensional case over an algebraically closed field of characteristic zero, actually $\mathrm{var}^\sharp(F \oplus F)$ and $\mathrm{var}^\sharp(M)$ are the only two $\sharp$-varieties of almost polynomial growth. In fact, the following theorem holds.

Theorem 1.4.2 ([45]). *Let A be a finite dimensional algebra with involution over an algebraically closed field F of characteristic zero. Then $var^\sharp(A)$ has polynomial growth if and only if $F \oplus F$, $M \notin var^\sharp(A)$.*

This result was improved by Giambruno and Mishchenko, that, in [20], were able to characterize the $\sharp$-varieties of polynomial growth as follows.

Theorem 1.4.3 (**Giambruno-Mishchenko, 2001**). *Let $\mathcal{V}$ be a $\sharp$-variety of algebras with involution over a field F of characteristic zero. Then $\mathcal{V}$ has polynomial growth if and only if $F \oplus F$, $M \notin \mathcal{V}$.*

Corollary 1.3. *The $\sharp$-varieties $var(F \oplus F)$ and $var(M)$ are the only $\sharp$-varieties of almost polynomial growth.*

1.5 Superalgebras

Let A be an algebra over a field F of characteristic zero and let $\mathbb{Z}_2$ be the cyclic group of order 2. We start by introducing the notion of $\mathbb{Z}_2$-graded algebra.

Definition 1.35. *The algebra A is $\mathbb{Z}_2$-graded if it can be written as the direct sum of subspaces $A = A_0 \oplus A_1$ such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$.*

The subspaces A_0 and A_1 are the homogeneous components of A and their elements are called homogeneous of degree zero (even elements) and of degree one (odd elements), respectively. If a is an homogeneous element we shall write $\deg a$ or $|a|$ to indicate its homogeneous degree. A subspace $B \subseteq A$ is graded if $B = (B \cap A_0) \oplus (B \cap A_1)$. In a similar way one can define graded subalgebra, graded ideals, and so on.

The $\mathbb{Z}_2$-graded algebras are simply called superalgebras.

Example 1.5. *Any algebra A can be view as a superalgebra with trivial grading by setting $A_0 = A$ and $A_1 = 0$.*

Example 1.6. *Let $A = M_n(F)$ be the algebra of $n \times n$ matrices over F and let $\mathbb{Z}_2 \cong \{0, 1\}$. Given any n-tuple $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$, one can define a $\mathbb{Z}_2$-grading on A, called elementary, by setting*

$$A_i = span_F \{e_{ij} \mid g_i + g_j = i\}, \quad i = 0, 1,$$

where e_{ij}s are the usual matrix units.

The free associative algebra $F\langle X\rangle$ on a countable set $X = \{x_1, x_2, \ldots\}$ has a natural structure of superalgebra as follows. We write $X = Y \cup Z$, the disjoint union of two sets and we denote by $\mathcal{F}_0$ the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of X having even degree in the variables of Z and by $\mathcal{F}_1$ the subspace spanned by all monomials of odd degree in Z. Then $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a superalgebra called the free superalgebra on $X = Y \cup Z$ over F.

Definition 1.36. *A superpolynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle Y \cup Z\rangle$ is a superidentity of the superalgebra $A = A_0 \oplus A_1$, and we write $f \equiv 0$, if, for all $v_1, \ldots, v_n \in A_0$, $u_1, \ldots, u_m \in A_1$, we have*

$$f(v_1, \ldots, v_n, u_1, \ldots, u_m) = 0.$$

We denote by $\mathrm{Id}^{sup}(A) = \{f \in F\langle Y \cup Z\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2-ideal of superpolynomial identities of A (i.e., an ideal invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$).

It is well known that in characteristic zero $\mathrm{Id}^{sup}(A)$ is completely determined by its multilinear polynomials and we denote by P_n^{sup} the vector space of all multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$.

Definition 1.37. *The non-negative integer*

$$c_n^{sup}(A) = \dim_F \frac{P_n^{sup}}{P_n^{sup} \cap Id^{sup}(A)}, \quad n \geq 1,$$

is called the n-th supercodimension of A.

Given a non-empty set $S \subseteq F\langle Y \cup Z\rangle$, the class of all superalgebras A such that $f \equiv 0$ on A, for all $f \in S$, is called the supervariety $\mathcal{V} = \mathcal{V}(S)$ determined by S. Moreover, if A is a superalgebra, we write $\mathrm{var}^{sup}(A)$ to denote the supervariety generated by A, i.e., the class of all superalgebras B such that $f \equiv 0$ on B, for all $f \in \mathrm{Id}^{sup}(A)$.

As in the case of associative algebras we have the following.

Definition 1.38. *Let A be a superalgebra. A has polynomial growth if its sequence of supercodimensions c_n^{sup}, $n = 1, 2, \ldots$, is polynomially bounded.*

At the same way we define the growth of a supervariety $\mathcal{V}$ as the growth of the sequence of supercodimensions of any superalgebra A generating $\mathcal{V}$. Then a supervariety $\mathcal{V}$ has polynomial growth if $c_n^{sup}(\mathcal{V})$, $n = 1, 2, \ldots$, is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^{sup}(\mathcal{V})$ is not polynomially bounded but any proper subvariety of $\mathcal{V}$ has polynomial growth.

We now present a list of superalgebras generating corresponding supervarieties of almost polynomial growth. Later we shall see that they generate the only supervarieties with this property.

First let $F \oplus F$ be the two-dimensional commutative algebra with exchange involution, defined in the previous section. Since $F \oplus F$ is commutative, the exchange involution is also an automorphism of order 2 and so $F \oplus F$ can be regarded as a superalgebra with grading $(F \oplus F)_0 = F(1,1)$ and $(F \oplus F)_1 = F(1,-1)$.

Recall that if A is a PI-algebra, then A can be viewed as a superalgebra with trivial grading, i.e., $A = A_0 \oplus A_1$, where $A_0 = A$ and $A_1 = 0$. Thus $\mathrm{var}(A)$ can be viewed as the supervariety defined by all the identities of the algebra A in even variables and by the identity $z \equiv 0$, z odd.

Let us consider the infinite dimensional Grassmann algebra G, i.e., the algebra generated by a countable set of elements $\{e_1, e_2, \ldots\}$ satisfying the relations $e_ie_j = -e_je_i$, for all i, j. It is well known that G has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where G_0 is the span of all monomials in the e_is of even length and G_1 is the span of all monomials in the e_is of odd length. Since we can define on G also the trivial grading, in order to distinguish between the two, we denote G^{sup} the Grassmann algebra with its natural $\mathbb{Z}_2$-grading and with G the Grassmann algebra with trivial grading.

In a similar way, we shall use UT_2 to denote the algebra of 2×2 upper triangular matrices over F with trivial grading whereas UT_2^{sup} denote the algebra of 2×2 upper triangular matrices with natural grading $(UT_2^{sup})_0 = Fe_{11} + Fe_{22}$ and $(UT_2^{sup})_1 = Fe_{12}$.

We now examine the structure of the superidentities of the above algebras. Recall that $\langle f_1, \ldots, f_n \rangle_{T_2}$ denotes the T_2-ideal generated by the polynomials $f_1, \ldots, f_n \in F\langle Y \cup Z \rangle$.

Since the two-dimensional commutative algebra $F \oplus F$ with exchange involution, can be regarded as a superalgebra, by Theorem 1.4.1, we have the following.

Proposition 1.9. *The superalgebra $F \oplus F$ generates a supervariety of almost polynomial growth. Moreover*

$$Id^{sup}(F \oplus F) = \langle [y_1, y_2], [y, z], [z_1, z_2] \rangle_{T_2}.$$

Let now consider the superalgebras G and UT_2 with trivial grading. Since $z \equiv 0$ on G and UT_2, then we are dealing with ordinary identity and, by Theorem 1.3.3 we get the following.

Proposition 1.10. *The superalgebras G and UT_2 generate corresponding supervarieties of almost polynomial growth. Moreover*

1. *$Id^{sup}(UT_2) = \langle [y_1, y_2][y_3, y_4], z \rangle_{T_2}$.*
2. *$Id^{sup}(G) = \langle [[y_1, y_2], y_3], z \rangle_{T_2}$.*

We now treat the case of G^{sup}. In [21] the authors proved the following.

Proposition 1.11. *The superalgebra G^{sup} generates a supervariety of almost polynomial growth. Moreover*

$$Id^{sup}(G^{sup}) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1 \rangle_{T_2}.$$

The following proposition collects the results, concerning the superalgebra UT_2^{sup}, proved by Valenti in [57].

Proposition 1.12. *The superalgebra UT_2^{sup} generates a supervariety of almost polynomial growth. Moreover*

$$Id^{sup}(UT_2^{sup}) = \langle [y_1, y_2], z_1 z_2 \rangle_{T_2}.$$

Finally, we are ready to present the following result in which the authors characterize the supervarieties of polynomial growth (see [21]).

Theorem 1.5.1 (Giambruno-Mishchenko-Zaicev, 2001). *Let $\mathcal{V}$ be a supervariety of superalgebras. Then $\mathcal{V}$ has polynomial growth if and only if $F \oplus F$, G, G^{sup}, UT_2, $UT_2^{sup} \notin \mathcal{V}$.*

Corollary 1.4. *The supervarieties $var(F \oplus F)$, $var(G)$, $var(G^{sup})$, $var(UT_2)$, $var(UT_2^{sup})$ are the only supervarieties of almost polynomial growth.*

1.6 Superalgebras with superinvolution

In this section we introduce the main objects of the thesis, that is the superalgebras with superinvolution. From now on, unless otherwise stated, the field F is of characteristic zero.

Recall that, if $A = A_0 \oplus A_1$, $B = B_0 \oplus B_1$ are two superalgebras, then a linear map $\varphi : A \to B$ is said to be graded if $\varphi(A_i) \subseteq B_i$, $i = 0, 1$.

Definition 1.39. *Let $A = A_0 \oplus A_1$ be a superalgebra. We say that A is a superalgebra with superinvolution $*$ if it is endowed with a graded linear map $* : A \to A$ with the following properties:*

1. $(a^*)^* = a$, *for all* $a \in A$,
2. $(ab)^* = (-1)^{|a||b|} b^* a^*$, *for any homogeneous elements* $a, b \in A_0 \cup A_1$.

Let $A = A_0 \oplus A_1$ be a superalgebra with superinvolution $*$. Since char$F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i : a^* = a\}$ and $A_i^- = \{a \in A_i : a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively.

From now on, we shall refer to a superalgebra with superinvolution simply as a $*$-algebra.

Our next step is to define a superinvolution on the free associative superalgebra $F\langle Y \cup Z \rangle$ which we have defined in the previous section. We write the sets Y and Z as the disjoint union of two other infinite sets of symmetric and skew elements, respectively. Hence the free algebra with superinvolution (free $*$-algebra), denoted by $F\langle Y \cup Z, *\rangle$, is generated by symmetric and skew elements of even and odd degree. We write

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

In order to simplify the notation, sometimes we denote by y any even variable, by z any odd variable and by x an arbitrary variable.

Definition 1.40. *A $*$-polynomial $f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_t^+, z_1^-, \ldots, z_s^-) \in F\langle Y \cup Z, *\rangle$ is a $*$-identity of the $*$-algebra $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, and we write $f \equiv 0$, if, for all $u_1^+, \ldots, u_n^+ \in A_0^+$, $u_1^-, \ldots, u_m^- \in A_0^-$, $v_1^+, \ldots, v_t^+ \in A_1^+$ and $v_1^-, \ldots, v_s^- \in A_1^-$, we have*

$$f(u_1^+, \ldots, u_n^+, u_1^-, \ldots, u_m^-, v_1^+, \ldots, v_t^+, v_1^-, \ldots, v_s^-) = 0.$$

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, *\rangle$ invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra $F\langle Y \cup Z\rangle$ commuting with the superinvolution $*$.

Given polynomials $f_1, \ldots, f_n \in F\langle Y \cup Z, *\rangle$, let us denote by $\langle f_1, \ldots, f_n\rangle_{T_2^*}$ the T_2^*-ideal generated by $f_1, \ldots, f_n$.

As in the ordinary case, it is easily seen that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by

$$P_n^* = \mathrm{span}_F\Big\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \ldots, n\Big\}$$

the space of multilinear polynomials of degree n in $y_1^+, y_1^-, z_1^+, z_1^-, \ldots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial at degree 1) the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$.

Definition 1.41. *The non-negative integer*

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap Id^*(A)},\ n \geq 1,$$

is called the n-th $*$-codimension of A.

Let $n \geq 1$ and write $n = n_1 + \cdots + n_4$ as a sum of non-negative integers. We denote by $P^*_{n_1,\ldots,n_4} \subseteq P_n^*$ the vector space of the multilinear $*$-polynomials in which the first n_1 variables are even symmetric, the next n_2 variables are even skew, the next n_3 variables are odd symmetric and the last n_4 variables are odd skew. The group $S_{n_1} \times \cdots \times S_{n_4}$ acts on the left on the vector space $P^*_{n_1,\ldots,n_4}$ by permuting the variables of the same homogeneous degree which are all even or all odd at the same time. Thus S_{n_1} permutes the variables $y_1^+, \ldots, y_{n_1}^+$, S_{n_2} permutes the variables $y_{n_1+1}^-, \ldots, y_{n_1+n_2}^-$, and so on. In this way $P^*_{n_1,\ldots,n_4}$ becomes a left $(S_{n_1} \times \cdots \times S_{n_4})$-module. Now $P^*_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)$ is invariant under this action and so the vector space

$$P^*_{n_1,\ldots,n_4}(A) = \frac{P^*_{n_1,\ldots,n_4}}{P^*_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)}$$

is a left $(S_{n_1} \times \cdots \times S_{n_4})$-module with the induced action.

Definition 1.42. *For $n = n_1 + \cdots + n_4 \geq 1$, the $(S_{n_1} \times \cdots \times S_{n_4})$-character of $P^*_{n_1,\ldots,n_4}(A)$, denoted by $\chi_{n_1,\ldots,n_4}(A)$, is called the $(n_1, \ldots, n_4)$-th cocharacter of A.*

If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of n, we write $\lambda \vdash n$. As we have seen in Proposition 1.4, there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. Hence if $\lambda \vdash n$, we denote by χ_λ the corresponding irreducible S_n-character. If $\lambda(1) \vdash n_1, \ldots, \lambda(4) \vdash n_4$ are partitions, we write $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(4)) \vdash (n_1, \ldots, n_4)$ or $\langle\lambda\rangle \vdash n$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_4$.

Since $\mathrm{char}\, F = 0$, by complete reducibility, $\chi_{n_1,\ldots,n_4}(A)$ can be written as a sum of irreducible characters

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\ldots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{1.2}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi_{n_1,\ldots,n_4}(A)$.

Now if we set $c^*_{n_1,\ldots,n_4}(A) = \dim_F P^*_{n_1,\ldots,n_4}(A)$ it is immediate to see that

$$c^*_n(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1, \ldots, n_4} c^*_{n_1,\ldots,n_4}(A) \tag{1.3}$$

where $\binom{n}{n_1,\ldots,n_4} = \frac{n!}{n_1! \cdots n_4!}$ stands for the multinomial coefficient.

Remark 1.1. *Let A be a $*$-algebra and $c_n(A)$ be its n-th ordinary codimension. For any $n = n_1 + \cdots + n_4$ then*

$$c^*_{n_1,n_2,n_3,n_4}(A) \leq c_n(A).$$

Proof. If $P_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ is the space of multilinear polynomials of degree n in the variables $x_1, \ldots, x_n$, and $\mathrm{Id}(A)$ is the T-ideal of ordinary polynomial identities of A then

$$c_n(A) = \dim_F \frac{P_{n_1+\cdots+n_4}}{P_{n_1+\cdots+n_4} \cap \mathrm{Id}(A)}.$$

Notice that $\mathrm{Id}(A) \subseteq \mathrm{Id}^*(A)$ and the map $\psi : P_{n_1+\cdots+n_4} \longrightarrow P^*_{n_1,\ldots,n_4}$ defined by

$$\psi\left(f(x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+\cdots+n_4})\right) =$$

$$f(y_1^+, \ldots, y_{n_1}^+, y_{n_1+1}^-, \ldots, y_{n_1+n_2}^-, z_{n_1+n_2+1}^+, \ldots, z_{n_1+n_2+n_3}^+, z_{n_1+n_2+n_3+1}^-, \ldots, z_{n_1+\cdots+n_4}^-)$$

is an isomorphism of vector spaces. Hence $\mathrm{Id}(A) \cap P_{n_1+\cdots+n_4} \hookrightarrow \mathrm{Id}^*(A) \cap P^*_{n_1,\ldots,n_4}$ and, so,

$$\dim_F \frac{P^*_{n_1,\ldots,n_4}}{P^*_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)} \leq \dim_F \frac{P_{n_1+\cdots+n_4}}{P_{n_1+\cdots+n_4} \cap \mathrm{Id}(A)}.$$

□

The relation between ordinary codimensions, supercodimensions and $*$-codimensions for a $*$-algebra A is given in the following.

Remark 1.2 ([22]). *Let A be a $*$-algebra satisfying an ordinary polynomial identity. Then*

$$c_n(A) \leq c_n^{sup}(A) \leq c_n^*(A) \leq 2^n c_n^{sup}(A) \leq 4^n c_n(A), \; n = 1, 2, \ldots.$$

As a consequence, we obtain an analogous result of Theorem 1.3.1, i.e. an exponential bound for the sequence of $*$-codimensions.

Corollary 1.5. *Let A be a $*$-algebra satisfying a non-trivial identity. Then $c_n^*(A)$, $n = 1, 2, \ldots,$ is exponentially bounded.*

1.7 Matrix algebras with superinvolution

In this section we focus our attention on matrix algebras with superinvolution. We shall see that it is possible to define on the algebra $M_n(F)$ of $n \times n$ matrices over F two different superinvolutions.

We start by introducing the matrix superalgebra $M_{k,h}(F)$. Recall that, given a n-tuple $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$, one can define on $M_n(F)$ the elementary $\mathbb{Z}_2$-grading by setting $(M_n(F))_i = \mathrm{span}_F \{e_{ij} \mid g_i + g_j = i\}$, $i = 0, 1$, where $e_{ij}s$ are the usual matrix units and $\mathbb{Z}_2 \cong \{0, 1\}$. Elementary gradings play a key role among gradings in matrix algebras. In fact the following result holds (see [4]).

Theorem 1.7.1. *Let F be an algebraically closed field. Then any $\mathbb{Z}_2$-grading on the matrix algebra $M_n(F)$ is an elementary grading.*

Let now $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$ be an n-tuple defining an elementary grading on $M_n(F)$. It is obvious that the n-tuple $(g + g_1, \ldots, g + g_n)$, $g \in \mathbb{Z}_2$, define the same grading. In particular, one may always assume that $g_1 = 0$. Furthermore, if we permute $g_1, \ldots, g_n$, according to some permutation $\sigma \in S_n$, then we get the isomorphic $\mathbb{Z}_2$-grading defined by the n-tuple $(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$. In fact, if A and B are the matrix algebra $M_n(F)$ with the grading defined by the n-tuple $(g_1, \ldots, g_n)$ and $(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$, respectively, then $\varphi_\sigma : A \to B$ defined by $\varphi(e_{ij}) = e_{\sigma(i)}e_{\sigma(j)}$ is an isomorphism of superalgebras. In conclusion, up to isomorphism, any elementary $\mathbb{Z}_2$-grading on $M_n(F)$ is defined by the n-tuple

$$(\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{h})$$

with $k+h=n$. In this case $M_n(F)$ is the superalgebra denoted by $M_{k,h}(F)$ where

$$(M_{k,h}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F),\ T \in M_h(F) \right\},$$
$$(M_{k,h}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

In [47], Racine proved that, up to isomorphism and if the field F is algebraically closed and of characteristic different from 2, it is possible to define on $M_{k,h}(F)$ only the following superinvolutions.

1. The *transpose* superinvolution denoted *trp*. In this case $h=k$ and for $\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in M_{k,k}(F)$, we have
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix},$$
where t is the usual transpose. Then the four sets of homogeneous symmetric and skew elements are the following:
$$(M_{k,k}(F),\ \mathrm{trp})_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \mid X \in M_k(F) \right\},$$
$$(M_{k,k}(F),\ \mathrm{trp})_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in M_k(F) \right\},$$
$$(M_{k,k}(F),\ \mathrm{trp})_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y = -Y^t,\ Z = Z^t,\ Y, Z \in M_k(F) \right\},$$
$$(M_{k,k}(F),\ \mathrm{trp})_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y = Y^t,\ Z = -Z^t,\ Y, Z \in M_k(F) \right\}.$$

2. The *orthosymplectic* superinvolution denoted *osp*. In this case $h=2l$ is even and
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} X^t & Z^tQ \\ QY^t & -QT^tQ \end{pmatrix},$$

where $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and I_r, I_l are the $r \times r$, $l \times l$ identity matrices, respectively. Thus, we have

$$(M_{k,2l}(F),\ \mathrm{osp})_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = X^t,\ T = -QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},$$

$$(M_{k,2l}(F),\ \mathrm{osp})_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = -X^t,\ T = QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},$$

$$(M_{k,2l}(F),\ \mathrm{osp})_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Z = QY^t,\ Y \in M_{k\times 2l}(F) \right\},$$

$$(M_{k,2l}(F),\ \mathrm{osp})_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Z = -QY^t,\ Y \in M_{k\times 2l}(F) \right\}.$$

1.8 Proper polynomials

1.8.1 Proper polynomials in the ordinary case

In this section we introduce a special kind of polynomials, the so-called proper polynomials. We refer to [10] and [12] for the results of this section.

Definition 1.43. *A polynomial $f \in F\langle X\rangle$ is called proper if it is a linear combination of products of commutators*

$$f(x_1, \dots, x_n) = \sum \alpha_{i,\dots,j}[x_{i_1}, \dots, x_{i_p}] \cdots [x_{j_1}, \dots, x_{j_q}],$$

where $\alpha_{i,\dots,j} \in F$ and $[x_1, \dots, x_n] = [[\dots[[x_1, x_2], x_3], \dots], x_n]$ is the Lie commutator with left normalized brackets.

We denote by B the set of all proper polynomials in $F\langle X\rangle$, by $B_m = B \cap F\langle x_1, \dots, x_m\rangle$ the set of the proper polynomials in m variables and by $\Gamma_n = B \cap P_n$, $n = 0, 1, 2, \dots$, the set of all proper multilinear polynomials of degree n.

Definition 1.44. *Let A be a unitary PI-algebra over a field of characteristic 0. We introduce the proper codimensions sequence*

$$\gamma_n(A) = \frac{\Gamma_n}{\Gamma_n \cap Id(A)},\ n = 0, 1, 2, \dots.$$

The special role that proper polynomials play in PI-theory, is underlined by the results that follow.

Proposition 1.13. *If A is an unitary PI-algebra over an infinite field F then all polynomial identities of A follow from the proper ones. Moreover, if $charF = 0$ then the polynomial identities follow from the proper multilinear identities.*

In order to present the next theorem, that is the main result of this section, we need to specify the notation as follows. If A is an algebra, for any S subset of $F\langle X\rangle$, we denote with $S(A)$ the image of S under the canonical homomorphisms

$$F\langle X\rangle \to F(A) = \text{var}(A) = \frac{F\langle X\rangle}{\text{Id}(A)}$$

Theorem 1.8.1. *Let A be an unitary PI-algebra over an infinite field F.*

1. *Let $\{w_j(x_1,\dots,x_m) \mid j = 1,2,\dots\}$ be a basis of the vector space $B_m(A)$ of the proper polynomials in the relatively free algebra $F_m(A)$ of rank m, i.e., $B_m(A) = \dfrac{B_m}{Id(A)\cap B_m}$. Then $F_m(A)$ has a basis*
$$\{x_1^{a_1}\cdots x_m^{a_m} w_j(x_1,\dots,x_m) \mid a_i \geq 0,\ j = 1,2,\dots\}.$$

2. *If $\{u_{jk}(x_1,\dots,x_k) \mid j = 1,2,\dots,\gamma_k(A)\}$ is a basis of the vector space $\Gamma_k(A)$ of the proper multilinear polynomials of degree k in $F(A)$, then $P_n(A)$ has a basis consisting of all multilinear polynomials of the form*
$$x_{p_1}\cdots x_{p_{n-k}} u_{jk}(x_{q_1},\dots,x_{q_k}),\ j = 1,2,\dots,\gamma_k(A),\ k = 0,1,\dots,n,$$
with $p_1 < \cdots < p_{n-k}$ and $q_1 < \cdots < q_k$.

The previous theorem tells that in order to compute a basis for the T-ideal of the polynomial identities for a unitary algebra, we need to study only the proper identities that are, of course, easier to deal with. Moreover, we get the following relationship among proper codimensions and ordinary codimensions sequence.

Corollary 1.6. *The codimensions sequence $c_n(A)$ of an algebra A and the corresponding proper codimensions $\gamma_k(A)$ are related by the condition*

$$c_n(A) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(A),\ n = 0,1,2\dots.$$

1.8.2 Proper $*$-polynomials

In this section we introduce the proper $*$-polynomials, i.e. the corresponding object, in the setting of unitary $*$-algebras, of the proper polynomials of the ordinary case. From now until the end of this section, F denotes a field of characteristic zero.

Definition 1.45. *A polynomial $f \in P_n^*$ is a proper $*$-polynomial if it is a linear combination of elements of the type*

$$y_{i_1}^- \cdots y_{i_s}^- z_{j_1}^+ \cdots z_{j_t}^+ z_{l_1}^- \cdots z_{l_r}^+ w_1 \cdots w_m$$

where $w_1, \ldots, w_m$ are left normed (long) Lie commutators in the variables from $Y \cup Z$ (here the symmetric even variables appear only inside the commutators).

We denote by Γ_n^* the subspace of P_n^* of proper $*$-polynomials and $\Gamma_0^* = \text{span}\{1\}$.

Definition 1.46. *The sequence of proper $*$-codimensions is defined as*

$$\gamma_n^*(A) = \dim \frac{\Gamma_n^*}{\Gamma_n^* \cap Id^*(A)}, \ n = 0, 1, 2, \ldots.$$

For a unitary $*$-algebra A, the relation between $*$-codimensions and proper $*$-codimensions is given in the following (see Corollary 1.6 for the analogous result in the ordinary case).

Theorem 1.8.2. *The $*$-codimensions sequence $c_n^*(A)$ of a $*$-algebra A and the corresponding proper $*$-codimensions $\gamma_i^*(A)$ are related by the condition*

$$c_n^*(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^*(A), \ n = 0, 1, 2 \ldots.$$

Given two sets of polynomials $S, S' \subseteq F\langle Y \cup Z, *\rangle$, we say that S' is a consequence of S if $S' \subseteq \langle S \rangle_{T_2^*}$. By following closely the proof of Lemma 2.2 in [40, 42] we get the next proposition.

Proposition 1.14. *For every $i \geq 1$, Γ_{k+i}^* is a consequence of Γ_k^*.*

As a consequence we have the following.

Corollary 1.7. *Let A be a $*$-algebra with 1. If for some $k \geq 2$, $\gamma_k^*(A) = 0$ then $\gamma_m^*(A) = 0$, for all $m \geq k$.*

Let $n = n_1 + \cdots + n_4 \geq 1$. We denote by $\Gamma_{n_1,\ldots,n_4}^* \subseteq P_{n_1,\ldots,n_4}^*$ the subspace of proper $*$-polynomials in which n_1 variables are even symmetric, n_2 variables are even skew, n_3 variables are odd symmetric and n_4 variables are odd skew. $\Gamma_{n_1,\ldots,n_4}^*$ is also an $(S_{n_1} \times \cdots \times S_{n_4})$-submodule of $P_{n_1,\ldots,n_4}^*$. Since $\Gamma_{n_1,\ldots,n_4}^* \cap \text{Id}^*(A)$ is invariant under the action of $S_{n_1} \times \cdots \times S_{n_4}$, the vector space

$$\Gamma_{n_1,\ldots,n_4}^*(A) = \frac{\Gamma_{n_1,\ldots,n_4}^*}{\Gamma_{n_1,\ldots,n_4}^* \cap \text{Id}^*(A)}$$

is a left $(S_{n_1} \times \cdots \times S_{n_4})$-module with the induced action. We denote by $\psi_{n_1,\ldots,n_4}(A)$ its character and it is called the $(n_1, \ldots, n_4)$-th proper cocharacter of A.

Since $\text{char}\, F = 0$, by complete reducibility, $\psi_{n_1,\ldots,n_4}(A)$ can be written as a sum of irreducible characters

$$\psi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \qquad (1.4)$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\psi_{n_1,\dots,n_4}(A)$.

Now if we set $\gamma^*_{n_1,\dots,n_4}(A) = \dim_F \Gamma^*_{n_1,\dots,n_4}(A)$ it is immediate to see that

$$\gamma^*_n(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\dots,n_4} \gamma^*_{n_1,\dots,n_4}(A). \tag{1.5}$$

Chapter 2

Characterization of finite dimensional $*$-algebras with polynomially bounded codimensions

In this chapter we shall give several characterizations concerning finite dimensional $*$-algebras with polynomial growth of the $*$-codimensions. From now on, unless otherwise stated, F denotes a field of characteristic zero.

Definition 2.1. *A $*$-algebra A has polynomial growth if its sequence of $*$-codimensions is polynomially bounded, i.e., $c_n^*(A) \leq an^k$, for some constants a and k.*

We now introduce the notion of $*$-variety, that is the analogous of the varieties in the ordinary case.

Definition 2.2. *Given an non-empty set $S \subseteq F\langle Y \cup Z, *\rangle$, the class of all $*$-algebras A such that $f \equiv 0$ on A, for all $f \in S$, is called the $*$-variety $\mathcal{V} = \mathcal{V}(S)$ determined by S.*

If A is a $*$-algebra, we write $\mathrm{var}^*(A)$ to denote the $*$-variety generated by A, i.e. the class of all $*$-algebras B such that $f \equiv 0$ on B, for all $f \in \mathrm{Id}^*(A)$.

The growth of a $*$-variety $\mathcal{V}$ is defined as the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$. Then we have the following definition.

Definition 2.3. *A $*$-variety $\mathcal{V}$ has polynomial growth if $c_n^*(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^*(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.*

If we consider the language of $*$-varieties, in this chapter we present results characterizing $*$-varieties of polynomial growth, generated by finite dimensional $*$-algebras.

2.1 Varieties of almost polynomial growth

In this section we shall construct finite dimensional $*$-algebras generating $*$-varieties of almost polynomial growth. We start with the following definition.

Definition 2.4. *A superalgebra $A = A_0 \oplus A_1$ is endowed with a graded involution if $\sharp : A \to A$ is an involution on A preserving the grading, i.e., $A_i^\sharp \subseteq A_i$, $i = 0, 1$.*

In this way we are ready to prove the following remark.

Remark 2.1. *Let $A = A_0 \oplus A_1$ be a superalgebra such that $A_1^2 = 0$. Then the superinvolutions on A coincide with the graded involutions on A. In particular, if $A_1 = 0$, then the superinvolutions on A coincide with the involutions on A.*

Proof. Let $* : A \to A$ be a graded linear map of order 2 on $A = A_0 \oplus A_1$. Since $A_1^2 = 0$ then, for all $a, b \in A_1$, we have that $b^*a^* = -b^*a^* = 0$. Hence if $*$ is a graded involution then it is also a superinvolution and vice versa. If $A_1 = 0$, then every involution on A is a graded involution and the result follows from the first part. □

In Theorem 1.4.3 we have seen that a variety of algebras with involution has polynomial growth if and only if it does not contain the two-dimensional commutative algebra $F \oplus F$ endowed with the exchange involution *ex* and $M = F(e_{11}+e_{44}) \oplus F(e_{22}+e_{33}) \oplus Fe_{12} \oplus Fe_{34}$, a subalgebra of UT_4 endowed with the reflection involution *ref*. Recall that the exchange involution is such that $(a, b)^{ex} = (b, a)$ whereas the reflection involution *ref* on the algebras of $n \times n$ upper triangular matrices UT_n is defined as follows: if $a = (a_{ij}) \in UT_n$ then $a^{ref} = (b_{ij})$ where $b_{ij} = a_{n+1-j\ n+1-i}$.

Let us consider $F \oplus F$ and M with trivial grading. In this way, by Remark 2.1, the exchange and the reflection involutions are also superinvolutions. Hence these algebras with involution can be viewed as algebras with superinvolution ($*$-algebras) and we can make the following definitions.

Definition 2.5. *$F \oplus F$ denotes the two-dimensional commutative algebra with trivial grading and exchange superinvolution.*

Definition 2.6. *M denotes the subalgebra of UT_4 with trivial grading and reflection superinvolution.*

In [20] the authors proved that $F \oplus F$ and M generate the only varieties of algebras with involution of almost polynomial growth (see Corollary 1.3). Since $F \oplus F$ and M have trivial grading ($z \equiv 0$ on $F \oplus F$ and M), by Theorem 1.4.1, we get the following result.

Theorem 2.1.1. *The $*$-algebras $F \oplus F$ with the exchange superinvolution and M with the reflection superinvolution generate $*$-varieties of almost polynomial growth. Moreover*

1. $Id^*(F \oplus F) = \langle [y_1, y_2], z^+, z^- \rangle_{T_2^*}$.

2. $Id^*(M) = \langle y_1^- y_2^-, z^+, z^- \rangle_{T_2^*}$.

Let now define a non-trivial grading on M: we denote by M^{sup} the algebra M with grading $(M^{sup})_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$ and $(M^{sup})_1 = Fe_{12} \oplus Fe_{34}$ and endowed with reflection involution.

It is easy to see that the reflection involution on M^{sup} is a graded involution. Since $(M^{sup})_1^2 = 0$, by Remark 2.1, the reflection involution on M^{sup} is also a superinvolution and we can make the following definition.

Definition 2.7. *M^{sup} denotes the subalgebra of UT_4 with grading $(M^{sup})_0 = F(e_{11}+e_{44}) \oplus F(e_{22}+e_{33})$ and $(M^{sup})_1 = Fe_{12} \oplus Fe_{34}$, endowed with reflection superinvolution.*

The algebra M^{sup} was extensively studied in [23]. In the following theorem we present these results.

Theorem 2.1.2. *The $*$-algebra M^{sup} with reflection superinvolution generate a $*$-variety of almost polynomial growth. Moreover $Id^*(M^{sup}) = \langle y^-, z_1 z_2 \rangle_{T_2^*}$ and, if*

$$\chi_{n_1,\dots,n_4}(M^{sup}) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$

is the $(n_1, \dots, n_4)$-th cocharacter of M^{sup}, $n_1 + \cdots + n_4 = n$, then

$$m_{\langle\lambda\rangle} = \begin{cases} 1 & \text{if } \langle\lambda\rangle = ((n), \emptyset, \emptyset, \emptyset) \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q, p), \emptyset, (1), \emptyset) \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q, p), \emptyset, \emptyset, (1)) \\ 0 & \text{otherwise} \end{cases},$$

where $p, q \geq 0$ and $2p + q + 1 = n$.

The final goal of this section is to prove that, in case $A \in \mathrm{var}^*(M^{sup})$ generates a $*$-variety of polynomial growth, then A satisfies the same $*$-identities as a finite dimensional $*$-algebra. We start with the following.

Theorem 2.1.3. *If $A \in var^*(M^{sup})$ then $var^*(A) = var^*(B)$, for some finitely generated $*$-algebra B.*

Proof. Let B be the relatively free algebra of $\mathrm{var}^*(A)$ with 2 even symmetric, 1 odd symmetric and 1 odd skew generators,

$$B = \frac{F\langle y_1^+, y_2^+, z^+, z^- \rangle}{\mathrm{Id}^*(A)}.$$

We shall prove that $\mathrm{var}^*(A) = \mathrm{var}^*(B)$. Clearly $\mathrm{var}^*(B) \subseteq \mathrm{var}^*(A)$.
In order to get the opposite inclusion we need to prove that $\mathrm{Id}^*(B) \subseteq \mathrm{Id}^*(A)$. Let f be a $*$-identity of B. Since $\mathrm{char}F = 0$, we may assume that $f = f(y_1^+, \dots, y_{n_1}^+, y_1^-, \dots, y_{n_2}^-, z_1^+, \dots, z_{n_3}^+, z_1^-, \dots, z_{n_4}^-)$ is multilinear. Let L be the left $(S_{n_1} \times \cdots \times S_{n_4})$-module generated by f and let $L = L_1 \oplus \cdots \oplus L_m$ be its decomposition into irreducible components with L_i generated by f_i as a left $(S_{n_1} \times \cdots \times S_{n_4})$-module, $i = 1, \dots, m$. If $f_i \equiv 0$ on A, for all $i = 1, \dots, m$, then also $f \equiv 0$ on A. Hence, without loss of generality, we may assume that L is irreducible.

Let $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ be the irreducible character of L, where $\lambda(i) \vdash n_i$, $i = 1, \dots, 4$ and let

$$e_{T_{\lambda(i)}} = \left(\sum_{\tau \in R_{T_{\lambda(i)}}} \tau \right) \left(\sum_{\sigma \in C_{T_{\lambda(i)}}} (sgn\sigma)\sigma \right), \ i = 1, \dots, 4,$$

be the corresponding essential idempotents.

Notice that, if $\lambda(1)_3 \neq 0$ (here $\lambda(1)_3$ means the length of the third row of the Young diagram corresponding to the partition $\lambda(1)$) or $\lambda(2) \neq \emptyset$ or $\lambda(3) \notin \{\emptyset, (1)\}$ or $\lambda(4) \notin \{\emptyset, (1)\}$ then, by Theorem 2.1.2 follows that $f \equiv 0$ on A.

Therefore, in order to complete the proof, we may assume that $\lambda(1)_3 = 0$, $\lambda(2) = \emptyset$ and $\lambda(3), \lambda(4) \in \{\emptyset, (1)\}$.
Now we consider $g = \left(\sum_{\tau \in R_{T_{\lambda(1)}}} \tau \right) f$. Since L is irreducible and $g \neq 0$ (see [26, Lemmas 2.5.1 and 2.5.2]) then $f \equiv 0$ on A if and only if $g \equiv 0$ on A. We shall prove that $g \equiv 0$ on A.

Notice that g is symmetric on at most 2 disjoint subsets Y_1, Y_2 of $\{y_1^+, y_2^+, \dots\}$. If we identify all variables of Y_1 with y_1^+ and all variables of Y_2 with y_2^+ we obtain a homogeneous polynomial $t = t(y_1^+, y_2^+, z^+, z^-)$ which is still a $*$-identity of B. From the definition of relatively free algebra, it follows that $t \equiv 0$ on A. But the complete linearization of t on all even symmetric variables is equal to $\gamma g(y_1^+, \dots, y_{n_1}^+, z^+, z^-)$ where $\gamma = \lambda(1)_1! \ \lambda(1)_2! \neq 0$. Hence $g \equiv 0$ on A and so $f \equiv 0$ on A follows. $\square$

In order to reach our goal we need to apply the following result.

Theorem 2.1.4 ([2]). *If A is a finitely generated $*$-algebra over an algebraically closed field F of characteristic zero then A satisfies the same $*$-identities as a finite dimensional $*$-algebra over F.*

As a consequence of Theorems 2.1.3 and 2.1.4 we get the following.

Corollary 2.1. *Let A be a $*$-algebra such that $A \in var^*(M^{sup})$. Then $Id^*(A) = Id^*(B)$, for some finite dimensional $*$-algebra B.*

2.2 Wedderburn-Malcev theorem for finite dimensional $*$-algebras

In this section we present a decomposition of finite dimensional $*$-algebras, by giving the analogous result than that Wedderburn-Malcev proved in the ordinary case. For a start we state the Wedderburn-Malcev theorem in the case of associative algebras and the corresponding result in the setting of superalgebras, proved by Giambruno and Zaicev.

Theorem 2.2.1. *Let A be a finite dimensional algebra over a field F of characteristic 0 and let $J(A)$ be its Jacobson radical. Then there exists a semisimple subalgebra B such that*

$$A = B + J(A).$$

Moreover if B and B' are semisimple subalgebras such that $A = B + J(A) = B' + J(A)$, then there exists $x \in J(A)$ such that $B' = (1+x)B(1+x)^{-1}$.

Let $A = A_0 \oplus A_1$ be a finite dimensional superalgebra over a field F of characteristic 0 and let φ be the automorphism of order ≤ 2 determined by the $\mathbb{Z}_2$-grading on A. We recall that $\varphi : A \to A$ is defined by $\varphi(a_0 + a_1) = a_0 - a_1$, for all $a_0 \in A_0$, $a_1 \in A_1$.

Definition 2.8. *An ideal (subalgebra) I of A is a graded ideal (subalgebra), if $I = (I \cap A_0) \oplus (I \cap A_1)$.*

The algebra A is a simple superalgebra if $A^2 \neq 0$ and A has no non-trivial graded ideals.

Theorem 2.2.2. *Let A be a finite dimensional superalgebra over a field F. Then*

1. *The Jacobson radical of A, $J(A)$ is a graded ideal of A.*
2. *If A is a simple superalgebra then either A is simple or $A = B \oplus B^{\varphi}$, for some simple subalgebra B.*
3. *If A is semisimple, then A is a finite direct sum of simple superalgebras.*
4. *If char$F = 0$, there exists a maximal semisimple subalgebra B of A such that $B^{\varphi} = B$.*

From now on $A = A_0 \oplus A_1$ denotes a finite dimensional $*$-algebra over a field F of characteristic zero, $J(A)$ its Jacobson radical and φ the automorphism of order ≤ 2 determined by the grading on A.

The following remark holds.

Remark 2.2. *Let A be a $*$-algebra with superinvolution $*$ and let $B \subseteq A$ be a subalgebra of A.*

1. *If $B = B_0 \oplus B_1$ is a graded subalgebra of A then $B^* = B_0^* \oplus B_1^*$ is a graded subalgebra of A.*

2. If $I = I_0 \oplus I_1 \subseteq B$ is a graded ideal of B then $I^* = I_0^* \oplus I_1^*$ is a graded ideal of B^*.

3. If I is a minimal graded ideal of B then I^* is a minimal graded ideal of B^*.

Lemma 2.1. *If B is a semisimple graded subalgebra of A, then B^* is a semisimple graded subalgebra of A.*

Proof. By Remark 2.2, B^* is a graded subalgebra of A and we are left to prove that B^* is semisimple, i.e, $J(B^*) = 0$, where $J(B^*)$ denotes the Jacobson radical of B^*. It is well known that $J(B^*)$ is a graded nilpotent ideal of B^*. We claim that $J(B^*)^*$ is a nilpotent ideal of B. Let m be the smallest positive integer such that $J(B^*)^m = 0$ and let $a_1, \ldots, a_m \in J(B^*)^*$. Since $J(B^*)$ is a graded ideal of B^* we get that, for all i, $a_i = (b_i + c_i)^*$, where b_i and c_i are homogeneous elements of $J(B^*)$ of degree zero and one, respectively. Then

$$a_1 \cdots a_m = \sum \alpha_j (d_1 \cdots d_m)^*$$

where either $d_i = b_i$ or $d_i = c_i$ and $\alpha_j = \pm 1$. But $J(B^*)^m = 0$ and, so, we get that $a_1 \cdots a_m = 0$ and $J(B^*)^*$ is nilpotent. Since the Jacobson radical of an algebra is the maximal nilpotent ideal of it, then $J(B^*)^* \subseteq J(B)$. But since B is semisimple we get that $J(B^*)^* = J(B) = 0$ and, so, $J(B^*) = 0$. □

By Theorem 2.2.2, we can write $A = B + J$, where B is a semisimple graded subalgebra of A and $J = J(A) = J_0 \oplus J_1$ is a graded ideal. Moreover $B = B_1 \oplus \cdots \oplus B_k$, with $B_1, \ldots, B_k$ simple superalgebras.

Lemma 2.2. *If B and B' are semisimple graded subalgebras of A such that $A = B + J = B' + J$, with $J^2 = 0$, then there exists $x_0 \in J_0$ such that*

$$B' = (1 + x_0)B(1 - x_0).$$

Proof. By Theorem 2.2.2, $B' = (1 + x)B(1 - x)$, for some $x \in J$. Therefore:

$$(1 + x)B(1 - x) = B' = (B')^\varphi = (1 + x^\varphi)B^\varphi(1 - x^\varphi) = (1 + x^\varphi)B(1 - x^\varphi),$$

where φ is the automorphism of order ≤ 2 determined by the grading. Hence:

$$B = (1 - x)(1 + x^\varphi)B(1 - x^\varphi)(1 + x).$$

This says that, for any $b \in B$, there exists $\bar{b} \in B$ such that $b = (1 - x + x^\varphi)\ \bar{b}\ (1 - x^\varphi + x)$ and, since $J \cap B = 0$, we obtain that $b = \bar{b}$ and $(x - x^\varphi)b = b(x - x^\varphi)$. Hence:

$$\begin{aligned} B' &= (1 + x)B(1 - x) \\ &= (1 + \frac{x + x^\varphi}{2} + \frac{x - x^\varphi}{2})B(1 - \frac{x + x^\varphi}{2} - \frac{x - x^\varphi}{2}) \\ &= (1 + \frac{x + x^\varphi}{2})B(1 - \frac{x + x^\varphi}{2}) \\ &= (1 + x_0)B(1 - x_0) \end{aligned}$$

where $x_0 = \dfrac{x + x^{\varphi}}{2} \in J_0$. □

Now we are in a position to prove the Wedderburn-Malcev theorem for $*$-algebras. First we recall the following definition.

Definition 2.9. *An ideal (subalgebra) I of A is a $*$-ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^* = I$. The algebra A is a simple $*$-algebra if $A^2 \neq 0$ and A has no non-trivial $*$-ideals.*

Theorem 2.2.3. *Let A be a finite dimensional $*$-algebra over a field F of characteristic 0. Then there exists a semisimple $*$-subalgebra C such that*

$$A = C + J(A)$$

and $J(A)$ is a $*$-ideal of A. Moreover $C = C_1 \oplus \cdots \oplus C_k$, where $C_1, \ldots, C_k$ are simple $*$-algebras.

Proof. By Theorem 2.2.2, we can write

$$A = B + J$$

where B is a semisimple graded subalgebra of A and $J = J(A)$, its Jacobson radical, is a graded ideal of A. Since J is nilpotent, as in the proof of Lemma 2.1, we have that J^* is a nilpotent ideal of A. But being J the maximal nilpotent ideal of A, we get $J^* \subseteq J$ and, so, $J = J^*$. Hence J is a $*$-ideal of A.

If $J = 0$ or $B = B^*$ then B is a semisimple $*$-algebra and we are done. So assume that $J \neq 0$ and $B \neq B^*$.

Suppose first that $J^2 = 0$.

By Lemma 2.1, B^* is a semisimple graded subalgebra of A. Hence, by Lemma 2.2, we have

$$B^* = (1 + x_0)B(1 - x_0),$$

for some $x_0 \in J_0$. For any homogeneous element $b \in B$, we have that $b^* = (1 + x_0)\bar{b}(1 - x_0)$, for some homogeneous element $\bar{b} \in B$ with the same homogeneous degree as b^* and b. Hence:

$$\begin{aligned} b &= (b^*)^* = \big((1 + x_0)\,\bar{b}\,(1 - x_0)\big)^* \\ &= (1 - x_0^*)\ \bar{b}^*\ (1 + x_0^*) \\ &= (1 - x_0^*)(1 + x_0)\ \tilde{b}\ (1 - x_0)(1 + x_0^*) \\ &= (1 - x_0^* + x_0)\ \tilde{b}\ (1 + x_0^* - x_0) \end{aligned}$$

for some $\tilde{b} \in B_0 \cup B_1$ with the same homogeneous degree as b. As in the proof of Lemma 2.2 we obtain that

$$(x_0 - x_0^*)b = b(x_0 - x_0^*).$$

It follows that, for any $b \in B_0 \cup B_1$,

$$\begin{aligned} b^* &= (1+x_0)\,\bar{b}\,(1-x_0) \\ &= (1+\frac{x_0+x_0^*}{2}+\frac{x_0-x_0^*}{2})\,\bar{b}\,(1-\frac{x_0+x_0^*}{2}-\frac{x_0-x_0^*}{2}) \\ &= (1+\frac{x_0+x_0^*}{2})\,\bar{b}\,(1-\frac{x_0+x_0^*}{2}) \\ &= (1+x_0^+)\,\bar{b}\,(1-x_0^+), \end{aligned}$$

where $x_0^+ = \dfrac{x_0+x_0^*}{2} \in J_0^+$.

Consider the subalgebra $C = (1+\frac{x_0^+}{2})B(1-\frac{x_0^+}{2})$ of A. Clearly C is a graded subalgebra of A and by the above C is a $*$-subalgebra. Also, since C is isomorphic to B, it is a semisimple $*$-subalgebra of A.

Suppose now that $J^2 \neq 0$ and choose $m \geq 2$ such that $J^m \neq 0$ and $J^{m+1} = 0$. It is easy to see that J^m is a $*$-ideal of A and, so, A/J^m is an algebra with induced superinvolution. Its Jacobson radical $J(A/J^m) = J(A)/J^m$ is such that $J(A/J^m)^m = 0$. Hence, by induction on m, we have that there exists a semisimple $*$-subalgebra B/J^m such that

$$A/J^m = B/J^m \oplus J/J^m.$$

From $J(B/J^m) = 0$ it follows that $J(B) = J^m$ and, so, we can write

$$B = C + J^m,$$

where C is a semisimple graded subalgebra of B. Since $(J^m)^2 \subseteq J^{2m} = 0$, by applying again induction on m we can assume $C^* = C$, i.e., C is a semisimple $*$-subalgebra of A and we are done since

$$A = B + J = C + J^m + J = C + J.$$

Finally we prove that C decomposes into the direct sum of simple $*$-algebras. By Theorem 2.2.2, we have

$$C = D_1 \oplus \cdots \oplus D_h,$$

where $D_1, \ldots, D_h$ are all the minimal graded ideals of C. Hence, by Remark 2.2, for every i, D_i^* is also a minimal graded ideal of C and, so, $D_i^* = D_j$, for some $j \in \{1, \ldots, h\}$. We now rename $D_1, \ldots, D_h$ and we write

$$C = C_1 \oplus \cdots \oplus C_k$$

where either $C_i = D_j$ with $D_j = D_j^*$ or $C_i = D_j \oplus D_j^*$, with $D_j \neq D_j^*$. Thus $C_1, \ldots, C_k$ are minimal $*$-ideals of C, i.e. simple $*$-algebras. $\square$

The structure of the simple $*$-algebras is given in the following.

Lemma 2.3. *If A is a finite dimensional simple $*$-algebra then either A is a simple superalgebra or $A = B \oplus B^*$, for some simple superalgebra B.*

Proof. Suppose that A is simple as a $*$-algebra but not as a superalgebra. Since $J(A)$ is a $*$-ideal of A, $J = 0$ and A is semisimple. If B is a minimal graded ideal of A, B^* is still a minimal graded ideal of A. Since A is a simple $*$-algebra we have that $B^* \neq B$. Then $B \oplus B^*$ is a $*$-ideal of A and hence $A = B \oplus B^*$ and we are done. □

We conclude this section by giving the classification of the finite dimensional simple $*$-algebras over an algebraically closed field F. We start by introducing the simple superalgebras which are involved in such a classification.

1. Given $n = k + h \geq 1$, we recall the $\mathbb{Z}_2$-grading on $M_n(F)$ introduced in Chapter 1:

$$\begin{aligned} M_{k,h}(F) &= \left\{ \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} : X, Y, Z, T \text{ are } k \times k,\ k \times h,\ h \times k,\ h \times h \text{ matrices, respectively} \right\} \\ &= \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \right\}. \end{aligned}$$

2. $Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$, with $c^2 = 1$.

As we have already seen in Section 1.7, if $h = 2l$ the superalgebra $M_{k,2l}(F)$ is endowed with the orthosymplectic superinvolution osp defined by:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix},$$

where t denotes the usual matrix transpose, $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and I_k, I_l are the identity matrices of orders k and l, respectively.

If $h = k$ the superalgebra $M_{k,k}(F)$ is endowed with the transpose superinvolution trp defined by:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix}.$$

If A is a superalgebra, we denote by A^{sop} the superalgebra which has the same graded vector space structure as A but the product in A^{sop} is given on homogeneous elements a, b by $a \circ b = (-1)^{(\deg a)(\deg b)} ba$. The direct sum $R = A \oplus A^{sop}$ is a superalgebra with $R_0 = A_0 \oplus A_0^{sop}$ and $R_1 = A_1 \oplus A_1^{sop}$ and it is endowed with the exchange superinvolution $(a, b)^* = (b, a)$.

Definition 2.10. *Let $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ be two algebras endowed with the superinvolutions $*$ and $\star$, respectively. We say that $(A, *)$ and $(B, \star)$ are isomorphic, as $*$-algebras, if there exists an isomorphism of algebras $\psi : A \to B$ such that*

1. $\psi(A_i) \subseteq B_i$, $i = 0, 1$ *(isomorphism of superalgebras),*

2. $\psi(x^*) = \psi(x)^\star$, *for all* $x \in A$.

The following theorem gives the classification of the finite dimensional simple $*$-algebras.

Theorem 2.2.4 ([5, 27, 47])**.** *Let A be a finite dimensional simple $*$-algebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic to one of the following:*

1. $M_{k,l}(F)$ *with the orthosymplectic or the transpose superinvolution,*

2. $M_{k,l}(F) \oplus M_{k,l}(F)^{sop}$ *with the exchange superinvolution,*

3. $Q(n) \oplus Q(n)^{sop}$ *with the exchange superinvolution.*

At the light of such classification, by Theorem 2.2.3 we get the following.

Corollary 2.2. *Let A be a finite dimensional $*$-algebra over an algebraically closed field F, $charF = 0$. Then*

$$A = B_1 \oplus \cdots \oplus B_k + J(A)$$

where $J(A)$ is a $*$-ideal of A and for every i, B_i is isomorphic to one of the algebras in the previous theorem.

2.3 Varieties of polynomial growth

In this section we shall characterize the $*$-varieties generated by finite dimensional $*$-algebras of polynomial growth. We start with the following definition.

Definition 2.11. *Let $A = A_0 \oplus A_1$ be a superalgebra. We say that A is endowed with the trivial superinvolution $*$ if $A_1 = 0$ and $*$ is the identity map. Clearly this says that A is commutative.*

Lemma 2.4. *Let A be a finite dimensional $*$-algebra over an algebraically closed field F of characteristic zero and suppose that $F \oplus F \notin var^*(A)$. Then $A = B + J(A)$, where $B \cong F \oplus \cdots \oplus F$ is endowed with trivial (induced) superinvolution.*

Proof. By Theorem 2.2.3

$$A = A_1 \oplus \cdots \oplus A_k + J,$$

where $A_1, \dots, A_k$ are finite dimensional simple $*$-algebras and J is the Jacobson radical of A.

We claim that, for every i, $A_i \cong F$ with trivial superinvolution. In fact, if not, there exists i such that $A_i \cong B$, where B is one of the $*$-algebras of Theorem 2.2.4, and B is not isomorphic to F.

Now, in all but one case, we shall construct a subalgebra $C = \langle a, b\rangle$ of B, generated by two elements, with induced superinvolution. Then the linear map $\psi : C \to F \oplus F$ which sends a to $(1, 0)$ and b to $(0, 1)$ will be an isomorphism of $*$-algebras so that $F \oplus F \in \text{var}^*(B) = \text{var}^*(A_i) \subseteq \text{var}^*(A)$, a contradiction. More precisely

- if $B = M_{r,s}(F) \oplus M_{r,s}(F)^{sop}$ with the exchange superinvolution, we let $a = (e_{11}, 0)$ and $b = (0, e_{11})$;
- if $B = Q(n) \oplus Q(n)^{sop}$, we let $a = (e_{11}, 0)$ and $b = (0, e_{11})$;
- if $B = M_{n,n}(F)$ with the transpose superinvolution, we let $a = e_{11}$ and $b = e_{n+1\ n+1}$;
- if $B = M_{r,2s}(F)$ with the orthosymplectic superinvolution and $s > 0$, we let $a = e_{r+1\ r+1}$ and $b = e_{r+s+1\ r+s+1}$.

We are left with the case $B = M_{r,0}(F)$ with the orthosymplectic superinvolution and $r > 1$. Notice that $M_{r,0}(F)$ is an ordinary algebra (superalgebra with trivial grading) with involution and, so, $\text{var}^*(M_{r,0}(F)) = \text{var}^\sharp((M_r(F), t))$, where t denotes the usual transpose involution and $\text{var}^\sharp$ denotes a variety of algebras with involution. By [43], $F \oplus F \in \text{var}^\sharp(M_2(F), t) \subseteq \text{var}^\sharp((M_r(F), t)) = \text{var}^*(A_i) \subseteq \text{var}^*(A)$ and we reach a contradiction.

Hence, for every i, we have $A_i \cong M_{1,0}(F) = F$ with trivial superinvolution, and this completes the proof. □

Lemma 2.5. *Let $A = A_1 \oplus \cdots \oplus A_m + J$ be a finite dimensional $*$-algebra over an algebraically closed field F of characteristic zero, where for every $i = 1, \dots, m$, $A_i \cong F$ is endowed with the trivial superinvolution. If $M, M^{sup} \notin var^*(A)$ then $A_iJA_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$.*

Proof. Suppose that there exist $i, k \in \{1, \dots, m\}$, $i \neq k$, such that $A_iJA_k \neq 0$. Then there exist elements $a \in A_i$, $b \in A_k$, $j \in J$ such that $ajb \neq 0$, with $a^2 = a = a^*$, $b^2 = b = b^*$ and $\deg a = \deg b = 0$. Without loss of generality we may assume that j is either symmetric or skew. In fact from $ajb \neq 0$ it follows that $2ajb = a(j + j^*)b + a(j - j^*)b \neq 0$ and at least one between $a(j + j^*)b$ and $a(j - j^*)b$ must be non-zero. Also we may clearly assume that j is homogeneous. Let C be the subalgebra of A generated by a, b, ajb, bja. Then C has an induced superinvolution and if I is the ideal generated by $ajbja, bjajb$, then I is a $*$-ideal. Thus the algebra $D = C/I$ has an induced superinvolution.

Consider the algebra M as a ordinary algebra and let $\psi : M \to D$ be the linear map defined by $\psi(e_{11}+e_{44}) = a+I$, $\psi(e_{22}+e_{33}) = b+I$, $\psi(e_{12}) = aj^*b+I$, $\psi(e_{34}) = bja+I$. Clearly ψ is an isomorphism of ordinary algebras. Moreover ψ can be regarded as an isomorphism between algebras with superinvolution $M \to D$ or $M^{sup} \to D$ according as $\deg j = 0$ or $\deg j = 1$, respectively. In both cases we reach a contradiction since we would have $\mathrm{var}^*(D) \subseteq \mathrm{var}^*(C) \subseteq \mathrm{var}^*(A)$. □

By following word by word the proof given in [34, Theorem 2.2] it is possible to prove the next theorem, characterizing the $*$-varieties of polynomial growth through the behaviour of their sequences of cocharacters.

Theorem 2.3.1. *Let A be a finite dimensional $*$-algebra over a field F of characteristic zero. Then $c_n^*(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if, for every $n_1, \ldots, n_4$, with $n_1+\cdots+n_4 = n$, it holds*

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where q is such that $J(A)^q = 0$ and $\lambda(1)_1$ denotes the length of the first row of the Young diagram corresponding to the partition $\lambda(1)$.

We are now in a position to prove the main result of this chapter.

Theorem 2.3.2. *Let $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ be a finite dimensional $*$-algebra over a field F of characteristic zero. Then the sequence $c_n^*(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $M, M^{sup}, F \oplus F \notin var^*(A)$.*

Proof. By Theorems 2.1.1 and 2.1.2, the algebras $F \oplus F$, M, M^{sup} generate varieties of exponential growth. Hence, if $c_n^*(A)$ is polynomially bounded, then $M, M^{sup}, F \oplus F \notin \mathrm{var}^*(A)$.

Conversely suppose that $M, M^{sup}, F \oplus F \notin \mathrm{var}^*(A)$. Since we are dealing with codimensions that do not change by extending the base field, we may assume that the field F is algebraically closed. Hence, by Lemmas 2.4 and 2.5,

$$A = A_1 \oplus \cdots \oplus A_m + J,$$

where for every $i = 1, \ldots, m$, $A_i \cong F$ is endowed with the trivial superinvolution and $A_i J A_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$. Hence $A_0^- \oplus A_1^+ \oplus A_1^- \subseteq J$ and, if q is the least positive integer such that $J^q = 0$, then $A_0^- \oplus A_1^+ \oplus A_1^-$ generates a nilpotent ideal of A of index of nilpotence $\leq q$. By Theorem 2.3.1, $c^*_{n_1,n_2,n_3,n_4}(A) = 0$ as soon as $n_2 + n_3 + n_4 \geq q$. Hence, by (1.3), we get:

$$c_n^*(A) = \sum_{\substack{n_1+\cdots+n_4 \\ n_2+n_3+n_4<q}} \binom{n}{n_1, \ldots, n_4} c^*_{n_1,\ldots,n_4}(A). \tag{2.1}$$

Notice that the number of non-zero summands in (2.1) is bounded by q^3 and that $\binom{n}{n_1,\ldots,n_4} < n^q$ (see [34, Proposition 2.2]). By Remark 1.1, $c^*_{r_1,r_2,r_3,r_4}(A) \le c_n(A)$ and, by Theorem 1.3.9, $c_n(A) \le an^t$, for some constants a and t. In this way we get the desired conclusion. □

As a consequence we have the following corollaries.

Corollary 2.3. *The algebras M, M^{sup} and $F \oplus F$ are the only finite dimensional $*$-algebras generating $*$-varieties of almost polynomial growth.*

Corollary 2.4. *If A is a finite dimensional $*$-algebra, the sequence $c^*_n(A)$, $n = 1, 2, \ldots,$ either is polynomially bounded or it grows exponentially.*

The next corollary follows directly from the proof of the theorem.

Corollary 2.5. *Let A be a finite dimensional $*$-algebra over an algebraically closed field F of characteristic zero. Then the sequence $c^*_n(A)$, $n = 1, 2, \ldots,$ is polynomially bounded if and only if*

$$A = A_1 \oplus \cdots \oplus A_m + J(A),$$

where for every $i = 1, \ldots, m$, $A_i \cong F$ is endowed with the trivial superinvolution and $A_i J(A) A_k = 0$, for all $1 \le i, k \le m$, $i \ne k$.

We remark that if A is a $*$-algebra having the above decomposition then $c^*_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded also if the field is not algebraically closed.

In order to get another characterization of finite dimensional $*$-algebras with polynomial growth of codimensions, we recall the following definition and prove the subsequent lemma.

Definition 2.12. *Given two $*$-algebras A and B, we say that A is T_2^*-equivalent to B and we write $A \sim_{T_2^*} B$ in case $Id^*(A) = Id^*(B)$.*

Lemma 2.6. *Let $\bar{F}$ be the algebraic closure of the field F and let A be a finite dimensional $*$-algebra over $\bar{F}$ such that $\dim_{\bar{F}} A/J(A) \le 1$. Then $A \sim_{T_2^*} B$, for some finite dimensional $*$-algebra B over F with $\dim_{\bar{F}} A/J(A) = \dim_F B/J(B)$.*

Proof. Since $\dim_{\bar{F}} A/J(A) \le 1$, it follows that either $A \cong \bar{F} + J(A)$ or $A = J(A)$ is a nilpotent algebra.

We now take an arbitrary $*$-basis $\{w_1, \ldots, w_p\}$ of $J(A)$ over $\bar{F}$ (i.e., a basis consisting of even and odd symmetric and even and odd skew elements) and we let B be the $*$-algebra over F generated by $\mathcal{B} = \{1_{\bar{F}}, w_1, \ldots, w_p\}$ or by $\mathcal{B} = \{w_1, \ldots, w_p\}$ according as $A \cong \bar{F} + J(A)$ or $A = J(A)$, respectively.

Clearly $\dim_F B/J(B) = \dim_{\bar{F}} A/J(A)$ and as F-algebras, $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(B)$. On the other hand, if f is a multilinear $*$-identity of B then f vanishes on the basis $\mathcal{B}$. But $\mathcal{B}$ is also a basis of A over $\bar{F}$. Hence $\mathrm{Id}^*(B) \subseteq \mathrm{Id}^*(A)$ and $A \sim_{T_2^*} B$. □

Theorem 2.3.3. *Let A be a finite dimensional $*$-algebra over a field F of characteristic zero. Then $c_n^*(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$, with $B_1, \ldots, B_m$ finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \ldots, m$.*

Proof. Suppose first that $A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$, with $B_1, \ldots, B_m$ finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \ldots, m$. Then $c_n^*(A) = c_n^*(B) \leq c_n^*(B_1) + \cdots + c_n^*(B_m)$ and the claim follows since, by the remark after Corollary 2.5, $c_n^*(B_i)$ is polynomially bounded, for all $i = 1, \ldots, m$.

Conversely, let $c_n^*(A)$ be polynomially bounded. Suppose first that F is algebraically closed. Then by Corollary 2.5, $A = A_1 \oplus \cdots \oplus A_l + J$, where for every $i = 1, \ldots, l$, $A_i \cong F$ is endowed with the trivial superinvolution and $A_i J A_k = 0$, for all $1 \leq i, k \leq l$, $i \neq k$.

Set $B_1 = A_1 + J, \ldots, B_l = A_l + J$. We claim that $A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_l \oplus J$. Clearly $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_l \oplus J)$. Now let $f \in \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_l \oplus J)$ and suppose that f is not a $*$-identity of A. We may clearly assume that f is multilinear. Moreover, by choosing a $*$-basis of A as the union of a basis of $A_1 \oplus \cdots \oplus A_l$ and a basis of J it is enough to evaluate f on this basis. Let $u_1, \ldots, u_t$ be elements of this basis such that $f(u_1, \ldots, u_t) \neq 0$. Since $f \in \mathrm{Id}^*(J)$ at least one element, say u_k, does not belong to J. Then $u_k \in A_i$, for some i. Recalling that $A_i A_j = A_j A_i = A_i J A_j = A_j J A_i = 0$, for all $j \neq i$, we must have that $u_1, \ldots, u_t \in A_i \cup J$. Thus $u_1, \ldots, u_t \in A_i + J = B_i$ and this contradicts the fact that f is a $*$-identity of B_i. This proves the claim. Now the proof is completed by noticing that $\dim B_i/J(B_i) = 1$.

In case F is arbitrary, we consider the algebra $\bar{A} = A \otimes_F \bar{F}$, where $\bar{F}$ is the algebraic closure of F and $\bar{A} = A \otimes_F \bar{F}$ is a $*$-algebra with the induced superinvolution $(a \otimes \alpha)^* = a^* \otimes \alpha$, for $a \in A, \alpha \in \bar{F}$. Clearly A is T_2^*-equivalent to $\bar{A}$. Moreover the $*$-codimensions of A over F coincide with the $*$-codimensions of $\bar{A}$ over $\bar{F}$. By the hypothesis it follows that the $*$-codimensions of $\bar{A}$ are polynomially bounded. But then by the first part of the proof, $\bar{A} = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras over $\bar{F}$ and $\dim_{\bar{F}} B_i/J(B_i) \leq 1$, for all $i = 1, \ldots, m$. By Lemma 2.6, there exist finite dimensional $*$-algebras $C_1, \ldots, C_m$ over F such that, for all i, $C_i \sim_{T_2^*} B_i$ and $\dim_F C_i/J(C_i) = \dim_{\bar{F}} B_i/J(B_i) \leq 1$. It follows that $\mathrm{Id}^*(A) = \mathrm{Id}^*(\bar{A}) = \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_m) = \mathrm{Id}^*(C_1 \oplus \cdots \oplus C_m)$ and we are done. □

The following theorem collects results about $*$-varieties of polynomial growth.

Theorem 2.3.4. *For a finite dimensional $*$-algebra A over a field F of characteristic zero the following conditions are equivalent:*

1) $c_n^*(A)$ is polynomially bounded;

2) $A \sim_{T_2^*} B$, *where* $B = B_1 \oplus \cdots \oplus B_m$, *with* $B_1, \ldots, B_m$ *finite dimensional* $*$*-algebras over* F *and* $\dim B_i/J(B_i) \leq 1$, *for all* $i = 1, \ldots, m$;

3) *for every* $n_1, \ldots, n_4$ *with* $n_1 + \cdots + n_4 = n$ *it holds*

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where q *is such that* $J(A)^q = 0$;

4) $M, M^{sup}, F \oplus F \notin var^*(A)$.

Chapter 3

Subvarieties of $*$-varieties of almost polynomial growth

In this chapter we completely classify all subvarieties and all minimal subvarieties of the $*$-varieties of almost polynomial growth, generated by a finite dimensional $*$-algebra. From now on, unless otherwise stated, F denotes a field of characteristic zero.

In Corollary 2.3 we have seen that the algebras M, M^{sup} and $F \oplus F$ are the only finite dimensional $*$-algebras generating $*$-varieties of almost polynomial growth. We then classify here all subvarieties and all minimal subvarieties of $\mathrm{var}^*(F \oplus F)$, $\mathrm{var}^*(M)$ and $\mathrm{var}^*(M^{sup})$, by giving a complete list of finite dimensional $*$-algebras generating them.

We start with the definition of minimal $*$-varieties.

Definition 3.1. *A $*$-variety $\mathcal{V}$ is minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$, for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n^*(\mathcal{U}) \approx q'n^t$, with $t < k$.*

In the first section of this chapter we deal with the subvarieties of $\mathrm{var}^*(F \oplus F)$ and $\mathrm{var}^*(M)$ whereas in the second section we shall focus our attention to the subvarieties of $\mathrm{var}^*(M^{sup})$. Finally we describe the $*$-algebras whose $*$-codimensions are bounded by a linear function.

3.1 Subvarieties of $\mathrm{var}^*(F \oplus F)$ and $\mathrm{var}^*(M)$

In this section we classify, up to T_2^*-equivalence, all the $*$-algebras contained in the $*$-variety generated by $F \oplus F$ or M. Recall that $F \oplus F$ is the two-dimensional commutative algebra endowed with the exchange involution ex and $M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34}$ is a subalgebra of UT_4 endowed with the reflection involution *ref*. The exchange involution is such that $(a, b)^{ex} = (b, a)$, for all $(a, b) \in F \oplus F$ whereas the reflection involution *ref* on the algebras of $n \times n$ upper triangular matrices UT_n is defined as follows: if $a = (a_{ij}) \in UT_n$ then $a^{ref} = (b_{ij})$ where $b_{ij} = a_{n+1-j\ n+1-i}$.

Since $F \oplus F$ is a $*$-algebra with trivial grading the following remark holds.

Remark 3.1. *The $*$-variety $var^*(F \oplus F)$, generated by $F \oplus F$, coincide with the variety of algebras with involution $var^\sharp(F \oplus F)$, where $F \oplus F$ is regarded as algebra with involution.*

At the same way, we can state the analogous result for M.

Remark 3.2. *The $*$-variety $var^*(M)$, generated by M, coincide with the variety of algebras with involution $var^\sharp(M)$, where M is regarded as algebra with involution.*

Remarks 3.1 and 3.2 assure us that the classification of the $*$-algebras inside $\text{var}^*(F \oplus F)$ or $\text{var}^*(M)$ is equivalent to the classification of the ordinary algebras with involution inside $\text{var}^\sharp(F \oplus F)$ or $\text{var}^\sharp(M)$. Such a classification was given in [41] by La Mattina and Martino. In what follows we present such results in the language of $*$-algebras.

We start by constructing, for any fixed $k \geq 1$, $*$-algebras belonging to the $*$-variety generated by $F \oplus F$ whose $*$-codimensions sequence grows polynomially as n^k.

For $k \geq 2$, let e_{ij}s be the usual matrix units, I_k the $k \times k$ identity matrix and $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$. We denote by

$$C_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\} \subseteq UT_k,$$

a commutative subalgebra of UT_k, the algebra of $k \times k$ upper triangular matrices over F. We also write C_k to mean the algebra C_k with trivial grading and superinvolution $*$ given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$

We next state the following result characterizing the $*$-identities and the $*$-codimensions of C_k (see [41]).

Theorem 3.1.1. *Let $k \geq 2$. Then*

1) $Id^*(C_k) = \langle [y_1, y_2], y_1^- \cdots y_k^-, z^+, z^- \rangle_{T_2^*}$.

2) $c_n^*(C_k) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}$.

The following result classifies all the subvarieties of the $*$-variety generated by $F \oplus F$.

Theorem 3.1.2. *[41] Let A be a $*$-algebra such that $A \in var^*(F \oplus F)$. Then either $A \sim_{T_2^*} F \oplus F$ or $A \sim_{T_2^*} N$ or $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} C_k \oplus N$, for some $k \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebra with trivial superinvolution.*

Next we exhibit finite dimensional $*$-algebras belonging to the $*$-variety generated by M whose $*$-codimensions sequence grows polynomially.

For $k \geq 2$, let

$$A_k = \text{span}_F \left\{ e_{11} + e_{2k,2k}, E, \ldots, E^{k-2}, e_{12}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-1,2k} \right\},$$

$$N_k = \text{span}_F \left\{ I, E, \ldots, E^{k-2}, e_{12} - e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k} \right\},$$

$$U_k = \text{span}_F \left\{ I, E, \ldots, E^{k-2}, e_{12} + e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k} \right\},$$

be subalgebras of UT_{2k}, the algebra of $2k \times 2k$ upper triangular matrices over F. Here I denotes the $2k \times 2k$ identity matrix and

$$E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1}.$$

We also write A_k, N_k and U_k to mean the above algebras with trivial grading and with reflection superinvolution. We next state the following results characterizing the $*$-identities and the $*$-codimensions of these algebras (see [41] for more details).

Theorem 3.1.3. *For every $k \geq 2$ we have:*

1) $Id^*(A_k) = \langle y_1^- y_2^-, z^+, z^-, y_1^+ \cdots y_{k-2}^+ St_3(y_{k-1}^+, y_k^+, y_{k+1}^+) y_{k+2}^+ \cdots y_{2k-1}^+, y_1^+ \cdots y_{k-1}^+ y^- y_k^+ \cdots y_{2k-2}^+ \rangle_{T_2^*}$;

2) $c_n^*(A_k) \approx qn^{k-1}$, *for some* $q > 0$.

Theorem 3.1.4. *The T_2^*-ideal $Id^*(N_k)$ is generated by the polynomials $[y_1^+, y_2^+]$, $[y^+, y^-]$, $y_1^- y_2^-$, z^+, z^-, in case $k = 2$ and by $[y_1^+, \ldots, y_{k-1}^+]$, $y_1^- y_2^-$, z^+, z^-, in case $k \geq 3$. Moreover*

$$c_n^*(N_k) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} (2i-1) + \binom{n}{k-1} (k-1) \approx qn^{k-1}, \text{ for some } q > 0.$$

Theorem 3.1.5. *The T_2^*-ideal $Id^*(U_k)$ is generated by the polynomials $[y_1^+, y_2^+]$, y^-, z^+, z^-, in case $k = 2$ and by $[y^-, y_1^+, \ldots, y_{k-2}^+]$, $y_1^- y_2^-$, z^+, z^-, in case $k \geq 3$. Moreover $c_n^*(U_2) = 1$ and*

$$c_n^*(U_k) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} (2i-1) + \binom{n}{k-1} (k-2) \approx qn^{k-1}, \text{ for some } q > 0, \text{ for } k \geq 3.$$

The following result classifies the subvarieties of var$^*(M)$.

Theorem 3.1.6. *[41, Theorem 6] If $A \in var^*(M)$ then A is T_2^*-equivalent to one of the following $*$-algebras:*

$$M,\ N,\ N_k \oplus N,\ U_k \oplus N,\ N_k \oplus U_k \oplus N,\ A_t \oplus N,\ N_k \oplus A_t \oplus N,\ U_k \oplus A_t \oplus N,\ N_k \oplus U_k \oplus A_t \oplus N,$$

for some $k, t \geq 2$, where N is a nilpotent $*$-algebra.

As a consequence of Theorems 3.1.2 and 3.1.6, we can also get the classification of all $*$-algebras generating minimal $*$-varieties.

Corollary 3.1. *A $*$-algebra $A \in var^*(F \oplus F)$ generates a minimal $*$-variety of polynomial growth if and only if $A \sim_{T_2^*} C_k$, for some $k \geq 2$.*

Corollary 3.2. *A $*$-algebra $A \in var^*(M)$ generates a minimal $*$-variety of polynomial growth if and only if either $A \sim_{T_2^*} U_r$ or $A \sim_{T_2^*} N_k$ or $A \sim_{T_2^*} A_k$, for some $r > 2$, $k \geq 2$.*

3.2 Subvarieties of var$^*(M^{sup})$

3.2.1 Unitary $*$-algebras inside var$^*(M^{sup})$

In this section we classify, up to T_2^*-equivalence, all the unitary $*$-algebras contained in the $*$-variety generated by M^{sup}.

We consider the algebras N_k and U_k we have defined before endowed with an elementary $\mathbb{Z}_2$-grading. Recall that if $\mathbf{g} = (g_1, \dots, g_{2k}) \in \mathbb{Z}_2^{2k}$ is an arbitrary $2k$-tuple of elements of $\mathbb{Z}_2$, then $\mathbf{g}$ defines an elementary $\mathbb{Z}_2$-grading on UT_{2k}, the algebra of $2k \times 2k$ upper triangular matrices over F, by setting

$$(UT_{2k})_0 = \text{span}\{e_{ij} \mid g_i + g_j = 0\} \text{ and } (UT_{2k})_1 = \text{span}\{e_{ij} \mid g_i + g_j = 1\}$$

(recall that equalities are taken modulo 2). If A is a graded subalgebra of UT_{2k} the induced grading on A is also called elementary.

Definition 3.2. *For $k \geq 2$, N_k^{sup} is the algebra N_k with elementary $\mathbb{Z}_2$-grading induced by $\mathbf{g} = (0, \underbrace{1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{k-1}, 1)$ and with reflection superinvolution.*

The following result characterizes the $*$-identities and the $*$-codimensions of N_k^{sup}.

Theorem 3.2.1. *Let $k \geq 2$. Then:*

1) $Id^*(N_k^{sup}) = \langle y^-,\ z_1 z_2,\ [z^+, y_1, \dots, y_{k-2}] \rangle_{T_2^*}$;

2) $c_n^*(N_k^{sup}) = 1 + \sum_{i=1}^{k-2} 2i\binom{n}{i} + \binom{n}{k-1}(k-1) \approx qn^{k-1}$, *for some $q > 0$.*

Proof. Let $I = \langle y^-,\ z_1z_2,\ [z^+, y_1, \dots, y_{k-2}]\rangle_{T_2^*}$. It is easy to see that $I \subseteq \mathrm{Id}^*(N_k^{sup})$. Let now f be a $*$-identity of N_k^{sup}. We may assume that f is multilinear and, since N_k^{sup} is an unitary algebra, we may take f proper (see Section 1.8). After reducing the polynomial f modulo I we obtain that f is the zero polynomial if $\deg f \geq k$, f is a linear combination of commutators

$$[z_i^-, y_{i_1}^+, \dots, y_{i_{k-2}}^+],\ i_1 < \cdots < i_{k-2},$$

in case $\deg f = k-1$ and f is a linear combination of commutators

$$[z_i^-, y_{i_1}^+, \dots, y_{i_{s-1}}^+],\ [z_j^+, y_{j_1}^+, \dots, y_{j_{s-1}}^+],\ i_1 < \cdots < i_{s-1},\ j_1 < \cdots < j_{s-1},$$

in case $\deg f = s < k-1$. Hence, for some $s = 1, \dots, k-1$,

$$f = \sum_{i=1}^{s} \alpha_i\ [z_i^-, y_{i_1}^+, \dots, y_{i_{s-1}}^+] + \sum_{j=1}^{s} \beta_j\ [z_j^+, y_{j_1}^+, \dots, y_{j_{s-1}}^+].$$

Suppose that there exists i such that $\alpha_i \neq 0$ (resp. $\beta_i \neq 0$). By making the evaluation $z_i^- = e_{12} - e_{2k-1,2k}$, $z_l^- = 0$, for all $l \neq i$, $z_j^+ = 0$, for $j = 1, \dots, s$ (resp. $z_i^+ = e_{13} + e_{2k-2,2k}$, $z_l^+ = 0$, for all $l \neq i$, $z_j^- = 0$, for $j = 1, \dots, s$) and $y_l = E$, for all $l = i_1, \dots, i_{s-1}$, we get that $\alpha_i = 0$ (resp. $\beta_i = 0$), a contradiction. Hence $\alpha_i = \beta_i = 0$, for all $i = 1, \dots, s$. This says that $f \in I$ and, so, $\mathrm{Id}^*(N_k^{sup}) = I$.

The argument above also proves the following fact concerning $\gamma_n^*(N_k^{sup})$, the sequence of proper $*$-codimensions of N_k^{sup}. We have that $\gamma_s^*(N_k^{sup}) = s$ for $s = k-1$, $\gamma_s^*(N_k^{sup}) = 2s$ for $s < k-1$ and $\gamma_s^*(N_k^{sup}) = 0$ for $s \geq k$. Then, by Theorem 1.8.2, we have

$$c_n^*(N_k^{sup}) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} 2i + \binom{n}{k-1}(k-1) \approx qn^{k-1},\ \text{for some } q > 0.$$

□

Definition 3.3. *For $k \geq 2$, U_k^{sup} is the algebra U_k with elementary $\mathbb{Z}_2$-grading induced by* $\mathbf{g} = (0, \underbrace{1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{k-1}, 1)$ *and with reflection superinvolution.*

The following results, characterizing the $*$-identities and the $*$-codimensions of U_k^{sup} and of $N_k^{sup} \oplus U_k^{sup}$, can be proved in a similar way as the previous theorem.

Theorem 3.2.2. *Let $k \geq 2$. Then:*

1) *$Id^*(U_k^{sup}) = \langle y^-,\ z_1z_2,\ [z^-, y_1, \dots, y_{k-2}]\rangle_{T_2^*}$;*

2) *$c_n^*(U_k^{sup}) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} 2i + \binom{n}{k-1}(k-1) \approx qn^{k-1}$, for some $q > 0$.*

Theorem 3.2.3. *Let $k \geq 2$. Then:*

1) $Id^*(N_k^{sup} \oplus U_k^{sup}) = \langle y^-,\ z_1z_2,\ [z, y_1, \ldots, y_{k-1}]\rangle_{T_2^*}$;

2) $c_n^*(N_k^{sup} \oplus U_k^{sup}) = 1 + \sum_{i=1}^{k-1} \binom{n}{i} 2i \approx qn^{k-1}$, *for some $q > 0$.*

The following remark is obvious.

Remark 3.3. *If $t > k$ then $U_t^{sup} \oplus N_k^{sup} \sim_{T_2^*} U_t^{sup}$ whereas $U_t^{sup} \oplus N_k^{sup} \sim_{T_2^*} N_k^{sup}$ if $t < k$.*

We recall that if $A = B + J$ is a finite dimensional $*$-algebra over F, where B is a semisimple $*$-subalgebra and $J = J(A)$ is its Jacobson radical, then J can be decomposed into the direct sum of B-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively and for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where δ_{kl} is the Kronecker delta (for a proof of this see Lemma 2 of [25]).

From now until the end of this section we assume that the field F of characteristic zero is algebraically closed.

Theorem 3.2.4. *For any $k \geq 2$, N_k^{sup} generates a minimal $*$-variety of polynomial growth.*

Proof. Let $A \in \mathrm{var}^*(N_k^{sup})$ be such that $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$. We shall prove that $A \sim_{T_2^*} N_k^{sup}$. Since $A \in \mathrm{var}^*(M^{sup})$, by Corollary 2.1, A satisfies the same $*$-identities as a finite dimensional $*$-algebra. Hence, since $c_n^*(A)$ is polynomially bounded, by Theorem 2.3.3 we may assume that

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \ldots, m$. This implies that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent $*$-algebra. Since $c_n^*(A) \leq c_n^*(B_1) + \cdots + c_n^*(B_m)$, then there exists B_i such that $c_n^*(B_i) \approx bn^{k-1}$, for some $b > 0$. Hence

$$\mathrm{var}^*(N_k^{sup}) \supseteq \mathrm{var}^*(A) \supseteq \mathrm{var}^*(F + J(B_i)) \supseteq \mathrm{var}^*(F + J_{11}(B_i)).$$

Hence, in order to complete the proof it is enough to show that $F + J_{11}(B_i) \sim_{T_2^*} N_k^{sup}$. Thus, without loss of generality, we may assume that A is a unitary $*$-algebra. Now since $c_n^*(A) \approx qn^{k-1}$, then $c_n^*(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^*(A)$ and, by Corollary 1.7, $\gamma_i^*(A) \neq 0$, for all $i = 0, \ldots, k-1$.

For $n_1 + \cdots + n_4$, let

$$\psi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)} \text{ and } \psi_{n_1,\ldots,n_4}(N_k^{sup}) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$

be the $(n_1, \ldots, n_4)$-th proper cocharacters of A and N_k^{sup}, respectively. Since $\mathrm{Id}^*(A) \supseteq \mathrm{Id}^*(N_k^{sup})$, we must have $m_{\langle\lambda\rangle} \leq m'_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash n = n_1 + \cdots + n_4$.

For any $i = 2, \ldots, k-1$, let $f_1 = [z_1^+, y_2^+, \ldots, y_2^+]$ and $f_2 = [z_1^-, y_2^+, \ldots, y_2^+]$ be highest weight vectors corresponding to the multipartitions $\langle\lambda\rangle = ((i-1),\ \emptyset,\ (1),\ \emptyset)$ and $\langle\mu\rangle = ((i-1),\ \emptyset,\ \emptyset,\ (1))$ (see Section 1.8 and [11, Chapter 12] for more details). It is easily seen that f_1 is not a $*$-identity of N_k^{sup}, for $i = 2, \ldots, k-2$ and f_2 is not a $*$-identity of N_k^{sup}, for $i = 2, \ldots, k-1$.

Thus for $i = 2, \ldots, k-2$, $\chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset$ participates in the $(i-1, 0, 1, 0)$-th proper cocharacter $\psi_{i-1,0,1,0}(N_k^{sup})$ with non-zero multiplicity. Also, for $i = 2, \ldots, k-1$, $\chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$ participates in the $(i-1, 0, 0, 1)$-th proper cocharacter $\psi_{i-1,0,0,1}(N_k^{sup})$ with non-zero multiplicity.

Hence, for $i = 2, \ldots, k-2$, since

$$\gamma_i^*(N_k^{sup}) = 2i = \binom{i}{i-1, 0, 1, 0} \deg \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset + \binom{i}{i-1, 0, 0, 1} \deg \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)},$$

by (1.5) we have that, for $n_1 + \cdots + n_4 = i$

$$\psi_{n_1,n_2,n_3,n_4}(N_k^{sup}) = \begin{cases} \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset & \text{if } (n_1, n_2, n_3, n_4) = (i-1, 0, 1, 0) \\ \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)} & \text{if } (n_1, n_2, n_3, n_4) = (i-1, 0, 0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, if $n_1 + \cdots + n_4 = k-1$, since $\gamma_{k-1}^*(N_k^{sup}) = k-1 = \binom{k-1}{k-2, 0, 0, 1} \deg \chi_{(k-2)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$, we get

$$\psi_{n_1,n_2,n_3,n_4}(N_k^{sup}) = \begin{cases} \chi_{(k-2)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)} & \text{if } (n_1, n_2, n_3, n_4) = (k-2, 0, 0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

We claim that $\psi_{n_1,n_2,n_3,n_4}(A) = \psi_{n_1,n_2,n_3,n_4}(N_k^{sup})$.

Suppose first that $n_1 + \cdots + n_4 = k-1$. Since $\gamma_{k-1}^*(A) \neq 0$ and $m_{\langle\lambda\rangle} \leq m'_{\langle\lambda\rangle}$, for any $\langle\lambda\rangle \vdash n_1 + \cdots + n_4$, then we get that

$$\psi_{n_1,n_2,n_3,n_4}(A) = \begin{cases} \chi_{(k-2)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)} & \text{if } (n_1, n_2, n_3, n_4) = (k-2, 0, 0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Let now $n_1 + \cdots + n_4 = i$, where $i = 2, \ldots, k-2$,. Since $\gamma_{k-1}^*(A) \neq 0$ and $m_{\langle\lambda\rangle} \leq m'_{\langle\lambda\rangle}$, for any $\langle\lambda\rangle \vdash n_1 + \cdots + n_4$, if $\psi_{i-1,0,0,1}(A) = 0$, for some $2 \leq i \leq k-2$, then the highest weight vector

$[z_1^-, \underbrace{y_2^+, \dots, y_2^+}_{i-1}]$ corresponding to the multipartition $((i-1),\ \emptyset,\ \emptyset,\ (1))$ would be a $*$-identity for A. But this implies that also $[z_1^+, \underbrace{y_2^+, \dots, y_2^+}_{k-1}]$ is a $*$-identity for A, and so $\psi_{k-2,0,0,1}(A) = 0$, a contradiction.
In a similar way one can prove that, if $\psi_{i-1,0,1,0}(A) = 0$, we would reach a contradiction and so the claim is proved. Hence

$$c_n^*(A) = \sum_{i=0}^{k-1} \binom{n}{i} \sum_{n_1+\cdots+n_4=i} \binom{i}{n_1, \dots, n_4} \gamma_{n_1,\dots,n_4}(A) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} 2i + \binom{n}{k-1}(k-1) = c_n^*(N_k^{sup}).$$

Thus A and N_k^{sup} have the same sequence of $*$-codimensions and, since $\mathrm{Id}^*(N_k^{sup}) \subseteq \mathrm{Id}^*(A)$, we get the equality $\mathrm{Id}^*(N_k^{sup}) = \mathrm{Id}^*(A)$ and the proof is complete. □

In a similar way it is possible to prove the following.

Theorem 3.2.5. *For any $k \geq 2$, U_k^{sup} generates a minimal $*$-variety of polynomial growth.*

In the following result we classify, up to T_2^*-equivalence, all the unitary $*$-algebras inside $\mathrm{var}^*(M^{sup})$.

Theorem 3.2.6. *Let $A \in var^*(M^{sup})$ be an unitary $*$-algebra such that $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$, $k \geq 1$. Then either $A \sim_{T_2^*} C$ or $A \sim_{T_2^*} U_k^{sup}$ or $A \sim_{T_2^*} N_k^{sup}$ or $A \sim_{T_2^*} N_k^{sup} \oplus U_k^{sup}$, where C is a commutative algebra with trivial superinvolution.*

Proof. If $k = 1$ it is immediate to see that A is a commutative algebra with trivial superinvolution.

Let now $k \geq 2$. Since $c_n^*(A) \approx qn^{k-1}$, by Theorem 1.8.2, $\gamma_{k-1}^*(A) \neq 0$. Hence at least one polynomial among $[z^+, y_1^+, \dots, y_{k-2}^+]$ and $[z^-, y_1^+, \dots, y_{k-2}^+]$ cannot be a $*$-identity for A, since otherwise we would have $\gamma_{k-1}^*(A) = 0$, a contradiction.
If $[z^-, y_1^+, \dots, y_{k-2}^+]$ is not a $*$-identity and $[z^+, y_1^+, \dots, y_{k-2}^+] \equiv 0$ on A then $\mathrm{Id}^*(N_k^{sup}) \subseteq \mathrm{Id}^*(A)$ and since $c_n^*(A) \approx qn^{k-1}$, by Theorem 3.2.4, one gets that $A \sim_{T_2^*} N_k^{sup}$. Similarly, if $[z^+, y_1^+, \dots, y_{k-2}^+]$ is not a $*$-identity and $[z^-, y_1^+, \dots, y_{k-2}^+] \equiv 0$ on A one gets that $A \sim_{T_2^*} U_k^{sup}$.
Finally, suppose that neither of the polynomials $[z^+, y_1^+, \dots, y_{k-2}^+]$ and $[z^-, y_1^+, \dots, y_{k-2}^+]$ are $*$-identities for A. Since $c_n^*(A) \approx qn^{k-1}$, then $\gamma_k^*(A) = 0$, and so $\mathrm{Id}^*(N_k^{sup} \oplus U_k^{sup}) \subseteq \mathrm{Id}^*(A)$. As in the proof of Theorem 3.2.4, for $i = 2, \dots, k-1$, we get

$$\psi_{i-1,0,1,0}(A) = \psi_{i-1,0,1,0}(N_k^{sup} \oplus U_k^{sup}) = \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset,$$

$$\psi_{i-1,0,0,1}(A) = \psi_{i-1,0,0,1}(N_k^{sup} \oplus U_k^{sup}) = \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$$

and

$$\psi_{n_1,n_2,n_3,n_4}(A) = \psi_{n_1,n_2,n_3,n_4}(N_k^{sup} \oplus U_k^{sup}) = 0,$$

if $(n_1, n_2, n_3, n_4) \notin \{(i-1,0,0,1),(i-1,0,1,0)\}, n_1+\cdots+n_4 = i$. Hence A and $N_k^{sup} \oplus U_k^{sup}$ have the same sequence of $*$-codimensions

$$c_n^*(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^*(A) = 1 + \sum_{i=1}^{k-1} 2i \binom{n}{i} = c_n^*(N_k^{sup} \oplus U_k^{sup}).$$

Since $\mathrm{Id}^*(N_k^{sup} \oplus U_k^{sup}) \subseteq \mathrm{Id}^*(A)$, we finally get the equality $\mathrm{Id}^*(N_k^{sup} \oplus U_k^{sup}) = \mathrm{Id}^*(A)$. □

3.2.2 Classifying the subvarieties of var$^*(M^{sup})$

In this section we can classify, up to T_2^*-equivalence, all $*$-algebras contained in the $*$-variety generated by M^{sup}. We start by constructing $*$-algebras without unit inside var$^*(M^{sup})$.

Definition 3.4. *For $k \geq 2$, A_k^{sup} is the algebra A_k with elementary $\mathbb{Z}_2$-grading induced by* $\mathbf{g} = (0, \underbrace{1,\ldots,1}_{k-1}, \underbrace{0,\ldots,0}_{k-1}, 1)$ *and with reflection superinvolution.*

Next we describe explicitly the $*$-identities of A_k^{sup}.

Theorem 3.2.7. *Let $k \geq 2$. Then*

1) $Id^*(A_k^{sup}) = \langle y^-, z_1z_2, y_1\cdots y_{k-1}zy_k\cdots y_{2k-2}\rangle_{T_2^*}$;

2) $c_n^*(A_k^{sup}) = 1 + 2\sum_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t) \approx qn^{k-1}$, for some $q > 0$.

Proof. Write $I = \langle y^-, z_1z_2, y_1\cdots y_{k-1}zy_k\cdots y_{2k-2}\rangle_{T_2^*}$. It is easily seen that $I \subseteq \mathrm{Id}^*(A_k^{sup})$. In order to prove the opposite inclusion, first we find a set of generators of P_n^* modulo $P_n^* \cap I$, for every $n \geq 1$. Any multilinear polynomial of degree n can be written, modulo I, as a linear combination of monomials of the type

$$y_1^+\cdots y_n^+, \quad y_{i_1}^+\cdots y_{i_t}^+ z_l^+ y_{j_1}^+\cdots y_{j_s}^+, \quad y_{r_1}^+\cdots y_{r_p}^+ z_l^- y_{s_1}^+\cdots y_{s_q}^+, \tag{3.1}$$

where $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $t < k-1$ or $s < k-1$, $r_1 < \cdots < r_p$, $s_1 < \cdots < s_q$ and $p < k-1$ or $q < k-1$.

We next show that the above elements are linearly independent modulo $\mathrm{Id}^*(A_k^{sup})$. Let $f \in \mathrm{Id}^*(A_k^{sup})$ be a linear combination of the above monomials:

$$f = \delta y_1^+\cdots y_n^+ + \sum_{\substack{t<k-1 \\ or \\ s<k-1}} \sum_{l,I,J} \alpha_{l,I,J}\, y_{i_1}^+\cdots y_{i_t}^+ z_l^+ y_{j_1}^+\cdots y_{j_s}^+ + \sum_{\substack{p<k-1 \\ or \\ q<k-1}} \sum_{m,R,S} \beta_{m,R,S}\, y_{r_1}^+\cdots y_{r_p}^+ z_m^- y_{s_1}^+\cdots y_{s_q}^+,$$

where $t+s = p+q = n-1$ and for any fixed t and p, $I = \{i_1,\ldots,i_t\}$, $J = \{j_1,\ldots,j_s\}$, $R = \{r_1,\ldots,r_p\}$ and $S = \{s_1,\ldots,s_q\}$.

By making the evaluation $y_1^+ = \cdots = y_n^+ = e_{11} + e_{2k,2k}$, and $z_l^+ = z_l^- = 0$, for all $l = 1, \ldots, n$, one gets $\delta(e_{11} + e_{2k,2k}) = 0$ and, so, $\delta = 0$.

For fixed $t < k-1, l, I, J$ the evaluation $z_l^+ = e_{12} + e_{2k-1,2k}$, $z_{l'}^+ = 0$, for all $l' \neq l$, $y_{i_1}^+ = \cdots = y_{i_t}^+ = E$, $y_{j_1}^+ = \cdots = y_{j_s}^+ = e_{11} + e_{2k,2k}$ and $z_m^- = 0$, for $m = 1, \ldots, n$, gives $\alpha_{l,I,J}\ e_{2k-t-1,2k} + \alpha_{l,J,I}\ e_{1,2+t} = 0$. Thus $\alpha_{l,I,J} = \alpha_{l,J,I} = 0$.

Similarly, for fixed $s < k-1, l, I, J$ the evaluation $z_l^+ = e_{12} + e_{2k-1,2k}$, $z_{l'}^+ = 0$, for all $l' \neq l$, $y_{i_1}^+ = \cdots = y_{i_t}^+ = e_{11} + e_{2k,2k}$, $y_{j_1}^+ = \cdots = y_{j_s}^+ = E$ and $z_m^- = 0$, for $m = 1, \ldots, n$, gives $\alpha_{l,I,J} = 0$.

In a similar way it is proved that the coefficients $\beta_{m,R,S} = 0$, for all m, R and S.

Therefore the elements in (3.1) are linearly independent modulo $P_n^* \cap \mathrm{Id}^*(A_k^{sup})$ and, since $P_n^* \cap \mathrm{Id}^*(A_k^{sup}) \supseteq P_n^* \cap I$, they form a basis of P_n^* modulo $P_n^* \cap \mathrm{Id}^*(A_k^{sup})$ and $\mathrm{Id}^*(A_k^{sup}) = I$. By counting, we obtain

$$c_n^*(A_k^{sup}) = 1 + 2 \sum_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t) \approx qn^{k-1},$$

for some $q > 0$. □

Remark 3.4. *Let $A = F + J \in var^*(M^{sup})$. Then $J_{10}J_{01} = J_{01}J_{10} = (J_{11})_1 J_{10} = J_{01}(J_{11})_1 = 0$. In particular, if $A \in var^*(A_k^{sup})$ then $(J_{11})_1 = 0$.*

Proof. We start by proving that $J_{10}J_{01} = J_{01}J_{10} = 0$. Let $a = a_0 + a_1 \in J_{10}$, $b = b_0 + b_1 \in J_{01}$. Notice that, since $A_0^- = 0$, $a - a^* = a_1 - a_1^*$ and $b - b^* = b_1 - b_1^*$. Then, because of $z_1 z_2 \equiv 0$, $(a - a^*)(b - b^*) = 0$ and, so, $ab = a^* b^* = 0$.
Now let $a \in (J_{11})_1$, $b = b_0 + b_1 \in J_{10}$. Then $a(b - b^*) = 0$ and, so, $ab = 0$.
Finally, if $A \in \mathrm{var}^*(A_k^{sup})$ then A satisfies the $*$-identity $y_1^+ \cdots y_{k-1}^+ z y_k^+ \cdots y_{2k-2}^+ \equiv 0$. Hence, since $(J_{11})_1 = \underbrace{F \cdots F}_{k-1} (J_{11})_1 \underbrace{F \cdots F}_{k-1}$, we get the desired result. □

Lemma 3.1. *Let $A = F + J \in var^*(A_k^{sup})$ with $J_{10} \neq 0$ (hence $J_{01} \neq 0$). If $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$, then $A \sim_{T_2^*} A_k^{sup}$.*

Proof. Since $A \in \mathrm{var}^*(A_k^{sup})$, by the previous remark we must have $(J_{11})_1 = J_{01}J_{10} = J_{10}J_{01} = 0$.

Suppose first that $(J_{10})_1((J_{00})_0^+)^{k-2} = 0$ and, so, $((J_{00})_0^+)^{k-2}(J_{01})_1 = 0$. Since J is a nilpotent ideal of A, then there exists m such that J^m. It can be proved that, for any $n \geq m$, the multilinear polynomial

$$f = y_{i_1} \cdots y_{i_l} y_1 \cdots y_{k-2} z y_{k-1} \cdots y_{2k-4} y_{j_1} \cdots y_{j_t} \in \mathrm{Id}^*(A),$$

where $l + t + 2k - 3 = n$.

Hence, if $Q \subseteq \mathrm{Id}^*(A)$ is the T_2^*-ideal generated by f plus the generators of the T_2^*-ideal $\mathrm{Id}^*(A_k^{sup})$, it is easy to see that for any $n \geq m$, a set of generators of $P_n^*(\bmod\ P_n^* \cap Q)$ is given by the polynomials

$$y_1^+ \cdots y_n^+,\ y_{i_1}^+ \cdots y_{i_t}^+\ z_l^+\ y_{j_1}^+ \cdots y_{j_s}^+,\ y_{i_1}^+ \cdots y_{i_t}^+\ z_l^-\ y_{j_1}^+ \cdots y_{j_s}^+,$$

where $t + s = n - 1$, $t < k - 2$ or $s < k - 2$, $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$. Hence

$$c_n^*(A) \leq 1 + 2 \sum_{\substack{t<k-2 \\ or \\ n-t<k-1}} \binom{n-1}{t} n \approx q n^{k-2},$$

a contradiction.

Therefore we must have $(J_{10})_1((J_{00})_0^+)^{k-2} \neq 0$. In order to complete the proof it is enough to show that $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(A_k^{sup})$. Let $f \in \mathrm{Id}^*(A)$ be a multilinear polynomial. By Theorem 3.2.7, we can write f, modulo $\mathrm{Id}^*(A_k^{sup})$ as

$$f = \delta y_1^+ \cdots y_n^+ + \sum_{\substack{t<k-1 \\ or \\ s<k-1}} \sum_{l,I,J} \alpha_{l,I,J}\ y_{i_1}^+ \cdots y_{i_t}^+ z_l^+ y_{j_1}^+ \cdots y_{j_s}^+ + \sum_{\substack{p<k-1 \\ or \\ q<k-1}} \sum_{m,R,S} \beta_{m,R,S}\ y_{r_1}^+ \cdots y_{r_p}^+ z_m^- y_{s_1}^+ \cdots y_{s_q}^+,$$

where $I = \{i_1, \ldots, i_t\}$, $J = \{j_1, \ldots, j_s\}$, $R = \{r_1, \ldots, r_p\}$, $S = \{s_1, \ldots, s_q\}$ are such that $I \cup J \cup \{l\} = R \cup S \cup \{m\} = \{1, \ldots, n\}$ and $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $r_1 < \cdots < r_p$ and $s_1 < \cdots < s_q$. It is easy to see that f must be the zero polynomial and so, $f \in \mathrm{Id}^*(A_k^{sup})$. This says that $\mathrm{Id}^*(A) = \mathrm{Id}^*(A_k^{sup})$ and the proof is complete. □

From now until the end of this section we assume that the field F of characteristic zero is also algebraically closed. We are now in a position to prove the following theorem.

Theorem 3.2.8. *For any $k \geq 2$, A_k^{sup} generates a minimal $*$-variety of polynomial growth.*

Proof. As in the proof of Theorem 3.2.4 we may assume that $A = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent algebra and there exists B_i such that $c_n^*(B_i) \approx b n^{k-1}$, for some $b > 0$. Since $k \geq 2$, we must have that $J_{10}(B_i) \neq 0$ (hence $J_{01}(B_i) \neq 0$). If not, $B_i \cong (F + J_{11}) \oplus J_{00}$ and $c_n^*(B_i) = c_n^*(F + J_{11})$, for n large enough. But since $C = F + J_{11} \in \mathrm{var}^*(A_k^{sup})$, we get that C is a commutative algebra with trivial superinvolution and, so, $c_n^*(F + J_{11}) = 1$, a contradiction. Therefore, since B_i satisfies the hypotheses of Lemma 3.1, we get that $B_i \sim_{T_2^*} A_k^{sup}$ and $A \sim_{T_2^*} A_k^{sup}$ follows. □

Lemma 3.2. *Let $A = F + J \in var^*(M^{sup})$ be a $*$-algebra. If $J_{10} \neq 0$ (hence $J_{01} \neq 0$) then A is T_2^*-equivalent to one of the following $*$-algebras*

$$A_k^{sup} \oplus N,\ N_u^{sup} \oplus A_k^{sup} \oplus N,\ U_u^{sup} \oplus A_k^{sup} \oplus N,\ N_u^{sup} \oplus U_u^{sup} \oplus A_k^{sup} \oplus N,$$

for some $k, u \geq 2$, where N is a nilpotent $*$-algebra.

Proof. Since the proof is very similar to that given in [41, Lemma 8] we shall just give a sketch of it.

Let $j \geq 0$ be the largest integer such that $J_{10}J_{00}^{j} \neq 0$ (hence $J_{00}^{j}J_{01} \neq 0$). We shall see that either $A \sim_{T_2^*} A_{j+2}^{sup} \oplus J_{00}$ or $A \sim_{T_2^*} A_{j+2}^{sup} \oplus N_u^{sup} \oplus J_{00}$ or $A \sim_{T_2^*} A_{j+2}^{sup} \oplus U_u^{sup} \oplus J_{00}$ or $A \sim_{T_2^*} A_{j+2}^{sup} \oplus N_u^{sup} \oplus U_u^{sup} \oplus J_{00}$, for some $u \geq 2$.

Suppose first that $(J_{11})_1 = 0$.
It is checked that $A_{j+2}^{sup} \sim_{T_2^*} A/J_{00}^{j+1}$ and so, $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(A_{j+2}^{sup} \oplus J_{00})$. In order to prove the opposite inclusion, it is taken $f \in \mathrm{Id}^*(A_{j+2}^{sup} \oplus J_{00})$ a multilinear polynomial of degree n. If $n \leq 2j+2$, since $f \in \mathrm{Id}^*(A_{j+2}^{sup})$, then f must be a consequence of $\langle y^-, z_1z_2\rangle_{T_2^*} \subseteq \mathrm{Id}^*(A)$. Hence $f \in \mathrm{Id}^*(A)$ and we are done in this case. Now let $n > 2j+2$. It is checked that f can be written modulo $\mathrm{Id}^*(A_{j+2}^{sup})$ as

$$f = \sum_{\substack{t\geq j+1 \\ and \\ s\geq j+1}} \sum_{l,I,J} \alpha_{l,I,J}\, y_{i_1}^+ \cdots y_{i_t}^+ z_l^+ y_{j_1}^+ \cdots y_{j_s}^+ + \sum_{\substack{p\geq j+1 \\ and \\ q\geq j+1}} \sum_{m,R,S} \beta_{m,R,S}\, y_{r_1}^+ \cdots y_{r_p}^+ z_m^- y_{s_1}^+ \cdots y_{s_q}^+ + g,$$

where $g \in \langle y^-, z_1z_2\rangle_{T_2^*}$ and $I = \{i_1, \ldots, i_t\}$, $J = \{j_1, \ldots, j_s\}$, $R = \{r_1, \ldots, r_p\}$, $S = \{s_1, \ldots, s_q\}$ with $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $r_1 < \cdots < r_p$ and $s_1 < \cdots < s_q$. It is easily seen that f is a $*$-identity of A and $\mathrm{Id}^*(A_{j+2}^{sup} \oplus J_{00}) \subseteq \mathrm{Id}^*(A)$. So $A \sim_{T_2^*} A_{j+2}^{sup} \oplus J_{00}$ follows.

Suppose now that $(J_{11})_1 \neq 0$.
Let $B = F + J_{10} + J_{01} + J_{00}$. By Remark 3.4 it follows that B is a subalgebra of A and, since $(J_{11}(B))_1 = 0$, by applying the first part of the lemma to B, we conclude that

$$B \sim_{T_2^*} A_{j+2}^{sup} \oplus J_{00}.$$

Now let $L = F + J_{11}$. By Theorem 3.2.6, either $L \sim_{T_2^*} F$ or $L \sim_{T_2^*} N_r^{sup}$ or $L \sim_{T_2^*} U_r^{sup}$ or $L \sim_{T_2^*} N_r^{sup} \oplus U_r^{sup}$, for some $r \geq 2$. It is proved that $A \sim_{T_2^*} L \oplus B$ and this complete the proof. □

Now we are in a position to classify all the subvarieties of $\mathrm{var}^*(M^{sup})$.

Theorem 3.2.9. *If $A \in var^*(M^{sup})$ then A is T_2^*-equivalent to one of the following $*$-algebras: M^{sup}, N, C, $N_k^{sup} \oplus N$, $U_k^{sup} \oplus N$, $N_k^{sup} \oplus U_k^{sup} \oplus N$, $A_t^{sup} \oplus N$, $N_k^{sup} \oplus A_t^{sup} \oplus N$, $U_k^{sup} \oplus A_t^{sup} \oplus N$, $N_k^{sup} \oplus U_k^{sup} \oplus A_t^{sup} \oplus N$, for some $k, t \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebra with trivial superinvolution.*

Proof. If $A \sim_{T_2^*} M^{sup}$ there is nothing to prove. Now let A generate a proper subvariety of M^{sup}. Since, by Theorem 2.1.2, M^{sup} generates a $*$-variety of almost polynomial growth, $\mathrm{var}^*(A)$ has polynomial growth. Hence by Corollary 2.1 and Theorem 2.3.3 we may assume that $A = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that $\dim B_i/J(B_i) \leq 1$. This means that for every i, either B_i is a nilpotent $*$-algebra or B_i has a decomposition of the type $B_i = F + J = F + J_{11} + J_{10} + J_{01} + J_{00}$. Now, by applying Theorem 3.2.6 and Lemma 3.2, we get the desired conclusion. □

As a consequence of the previous theorem and of Theorems 3.2.4, 3.2.5, 3.2.8 we can also get the classification of all $*$-algebras generating minimal $*$-varieties.

Corollary 3.3. *A $*$-algebra $A \in var^*(M^{sup})$ generates a minimal $*$-variety of polynomial growth if and only if either $A \sim_{T_2^*} U_k^{sup}$ or $A \sim_{T_2^*} N_k^{sup}$ or $A \sim_{T_2^*} A_k^{sup}$, for some $k \geq 2$.*

3.3 Classifying $*$-varieties of at most linear growth

In this section we present a classification, up to T_2^*-equivalence, of the finite dimensional $*$-algebras generating varieties of at most linear growth.

Throughout this section F denotes a field of characteristic zero.

The next theorem can be proved by following word by word the proof of [34, Theorem 5.1].

Theorem 3.3.1. *Let A be a $*$-algebra. Then $c_n^*(A) \leq an^p$, for some constants a and p, if and only if for every $n_1 + \cdots + n_4 = n$ it holds*

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)\\ n-\lambda(1)_1\leq p}} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}.$$

As a consequence, we get the following.

Lemma 3.3. *Let A be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then A satisfies the polynomial identities $x_1x_2 \equiv 0$, with $x_1, x_2 \in X \setminus Y^+$, where $Y^+ = \{y_1^+, y_2^+, \ldots\}$.*

Lemma 3.4. *Let $A = F + J$ be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} (F + J_0) \oplus \left(F + J_1^+\right) \oplus \left(F + J_1^-\right).$$

Proof. Since $c_n^*(A) \leq an$, by Lemma 3.3, A satisfies the polynomial identities $x_1x_2 \equiv 0$. Hence $F + J_0$, $F + J_1^+$ and $F + J_1^-$ are $*$-subalgebras of A and obviously

$$\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*\left((F + J_0) \oplus (F + J_1^+) \oplus (F + J_1^-)\right).$$

Conversely, let $f \in \mathrm{Id}^*\left((F + J_0) \oplus (F + J_1^+) \oplus (F + J_1^-)\right)$ be a multilinear polynomial of degree n. By multihomogeneity of T_2^*-ideals we may assume, modulo $\mathrm{Id}^*(A)$, that either

$$f = \sum_{\sigma\in S_n} \alpha_\sigma y^+_{\sigma(1)}\cdots y^+_{\sigma(n)} \quad \text{or} \quad f = \sum_{\substack{i=1,\ldots,n\\ \sigma\in S_n}} \beta_\sigma y^+_{\sigma(1)}\cdots y^+_{\sigma(i-1)}x_{\sigma(i)}y^+_{\sigma(i+1)}\cdots y^+_{\sigma(n)},$$

where $x_i \in X \setminus Y^+$, $i = 1, \ldots, n$.

If f is of the first type, in order to get a non-zero value, we should evaluate f on $F + J_0$. But $f \in \mathrm{Id}^*(F + J_0)$ by the hypothesis, and so we get that $f \equiv 0$ on A. Similarly, if f is of the second type we get that $f \equiv 0$ on A. Hence $\mathrm{Id}^*\left((F + J_0) \oplus (F + J_1^+) \oplus (F + J_1^-)\right) \subseteq \mathrm{Id}^*(A)$ and we are done. $\square$

Since it is easily checked that $F + J_0 \in \mathrm{var}^*(M)$, $F + J_1^+$, $F + J_1^- \in \mathrm{var}^*(M^{sup})$, we get the following.

Corollary 3.4. *Let $A = F + J$ be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then $A \sim_{T_2^*} B_1$ or $A \sim_{T_2^*} B_2$ or $A \sim_{T_2^*} B_1 \oplus B_2$, where $B_1 \in var^*(M)$ and $B_2 \in var^*(M^{sup})$.*

Now we are ready to present the main result of this section.

Theorem 3.3.2. *Let A be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where $B_i \in var^*(M)$ or $B_i \in var^*(M^{sup})$, for all $i = 1, \ldots, m$ and N is a nilpotent $*$-algebra.

Proof. Since $c_n^*(A) \leq an$, for some constant a, by Theorem 2.3.3, we may assume that

$$A = A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are finite dimensional $*$-algebras with $\dim A_i/J(A_i) \leq 1$, $1 \leq i \leq m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent $*$-algebra. Since $c_n^*(A_i) \leq c_n^*(A)$ then $c_n^*(A_i) \leq an$, for all $i = 1, \ldots, m$. Now the result follows by applying Corollary 3.4 to each non-nilpotent A_i. $\square$

Finally, by putting together Theorem 3.3.2 and Theorems 3.1.6 and 3.2.9, we get a finer classification of the $*$-algebras of at most linear codimension growth.

Theorem 3.3.3. *Let A be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where N is a nilpotent $*$-algebra and for all $i = 1, \ldots, m$, B_i is T_2^*-equivalent to one of the following algebras:

$$N_i,\ C \oplus N_i,\ N_2 \oplus N_i,\ A_2 \oplus N_i,\ N_2 \oplus A_2 \oplus N_i,$$

$$N_2^{sup} \oplus N_i,\ U_2^{sup} \oplus N_i,\ A_2^{sup} \oplus N_i,\ N_2^{sup} \oplus U_2^{sup} \oplus N_i,\ N_2^{sup} \oplus A_2^{sup} \oplus N_i,\ U_2^{sup} \oplus A_2^{sup} \oplus N_i,\ N_2^{sup} \oplus U_2^{sup} \oplus A_2^{sup} \oplus N_i,$$

where C is a commutative algebra with trivial superinvolution and N_i is a nilpotent $*$-algebra.

Chapter 4

Standard identities on matrices with superinvolution

In this chapter we focus our attention on particular identities on the matrix algebras with superinvolution we have introduced in Section 1.7. We shall give results inspired from the celebrated theorem of Amitsur and Levitski. Recall that the standard polynomial of degree r in the non-commutative variables $x_1, \ldots, x_r$ is defined as

$$St_r(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} (\mathrm{sgn}\sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}.$$

Theorem 4.0.4 (Amitsur-Levitzki, 1950). *For each $n \geq 1$, the standard polynomial St_{2n} is an identity for $M_n(F)$, the algebra of $n \times n$ matrices over the field F. Moreover, if char$F \neq 2$, then St_{2n} is an identity of minimal degree for $M_n(F)$.*

The general question whether the Amitsur-Levitski theorem could be improved by considering only certain kinds of matrices was positively solved by Kostant and Rowen, which proved some powerful results in the setting of matrix algebras with involution. The following theorem clarify the situation in this area (for a proof of it, see for instance, [53, Theorem 3.1.62]).

Theorem 4.0.5. *For any infinite field F and any involution $\sharp$ on $M_n(F)$, either Id$(M_n(F), \sharp) =$ Id$(M_n(F), t)$ or Id$(M_n(F), \sharp) =$ Id$(M_n(F), s)$, where t and s denote the transpose and the symplectic involution and $(M_n(F), \ddagger)$ denotes the matrix algebra $M_n(F)$ endowed with the involution $\ddagger$.*

In practice, up to isomorphism, we have only the transpose and the symplectic involution on $M_n(F)$. Recall that s is defined only when $n = 2l$ is even by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where the original matrix is partitioned into $l \times l$ blocks.

Theorem 4.0.6 (Kostant, 1958). *Let $(M_n(F), t)$ be the algebra of $n \times n$ matrices endowed with the transpose involution. If n is even, then $St_{2n-2}(x_1, \ldots, x_{2n-2})$ is a standard identity in skew variables.*

Theorem 4.0.7 (Rowen, 1974). *Let $(M_n(F), t)$ be the algebra of $n \times n$ matrices endowed with the transpose involution. Then $St_{2n-2}(x_1, \ldots, x_{2n-2})$ is a standard identity of minimal degree in skew variables.*

In order to complete the case of the transpose involution, we present the following well known result.

Theorem 4.0.8. *Let $(M_n(F), t)$ be the algebra of $n \times n$ matrices endowed with the transpose involution. Then $St_{2n}(x_1, \ldots, x_{2n})$ is a standard identity of minimal degree in symmetric variables.*

Another famous result of Rowen concerns the symplectic involution.

Theorem 4.0.9 (Rowen, 1982). *Let $(M_{2l}(F), s)$ be the algebra of $2l \times 2l$ matrices endowed with the symplectic involution. Then $St_{4l-2}(x_1, \ldots, x_{4l-2})$ is a standard identity in symmetric variables.*

We remark that in this case we have no results concerning the minimal degree. In fact, although Rowen proved that $4l-2$ is the minimal degree of a standard polynomial in symmetric matrices with respect to s when $l = 2$ and Adamsson, in [1], extended this result for $l = 3, 4$, the proof of the general case has not been found yet (see [29] for more details). Then $4l-2$ appears to be the minimal degree of a standard identity on $2l \times 2l$ symmetric matrices with respect to the symplectic involution s but we have no proof of that.

Let now focus our attention to skew-symmetric matrices with respect to the symplectic involution s. The following lemma holds.

Lemma 4.1. *Let $(M_{2l}(F), s)$ be the algebra of $2l \times 2l$ matrices endowed with the symplectic involution. Then the polynomial $St_{4l}(x_1, \ldots, x_{4l})$ is a standard identity of minimal degree in skew variables.*

Proof. Let $B_1, \ldots, B_r$, be $2l \times 2l$ skew-symmetric matrices with respect to s. By Theorem 4.0.4, it is obvious that $St_r(B_1, \ldots, B_r) = 0$, if $r \geq 4l$. In order to complete the proof, let us consider the set of skew-symmetric matrices $\alpha_i = e_{i,i} - e_{l+i,l+i}$, $\beta_j = e_{j,j+l}$, $\gamma_m = e_{m+l,m}$, $1 \leq i, j, m \leq l$ and $\delta_n = e_{n,n+1} - e_{n+1+l,n+l}$, $1 \leq n \leq l-1$. We claim that

$$e_{2l,2l} St_{4l-1}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l, \gamma_1, \ldots, \gamma_l, \delta_1, \ldots, \delta_{l-1}) e_{l+1,l+1} = 3^l \; e_{2l,l+1}.$$

Now, if g is a monomial of St_{4l-1} whose evaluation is a multiple of $e_{2l,l+1}$, then α_l, γ_l and δ_{l-1} are the only elements that can lie in the first position of g. Moreover, α_l, β_l, γ_l and δ_{l-1} are the only ones where the index $2l$ or l appears, so if the first element of g is δ_{l-1}, we will obtain 0, since we can no longer insert the matrices $\alpha_l, \beta_l, \gamma_l$. Thus the first element of g must be either α_l or γ_l and there are only 3 non-zero possibilities:

(1) $\alpha_l \ \gamma_l \ \beta_l \ \delta_{l-1} \ g' = e_{2l,2l-1} \ g'$,

(2) $\gamma_l \ \beta_l \ \alpha_l \ \delta_{l-1} \ g' = e_{2l,2l-1} \ g'$,

(3) $\gamma_l \ \alpha_l \ \beta_l \ \delta_{l-1} \ g' = -e_{2l,2l-1} \ g'$,

where g' is a monomial in the remaining matrices. We notice that (1) (resp. (2)) and (3) appear with opposite sign in the standard polynomial.

Now we have to insert in g the matrices $\alpha_{l-1}, \beta_{l-1}, \gamma_{l-1}$ and δ_{l-2}, that are the only ones where the index $2l-1$ or $l-1$ appears. We have only 3 non-zero possibilities

(1) $g_1' = \alpha_{l-1} \ \gamma_{l-1} \ \beta_{l-1} \ \delta_{l-2} \ g'' = e_{2l-1,2l-2} \ g''$,

(2) $g_2' = \gamma_{l-1} \ \beta_{l-1} \ \alpha_{l-1} \ \delta_{l-2} \ g'' = e_{2l-1,2l-2} \ g''$,

(3) $g_3' = \gamma_{l-1} \ \alpha_{l-1} \ \beta_{l-1} \ \delta_{l-2} \ g'' = -e_{2l-1,2l-2} \ g''$.

As before (1) (resp. (2)) and (3) appear with opposite sign in the standard polynomial. By iterating the process, the result follows. □

We now are ready to treat the so-called standard $*$-identities.

Definition 4.1. *Let $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ be a $*$-algebra. We say that A satisfies a standard $*$-identity if*

$$St_r\left(u_1^+, \ldots, u_n^+, u_1^-, \ldots, u_m^-, v_1^+, \ldots, v_t^+, v_1^-, \ldots, v_s^-\right) = 0,$$

for all $u_1^+, \ldots, u_n^+ \in A_0^+$, $u_1^-, \ldots, u_m^- \in A_0^-$, $v_1^+, \ldots, v_t^+ \in A_1^+$ and $v_1^-, \ldots, v_s^- \in A_1^-$.

In this chapter we focus our attention on standard $*$-identities on the matrix algebras $M_n(F)$ endowed with a superinvolution. As we have seen in Section 1.7, on the matrix algebra $M_n(F)$ it is possible to define the following $\mathbb{Z}_2$-grading. Let be $n = k+h$, then $A = M_n(F)$ becomes a superalgebra $A = A_0 \oplus A_1$, where

$$A_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F), \ T \in M_h(F) \right\},$$

$$A_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F), \ Z \in M_{h\times k}(F) \right\}.$$

Such superalgebra is denoted by $M_{k,h}(F)$ and, if F is an algebraically closed field of characteristic different from 2, any non trivial $\mathbb{Z}_2$-grading on $M_n(F)$ is isomorphic to $M_{k,h}(F)$, for some k and h.

Moreover, in [47], Racine showed that, up to isomorphism, it is possible to define on $M_{k,h}(F)$ only the orthosymplectic and the transpose superinvolution.

In this chapter we find the minimal degree for which the standard polynomial vanishes when evaluated in homogeneous symmetric or skew-symmetric matrices of $M_{k,h}(F)$ in the case of both transpose or orthosymplectic superinvolution, respectively. In the last section of the chapter we make a systematic study of the identities of the algebra $M_{1,1}(F)$ endowed with the transpose superinvolution. We compute a generating set of the T_2^*-ideal of identities and we find the decomposition of the corresponding character into irreducibles.

From now on, unless otherwise stated, F denotes an algebraically closed field of characteristic zero.

4.1 Standard identities on $(M_{k,k}(F), trp)$

In this section we consider $2k \times 2k$ matrices endowed with the transpose superinvolution *trp*. Recall that *trp* is defined by the formula

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix},$$

where t is the usual transpose. The four sets of homogeneous symmetric and skew elements are the following:

$$(M_{k,k}(F),\ \mathrm{trp})_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \ \mid\ X \in M_k(F) \right\},$$

$$(M_{k,k}(F),\ \mathrm{trp})_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \ \mid\ X \in M_k(F) \right\},$$

$$(M_{k,k}(F),\ \mathrm{trp})_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \ \mid\ Y = -Y^t,\ Z = Z^t,\ Y, Z \in M_k(F) \right\},$$

$$(M_{k,k}(F),\ \mathrm{trp})_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \ \mid\ Y = Y^t,\ Z = -Z^t,\ Y, Z \in M_k(F) \right\}.$$

The main result of this section states that the polynomial $St_{2k}(x_1, \dots, x_{2k})$ is a standard $*$-identity of minimal degree of $(M_{k,k}(F), trp)$, where $x_1, \dots, x_{2k}$ are all symmetric or skew variables of the same homogeneous degree (i.e. the $x_i s$ are all $y_i^+ s$ or $y_i^- s$ or $z_i^+ s$ or $z_i^- s$). We shall prove the theorem in several steps. For a start, the following remark holds.

Remark 4.1. *Let $r \geq 1$ and let t be the usual transpose. If $B_1, \ldots, B_r \in M_k(F)$ then*

$$St_r\left(B_1^t, \ldots, B_r^t\right) = (-1)^{f(r)}\left(St_r\left(B_1, \ldots, B_r\right)\right)^t,$$

where

$$f(r) = \begin{cases} 0 & \text{if } r \equiv 0, 1\ (mod\ 4) \\ 1 & \text{if } r \equiv 2, 3\ (mod\ 4) \end{cases}.$$

In the next proposition we find the minimal degree of a standard $*$-identity in the y_i^+s and y_i^-s, respectively.

Proposition 4.1. *The polynomials $St_{2k}(y_1^+, \ldots, y_{2k}^+)$ and $St_{2k}(y_1^-, \ldots, y_{2k}^-)$ are standard $*$-identities of $(M_{k,k}(F), trp)$ of minimal degree in symmetric and skew variables of homogeneous degree zero, respectively.*

Proof. Let $U_1^+, \ldots, U_r^+ \in (M_{k,k}(F), trp)_0^+$, where $U_i^+ = \begin{pmatrix} X_i & 0 \\ 0 & X_i^t \end{pmatrix}$ and $X_i \in M_k(F)$, for all $i = 1, \ldots, r$. By Remark 4.1 we get

$$\begin{aligned} St_r\left(U_1^+, \ldots, U_r^+\right) &= St_r\left(\begin{pmatrix} X_1 & 0 \\ 0 & X_1^t \end{pmatrix}, \ldots, \begin{pmatrix} X_r & 0 \\ 0 & X_r^t \end{pmatrix}\right) \\ &= \begin{pmatrix} St_r\left(X_1, \ldots, X_r\right) & 0 \\ 0 & St_r\left(X_1^t, \ldots, X_r^t\right) \end{pmatrix} \\ &= \begin{pmatrix} St_r\left(X_1, \ldots, X_r\right) & 0 \\ 0 & (-1)^{f(r)}\left(St_r\left(X_1, \ldots, X_r\right)\right)^t \end{pmatrix}. \end{aligned}$$

Similarly, for all $U_1^-, \ldots, U_r^- \in (M_{k,k}(F), trp)_0^-$, where $U_i^- = \begin{pmatrix} T_i & 0 \\ 0 & -T_i^t \end{pmatrix}$ and $T_i \in M_k(F)$, for all $i = 1, \ldots, r$, we get

$$\begin{aligned} St_r\left(U_1^-, \ldots, U_r^-\right) &= St_r\left(\begin{pmatrix} T_1 & 0 \\ 0 & -T_1^t \end{pmatrix}, \ldots, \begin{pmatrix} T_r & 0 \\ 0 & -T_r^t \end{pmatrix}\right) \\ &= \begin{pmatrix} St_r(T_1, \ldots, T_r) & 0 \\ 0 & (-1)^r St_r(T_1^t, \ldots, T_r^t) \end{pmatrix} \\ &= \begin{pmatrix} St_r(T_1, \ldots, T_r) & 0 \\ 0 & (-1)^{f(r)+r}\left(St_r(T_1, \ldots, T_r)\right)^t \end{pmatrix}. \end{aligned}$$

Thus, $St_r(y_1^+, \ldots, y_r^+)$ and $St_r(y_1^-, \ldots, y_r^-)$ are polynomial identities if and only if $St_r(X_1, \ldots, X_r) = 0$ and $St_r(T_1, \ldots, T_r) = 0$, where the X_is and the T_js are $k \times k$ matrices. By Theorem 4.0.4, we obtain that in both cases, $St_r(y_1^+, \ldots, y_r^+)$ and $St_r(y_1^-, \ldots, y_r^-)$ are polynomial identities if and only if $r \geq 2k$. □

In order to simplify the notation, for $i \geq 1$, let S_i, K_i be $k \times k$ symmetric and skew-symmetric matrices with respect to the transpose involution, respectively. Our next goal is to find the minimal degree of a standard $*$-identity in the z_i^+s and z_i^-s. The following lemma shows that the two cases are strictly related.

Lemma 4.2. *In $(M_{k,k}, trp)$, $St_r\left(z_1^+, \ldots, z_r^+\right) \equiv 0$ if and only if $St_r\left(z_1^-, \ldots, z_r^-\right) \equiv 0$, for all $r \geq 1$.*

Proof. Let $V_1^+, \ldots, V_r^+ \in (M_{k,k}(F), trp)_1^+$ where $V_i^+ = \begin{pmatrix} 0 & K_i \\ S_i & 0 \end{pmatrix}$, for all $i = 1, \ldots, r$. If we set

$$C = \sum_{\sigma \in S_r} (\text{sgn}\sigma) K_{\sigma(1)} S_{\sigma(2)} \cdots K_{\sigma(r-1)} S_{\sigma(r)}, \quad \text{if } r \text{ is even},$$

$$D = \sum_{\sigma \in S_r} (\text{sgn}\sigma) S_{\sigma(1)} K_{\sigma(2)} \cdots S_{\sigma(r-1)} K_{\sigma(r)}, \quad \text{if } r \text{ is even},$$

$$C' = \sum_{\sigma \in S_r} (\text{sgn}\sigma) K_{\sigma(1)} S_{\sigma(2)} \cdots K_{\sigma(r-2)} S_{\sigma(r-1)} K_{\sigma(r)}, \quad \text{if } r \text{ is odd},$$

$$D' = \sum_{\sigma \in S_r} (\text{sgn}\sigma) S_{\sigma(1)} K_{\sigma(2)} \cdots S_{\sigma(r-2)} K_{\sigma(r-1)} S_{\sigma(r)}, \quad \text{if } r \text{ is odd},$$

a simple computation shows that $St_r(V_1^+, \ldots, V_r^+)$ is equal to $\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ if r is even and $\begin{pmatrix} 0 & C' \\ D' & 0 \end{pmatrix}$ if r is odd.

Similarly, let $V_1^-, \ldots, V_r^- \in (M_{k,k}(F), trp)_1^-$ where $V_i^- = \begin{pmatrix} 0 & S_i \\ K_i & 0 \end{pmatrix}$, for all $i = 1, \ldots, r$. Then $St_r(V_1^-, \ldots, V_r^-)$ is equal to $\begin{pmatrix} D & 0 \\ 0 & C \end{pmatrix}$ if r is even and $\begin{pmatrix} 0 & D' \\ C' & 0 \end{pmatrix}$ if r is odd.

Therefore, it is clear that $St_r(z_1^+, \ldots, z_r^+) \equiv 0$ if and only if $C = D = 0$ if r is even, (resp. $C' = D' = 0$ if r is odd), if and only if $St_r(z_1^-, \ldots, z_r^-) \equiv 0$. □

The following remark holds.

Remark 4.2. *For $r \geq 2$ even, we have that $C^t = D$.*

Proof. $C^t = \left(\sum_{\sigma \in S_r} (\text{sgn}\sigma) K_{\sigma(1)} S_{\sigma(2)} \cdots K_{\sigma(r-1)} S_{\sigma(r)} \right)^t = \sum_{\sigma \in S_r} (\text{sgn}\sigma) S_{\sigma(r)}^t K_{\sigma(r-1)}^t \cdots S_{\sigma(2)}^t K_{\sigma(1)}^t =$
$= \sum_{\sigma \in S_r} (\text{sgn}\sigma) S_{\sigma(r)} (-K_{\sigma(r-1)}) \cdots S_{\sigma(2)} (-K_{\sigma(1)}) = \sum_{\sigma \in S_r} (\text{sgn}\sigma) S_{\sigma(1)} K_{\sigma(2)} \cdots S_{\sigma(r-1)} K_{\sigma(r)} =$
$= D$ □

In the proof of the next proposition we shall use Rosset's approach to the proof of the Amitsur-Levitzki theorem (see [51]). Recall that the infinite dimensional Grassmann algebra G over F is the algebra generated by a countable set of elements $\{e_1, e_2, \ldots\}$ satisfying the relations $e_ie_j = -e_je_i$, for all i, j. It is well known that G has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where G_0 is the span of all monomials in the e_is of even length and G_1 is the span of all monomials in the e_is of odd length.

We recall the following fact (see for instance [26, Lemma 1.7.4]).

Lemma 4.3. *Let $\mathcal{A}$ be a commutative algebra over $\mathbb{Q}$. If $b \in M_k(\mathcal{A})$ is such that $tr(b) = tr(b^2) = \cdots = tr(b^k) = 0$ then $b^k = 0$.*

Proposition 4.2. *The polynomials $St_{2k}(z_1^+, \ldots, z_{2k}^+)$ and $St_{2k}(z_1^-, \ldots, z_{2k}^-)$ are standard $*$-identities of the $*$-algebra $(M_{k,k}(F), trp)$ of minimal degree in symmetric and skew variables of degree one, respectively.*

Proof. By Lemma 4.2 we need to deal only with $St_r(z_1^+, \ldots, z_r^+)$.

We start by proving that $St_r(z_1^+, \ldots, z_r^+)$ is a polynomial identity if $r \geq 2k$. Since it is well-known that

$$St_{r+1}(x_1, \ldots, x_{r+1}) = \sum_{i=1}^{r+1}(-1)^{i+1}x_iSt_r(x_1, \ldots, \widehat{x}_i, \ldots, x_{r+1}),$$

where $\widehat{x}_i$ means that the variable x_i is omitted (see for instance [26, Proposition 1.5.7]), it suffices to work with standard polynomials of degree $2k$.

For $1 \leq i \leq 2k$, let $V_i^+ = \begin{pmatrix} 0 & K_i \\ S_i & 0 \end{pmatrix} \in (M_{k,k}(F), trp)_1^+$, $e_i \in G$, the infinite dimensional Grassmann algebra over F and define

$$a = \sum_{i=1}^{2k} V_i^+ e_i \in M_{2k}(G).$$

Recalling that, for all $\sigma \in S_{2k}$, $e_{\sigma(1)} \cdots e_{\sigma(2k)} = (\text{sgn}\sigma)e_1 \cdots e_{2k}$, we get that

$$a^{2k} = St_{2k}\left(V_1^+, \ldots, V_{2k}^+\right) e_1 \cdots e_{2k}.$$

Hence, in order to prove that $St_{2k}\left(z_1^+, \ldots, z_{2k}^+\right) \equiv 0$ it is sufficient to show that $a^{2k} = 0$. For all $1 \leq i \leq k$,

$$a^{2i} = \sum St_{2i}(V_{j_1}^+, \ldots, V_{j_{2i}}^+)e_{j_1} \cdots e_{j_{2i}} \in M_{2k}(G_0),$$

where the sum runs on $j_1, \ldots, j_{2i} \in \{1, \ldots, 2k\}$ such that $j_1 < \cdots < j_{2i}$. Since the trace of a standard polynomial of even degree evaluated in any set of matrices is zero (see for instance [26, Corollary 1.7.6]), we get that, for all $1 \leq i \leq k$,

$$tr(a^{2i}) = tr\left(\sum_{j_1 < \cdots < j_{2i}} St_{2i}(V_{j_1}^+, \ldots, V_{j_{2i}}^+)e_{j_1} \cdots e_{j_{2i}}\right) = \sum_{j_1 < \cdots < j_{2i}} tr\left(St_{2i}(V_{j_1}^+, \ldots, V_{j_{2i}}^+)\right) e_{j_1} \cdots e_{j_{2i}} = 0.$$

Now, by Remark 4.2, it is clear that $a^{2i} = \begin{pmatrix} C_i & 0 \\ 0 & C_i^t \end{pmatrix}$, where

$$C_i = \sum_{j_1 < \cdots < j_{2i}} \sum_{\sigma \in S_{2i}} (\mathrm{sgn}\sigma) K_{\sigma(j_1)} S_{\sigma(j_2)} \cdots K_{\sigma(j_{2i-1})} S_{\sigma(j_{2i})}\ e_{j_1} \cdots e_{j_{2i}} \in M_k(G_0).$$

Hence $0 = tr(a^{2i}) = tr\left(\begin{pmatrix} C_i & 0 \\ 0 & C_i^t \end{pmatrix}\right) = 2tr(C_i)$ and, so, $tr(C_i) = 0$, for all $1 \leq i \leq k$.
Since G_0 is commutative, Lemma 4.3 applies and we get that $C_k = 0$. But then $a^{2k} = 0$ and we are done.

In order to complete the proof we have to show that $St_{2k-1}(z_1^+, \ldots, z_{2k-1}^+)$ does not vanish in $(M_{k,k}(F), trp)$.
If $k = 1$ then $2k - 1 = 1$ and clearly $z^+ \not\equiv 0$.
Let now $k \geq 2$. For $V_1^+, \ldots, V_{2k-1}^+ \in (M_{k,k}(F), trp)_1^+$ we have

$$St_{2k-1}(V_1^+, \ldots, V_{2k-1}^+) = St_{2k-1}\left(\begin{pmatrix} 0 & K_1 \\ S_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & K_{2k-1} \\ S_{2k-1} & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & C' \\ D' & 0 \end{pmatrix},$$

where

$$C' = \sum_{\sigma \in S_{2k-1}} (\mathrm{sgn}\sigma) K_{\sigma(1)} S_{\sigma(2)} \cdots K_{\sigma(2k-3)} S_{\sigma(2k-2)} K_{\sigma(2k-1)},$$

$$D' = \sum_{\sigma \in S_{2k-1}} (\mathrm{sgn}\sigma) S_{\sigma(1)} K_{\sigma(2)} \cdots S_{\sigma(2k-3)} K_{\sigma(2k-2)} S_{\sigma(2k-1)}.$$

We now specialize the matrices $S_1, \ldots, S_{2k-1}$ and $K_1, \ldots, K_{2k-1}$ as follows: $S_1 = e_{11}$, $S_2 = 0$, $S_3 = e_{22}$, $S_4 = 0, \ldots,$ $S_{2k-3} = e_{k-1k-1}$, $S_{2k-2} = 0$, $S_{2k-1} = e_{kk}$, $K_1 = 0$, $K_2 = e_{12} - e_{21}$, $K_3 = 0$, $K_4 = e_{23} - e_{32}, \ldots,$ $K_{2k-3} = 0$, $K_{2k-2} = e_{k-1k} - e_{kk-1}$. It turns out that

$$D' = \sum_{\sigma \in S_{2k-1}} (\mathrm{sgn}\sigma) S_{\sigma(1)} K_{\sigma(2)} \cdots S_{\sigma(2k-3)} K_{\sigma(2k-2)} S_{\sigma(2k-1)} = e_{1k} + (-1)^k e_{k1} \neq 0$$

and the proof is complete. □

By putting together these results we get the main theorem of this section.

Theorem 4.1.1. *The polynomial $St_{2k}(x_1, \ldots, x_{2k})$ is a standard $*$-identity of minimal degree of $(M_{k,k}(F), trp)$ where $x_1, \ldots, x_{2k}$ are all symmetric or skew variables of the same homogeneous degree.*

4.2 Standard identities on $(M_{k,2l}(F), osp)$

Let us now consider the orthosymplectic superinvolution on $M_{k,2l}(F)$. We will see that the situation in this case is much more complicated than in the previous section. First, recall that the orthosymplectic superinvolution, denoted *osp*, is defined by

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} X^t & Z^tQ \\ QY^t & -QT^tQ \end{pmatrix},$$

where $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and I_r, I_l are the $r \times r$, $l \times l$ identity matrices, respectively. Thus, we have

$$(M_{k,2l}(F),\ \text{osp})_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = X^t,\ T = -QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},$$

$$(M_{k,2l}(F),\ \text{osp})_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = -X^t,\ T = QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},$$

$$(M_{k,2l}(F),\ \text{osp})_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Z = QY^t,\ Y \in M_{k\times 2l}(F) \right\},$$

$$(M_{k,2l}(F),\ \text{osp})_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Z = -QY^t,\ Y \in M_{k\times 2l}(F) \right\}.$$

Now we can prove the following.

Proposition 4.3. *The polynomial $St_r(y_1^-, \ldots, y_r^-)$ is a standard $*$-identity of the $*$-algebra $(M_{k,2l}(F), osp)$ if and only if $r \geq \max\{2k-2,\ 4l\}$.*

Proof. Let $U_1^-, \ldots, U_r^- \in (M_{k,2l}(F), osp)_0^-$, where $U_i^- = \begin{pmatrix} X_i & 0 \\ 0 & T_i \end{pmatrix}$, $X_i = -X_i^t$ and $T_i = QT_i^tQ$, for all $i = 1, \ldots, r$. An easy computation shows that

$$St_r(U_1^-, \ldots, U_r^-) = St_r\left(\begin{pmatrix} X_1 & 0 \\ 0 & T_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_r & 0 \\ 0 & T_r \end{pmatrix} \right) = \begin{pmatrix} St_r(X_1, \ldots, X_r) & 0 \\ 0 & St_r(T_1, \ldots, T_r) \end{pmatrix}.$$

Since the X_is are $k \times k$ skew-symmetric matrices under the transpose involution, by Theorem 4.0.7, we get that $St_r(X_1, \ldots, X_r) = 0$ if and only if $r \geq 2k-2$.

Moreover the T_is are $2l \times 2l$ matrices of the kind $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ where $A, B, C \in M_l(F)$, $B = B^t$ and $C = C^t$. Hence they are skew-symmetric matrices with respect to the symplectic involution and by Lemma 4.1, $St_r(T_1, \ldots, T_r) = 0$ if and only if $r \geq 4l$.

It clearly follows that $St_r(y_1^-, \ldots, y_r^-) \equiv 0$ if and only if $r \geq \max\{2k-2,\ 4l\}$ and the proof is complete. □

In a similar way it is possible to prove the following lemma.

Lemma 4.4. *The polynomial* $St_r(y_1^+, \dots, y_r^+)$ *is a* $*$*-identity of* $(M_{k,2l}(F), osp)$ *if* $r \geq \max\{2k,\ 4l-2\}$.

Proof. Let $U_1^+, \dots, U_r^+ \in (M_{k,2l}(F), osp)_0^+$, where $U_i^+ = \begin{pmatrix} X_i & 0 \\ 0 & T_i \end{pmatrix}$, $X_i = X_i^t$ and $T_i = -QT_i^tQ$, for all $i = 1, \dots, r$. As in the previous proposition,

$$St_r(U_1^+, \dots, U_r^+) = St_r\left(\begin{pmatrix} X_1 & 0 \\ 0 & T_1 \end{pmatrix}, \dots, \begin{pmatrix} X_r & 0 \\ 0 & T_r \end{pmatrix}\right) = \begin{pmatrix} St_r(X_1, \dots, X_r) & 0 \\ 0 & St_r(T_1, \dots, T_r) \end{pmatrix}.$$

Since the X_is are $k \times k$ symmetric matrices under the transpose involution, by Theorem 4.0.8,

$$St_r(X_1, \dots, X_r) = 0 \text{ if and only if } r \geq 2k.$$

Moreover, from $T_i = -QT_i^tQ$, we get that the T_is are matrices of the type $\begin{pmatrix} A & B \\ C & A^t \end{pmatrix}$, where $B = -B^t$ and $C = -C^t$. Hence they are symmetric with respect to the symplectic involution. By Theorem 4.0.9, if $r \geq 4l - 2$, then $St_r(T_1, \dots, T_r) = 0$. Thus it is clear that $St_r(y_1^+, \dots, y_r^+) \equiv 0$ if $r \geq \max\{2k,\ 4l-2\}$. □

Since, as we have remarked before, $4l - 2$ appears to be the minimal degree of a standard identity on $2l \times 2l$ symmetric matrices with respect to the symplectic involution s but we have no proof of that, we can only state the following conjecture.

Conjecture 4.1. *If* $St_r(y_1^+, \dots, y_r^+)$ *is a* $*$*-identity of the algebra* $(M_{k,2l}(F), osp)$ *then* $r \geq \max\{2k,\ 4l-2\}$.

Next we analyse identities in homogeneous degree one variables of the $*$-algebra $(M_{k,2l}(F), osp)$. The following lemma shows that standard $*$-identities in z^+s and z^-s, respectively, are strictly related.

Lemma 4.5. *In* $(M_{k,2l}(F), osp)$, $St_r\left(z_1^+, \dots, z_r^+\right) \equiv 0$ *if and only if* $St_r\left(z_1^-, \dots, z_r^-\right) \equiv 0$, *for all* $r \geq 1$.

Proof. For $1 \leq i \leq r$, let $V_i^+ = \begin{pmatrix} 0 & Y_i \\ QY_i^t & 0 \end{pmatrix} \in (M_{k,2l}(F), osp)_1^+$, where $Y_i \in M_{k \times 2l}(F)$. If we set

$$C = \sum_{\sigma \in S_r} (\mathrm{sgn}\sigma) Y_{\sigma(1)} QY_{\sigma(2)}^t \cdots Y_{\sigma(r-1)} QY_{\sigma(r)}^t, \quad \text{for } r \text{ even},$$

$$D = \sum_{\sigma \in S_r} (\mathrm{sgn}\sigma) QY_{\sigma(1)}^t Y_{\sigma(2)} \cdots QY_{\sigma(r-1)}^t Y_{\sigma(r)}, \quad \text{for } r \text{ even},$$

$$C' = \sum_{\sigma \in S_r} (\mathrm{sgn}\sigma) Y_{\sigma(1)} QY_{\sigma(2)}^t \cdots Y_{\sigma(r-2)} QY_{\sigma(r-1)}^t Y_{\sigma(r)}, \quad \text{for } r \text{ odd},$$

$$D' = \sum_{\sigma \in S_r} (\mathrm{sgn}\sigma) QY_{\sigma(1)}^t Y_{\sigma(2)} \cdots QY_{\sigma(r-2)}^t Y_{\sigma(r-1)} QY_{\sigma(r)}^t, \quad \text{for } r \text{ odd},$$

then an easy computation shows that $St_r(V_1^+, \dots, V_r^+)$ equals $\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ if r is even and $\begin{pmatrix} 0 & C' \\ D' & 0 \end{pmatrix}$ if r is odd.

Similarly, for $V_1^-, \dots, V_r^- \in (M_{k,2l}(F), osp)_1^-$, where $V_i^- = \begin{pmatrix} 0 & Y_i \\ -QY_i^t & 0 \end{pmatrix}$, $Y_i \in M_{k\times 2l}(F)$, $1 \le i \le r$, we get that $St_r(V_1^-, \dots, V_r^-)$ equals $\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ if $r \equiv 0 \pmod 4$ or $\begin{pmatrix} 0 & C' \\ -D' & 0 \end{pmatrix}$ if $r \equiv 1 \pmod 4$ or $\begin{pmatrix} -C & 0 \\ 0 & -D \end{pmatrix}$ if $r \equiv 2 \pmod 4$ or $\begin{pmatrix} 0 & -C' \\ D' & 0 \end{pmatrix}$ if $r \equiv 3 \pmod 4$.

Then $St_r(z_1^+, \dots, z_r^+) \equiv 0$ if and only if $C = D = 0$ if r is even, (resp. $C' = D' = 0$ if r is odd), if and only if $St_r(z_1^-, \dots, z_r^-) \equiv 0$. □

Proposition 4.4. *The polynomials $St_r(z_1^+, \dots, z_r^+)$ and $St_r(z_1^-, \dots, z_r^-)$ are $*$-identities of the algebra $(M_{k,2l}(F), osp)$ if $r \ge \min\{2kl+1,\ 2k+4l\}$.*

Proof. Due to Lemma 4.5, in order to prove the statement it suffices only to study $St_r(z_1^+, \dots, z_r^+)$.

Let $V_1^+, \dots, V_r^+ \in (M_{k,2l}(F), osp)_1^+$ as before. By Theorem 4.0.4,

$$St_r(V_1^+, \dots, V_r^+) = 0 \quad \text{for all } r \ge 2(k+2l) = 2k+4l.$$

Moreover, since $\dim_F (M_{k,2l}(F),\ osp)_1^+ = 2kl$ and the standard polynomial is alternating on all of its variables, by Proposition 1.1, we get that

$$St_r(V_1^+, \dots, V_r^+) = 0 \quad \text{for all } r \ge 2kl+1.$$

It follows that $St_r(z_1^+, \dots, z_r^+) \equiv 0$ if $r \ge \min\{2kl+1,\ 2k+4l\}$ and the proof is complete. □

Next we are searching for the minimal degree of a standard $*$-identity $St_r(z_1^+, \dots, z_r^+) \equiv 0$. In this case we have only partial results. First we observe that $2kl+1 \le 2k+4l$ if and only if $l = 1$ and $k \ge 1$ or $k = 1$ and $l \ge 1$ or $k = 2$ and $l \ge 1$ or $k = 3$ and $l = 2$.

We fix the following basis for $(M_{k,2l}(F), osp)_1^+$

$$\Big\{\ e_{i,j} - e_{j+l,i},\ e_{p,q} + e_{q-l,p} \mid\ i,p = 1,\dots,k,\ j = k+1,\dots,k+l,\ q = k+l+1,\dots,k+2l\Big\}.$$

The following lemma holds.

Lemma 4.6. *Let $l = 1$, $k \ge 1$ and let $f_1, \dots, f_{2k}$ be the following elements of $(M_{k,2}(F), osp)_1^+$*

$$f_1 = e_{1,k+1} - e_{k+2,1}, \dots,\ f_k = e_{k,k+1} - e_{k+2,k},\ f_{k+1} = e_{1,k+2} + e_{k+1,1}, \dots,\ f_{2k} = e_{k,k+2} + e_{k+1,k}.$$

Then $e_{k+2,k+2} St_{2k}(f_1, \dots, f_{2k}) e_{k+2,k+2} = (-1)^k\ k!\ e_{k+2,k+2}$.

Proof. In order to obtain $e_{k+2,k+2}$ we have to start with one f_i, $i = 1,\ldots,k$ and end with one f_j, $j = k+1,\ldots,2k$. Let $m = f_1 f_{k+1} f_2 f_{k+2} \cdots f_k f_{2k}$ be the first combination for which we obtain $e_{k+2,k+2}$. It is easy to see that $f_1 f_{k+1} \cdots f_k f_{2k} = (-1)^k \; e_{k+2,k+2}$. Moreover, any other combination that gives $e_{k+2,k+2}$ will be of the type

$$m_\sigma = f_{\sigma(1)} f_{\sigma(1)+k} f_{\sigma(2)} f_{\sigma(2)+k} \cdots f_{\sigma(k)} f_{\sigma(k)+k}$$

with $\sigma \in S_k$, since in m we can permute only groups of $f_i f_{k+i}$ among each other. We remark that each m_σ has the same sign and so the proof is complete. □

With a similar argument we also get the following lemma.

Lemma 4.7. *Let $k = 1$, $l \geq 1$ and $f_1,\ldots,f_{2l}$ be the following elements of $(M_{1,2l}(F), osp)_1^+$*

$$f_1 = e_{1,2} - e_{l+2,1},\ldots,\; f_l = e_{1,l+1} - e_{2l+1,1},\; f_{l+1} = e_{1,l+2} + e_{2,1},\ldots,\; f_{2l} = e_{1,2l+1} + e_{l+1,1}.$$

Then $e_{l+2,l+2} St_{2l}(f_1,\ldots,f_{2l}) e_{l+2,l+2} = -2^{l-1} \; (l-1)! \; e_{l+2,l+2}$.

Lemmas 4.6 and 4.7 show that the minimal degree of the standard $*$-identity in case of symmetric or skew variables of odd degree is $2kl + 1$ if $l = 1$ and $k \geq 1$ or $k = 1$ and $l \geq 1$.

When $k = 3$ and $l = 2$ a straightforward computation shows that

$$St_r(z_1^+,\ldots,z_r^+) \equiv 0 \;\text{ if and only if }\; r \geq 11.$$

If $k = 2$ and $l \geq 1$ and when $2kl + 1 \geq 2k + 4l$ some computational difficulties arise. Then in general, we have no information about the minimal degree of standard $*$-identities in odd variables.

Finally, in the next theorem we resume the results of this section.

Theorem 4.2.1. *In the $*$-algebra $(M_{k,2l}(F), osp)$ hold*

1. $St_r(y_1^+,\ldots,y_r^+) \equiv 0$ *if* $r \geq \max\{2k,\; 4l-2\}$,

2. $St_r(y_1^-,\ldots,y_r^-) \equiv 0$ *if and only if* $r \geq \max\{2k-2,\; 4l\}$,

3. $St_r(z_1^+,\ldots,z_r^+) \equiv 0$ *if* $r \geq \min\{2kl+1,\; 2k+4l\}$,

4. $St_r(z_1^-,\ldots,z_r^-) \equiv 0$ *if* $r \geq \min\{2kl+1,\; 2k+4l\}$.

4.3 Polynomial identities and cocharacters of $(M_{1,1}(F), trp)$

In this section we consider the case of 2×2 matrices endowed with the transpose superinvolution. We shall find generators for the T_2^*-ideal of identities of $(M_{1,1}(F), trp)$ and we shall compute the $(n_1, \ldots, n_4)$-th cocharacter and the codimensions sequence $c_n^*((M_{1,1}(F), trp))$, where F denotes an algebraically closed field of characteristic zero. From now on, we write $(M_{1,1}(F), trp)$ as $M_{1,1}(F)$. Recall that the transpose superinvolution on $M_{1,1}(F)$ is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{trp} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}.$$

Clearly $(M_{1,1}(F))_0^+ = F(e_{11} + e_{22})$, $(M_{1,1}(F))_0^- = F(e_{11} - e_{22})$, $(M_{1,1}(F))_1^+ = Fe_{21}$ and $(M_{1,1}(F))_1^- = Fe_{12}$.

In the following lemma we compute a basis for the T_2^*-ideal of identities of $M_{1,1}(F)$. In order to simplify the notation we denote by z any odd variable and by x an arbitrary variable.

Lemma 4.8. *The T_2^*-ideal of identities of $M_{1,1}(F)$ is generated by the following polynomials*

$$[y^+, x],\ [y_1^-, y_2^-],\ z_1^+ z_2^+,\ z_1^- z_2^-,\ y^- z + z y^-.$$

Proof. Let J be the T_2^*-ideal generated by the above polynomials. It is easy to prove that $J \subseteq \mathrm{Id}^*(M_{1,1}(F))$.

In order to prove the opposite inclusion, let $f \in \mathrm{Id}^*(M_{1,1}(F))$, $\deg f = n$, and assume, as we may, that f is multilinear and $f \in P^*_{n_1,\ldots,n_4}$, $n = n_1 + \cdots + n_4$. We want to show that f is the zero polynomial modulo J. To this end, we notice that, since $[y^+, x] \equiv 0$, $[y_1^-, y_2^-] \equiv 0$ and $y^- z + z y^- \equiv 0$, f is a linear combination (modulo J) of monomials of the type

$$y_1^+ \cdots y_{n_1}^+\ y_1^- \cdots\ y_{n_2}^-\ z_{i_1} \cdots z_{i_{n_3+n_4}},$$

where, for $k = 1, \ldots, n_3 + n_4$, $z_{i_k} \in \{z_1^+, \ldots, z_{n_3}^+, z_1^-, \ldots, z_{n_4}^-\}$. Since $z_1^+ z_2^+ \equiv 0$ and $z_1^- z_2^- \equiv 0$ we cannot have monomials with two adjacent z^+ or two adjacent z^-. Hence

$$|n_3 - n_4| \leq 1.$$

Moreover, from $z_1^+ z_2^+ \equiv 0$, $z_1^- z_2^- \equiv 0$ and $y^- z + z y^- \equiv 0$ the identities $z_1^+ z_2^- z_3^+ - z_3^+ z_2^- z_1^+ \equiv 0$ and $z_1^- z_2^+ z_3^- - z_3^- z_2^+ z_1^- \equiv 0$ follow. Hence in every monomial of f it is possible to reorder the variables z^+ and z^-. Thus, if $n_3 = n_4$ we obtain

$$f \equiv \alpha y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- z_1^+ z_1^- z_2^+ z_2^- \cdots z_{n_3}^+ z_{n_3}^- + \beta y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- z_1^- z_1^+ z_2^- z_2^+ \cdots z_{n_3}^- z_{n_3}^+ \pmod{J}. \tag{4.1}$$

Whereas if $|n_3 - n_4| = 1$, we get

1. $f \equiv \gamma y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- z_1^+ z_1^- z_2^+ z_2^- \cdots z_{n_3-1}^+ z_{n_4}^- z_{n_3}^+ \pmod J$, if $n_3 = n_4 + 1$,

2. $f \equiv \gamma y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- z_1^- z_1^+ z_2^- z_2^+ \cdots z_{n_3-1}^- z_{n_4}^+ z_{n_3}^- \pmod J$, if $n_4 = n_3 + 1$.

Suppose that f is as in (4.1). By making the evaluation $y_i^+ = e_{11} + e_{22}$, $1 \le i \le n_1$, $y_j^- = e_{11} - e_{22}$, $1 \le j \le n_2$, $z_r^+ = e_{21}$, $1 \le r \le n_3$, and $z_s^+ = e_{12}$, $1 \le s \le n_3$, we get $\pm\alpha e_{22} + \beta e_{11} = 0$. Thus $\alpha = \beta = 0$ and f is the zero polynomial modulo J. One can deal in a similar way with the other two cases. Thus $\mathrm{Id}^*(M_{1,1}(F)) = J$ and we are done. □

We remark that Lemma 4.8 works as far as the field F is infinite and has characteristic different from 2. We should also mention that Di Vincenzo in [6] determined a set of generators of the T_2-ideal of $\mathbb{Z}_2$-graded identities of $M_{1,1}(F)$.

Let $\langle\lambda\rangle$ be a multipartition of n, $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(4))$, where $\lambda(i) \vdash n_i$, $1 \le i \le 4$. We shall next compute the $(n_1, \ldots, n_4)$-th cocharacter $\chi_{n_1,\ldots,n_4}(M_{1,1}(F))$ of $M_{1,1}(F)$. Since $\operatorname{char} F = 0$, by complete reducibility, $\chi_{n_1,\ldots,n_4}(M_{1,1}(F))$ can be written as a sum of irreducible characters

$$\chi_{n_1,\ldots,n_4}(M_{1,1}(F)) = \sum_{\langle\lambda\rangle \vdash (n_1,\ldots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{4.2}$$

where $m_{\langle\lambda\rangle} \ge 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi_{n_1,\ldots,n_4}(M_{1,1}(F))$.

In the following lemmas we compute the non-zero multiplicities of such cocharacter. To this end, we recall that the multiplicities in the cocharacter sequence are equal to the maximal number of linearly independent highest weight vectors, according to the representation theory of GL_n. Moreover a highest weight vector is obtained from the polynomial corresponding to an essential idempotent by identifying the variables whose indices lie in the same row of the corresponding Young tableaux (see Section 1.2.3 and [11, Chapter 12] for more details).

Lemma 4.9. *If $\langle\lambda\rangle = ((n_1), (n_2), \emptyset, \emptyset)$, with $n_1 + n_2 > 0$, then $m_{\langle\lambda\rangle} = 1$ in (4.2).*

Proof. Define the following tableaux

$$T_{\lambda(1)} = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{n_1}\,,\quad T_{\lambda(2)} = \boxed{n_1+1}\boxed{n_1+2}\boxed{\cdots}\boxed{n_1+n_2}\,,\quad T_{\lambda(3)} = T_{\lambda(4)} = \emptyset.$$

We notice that the polynomial $a = (y_1^+)^{n_1}(y_1^-)^{n_2}$ is a corresponding highest weight vector which clearly is not a polynomial identity for $M_{1,1}(F)$. Thus $m_{\langle\lambda\rangle} \ge 1$.

It is clear that, up to a scalar, a is the only highest weight vector in variables of homogeneous degree 0 which is not an identity of $M_{1,1}(F)$. Hence $m_{\langle\lambda\rangle} = 1$ and the proof is complete. □

Lemma 4.10. *If $\langle\lambda\rangle = ((n_1), (n_2), (n_3), (n_3))$, with $n_3 > 0$, then $m_{\langle\lambda\rangle} = 2$ in (4.2).*

Proof. We consider the following standard tableaux

$$T_{\lambda(1)} = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{n_1}\,,\quad T_{\lambda(2)} = \boxed{n_1+1}\boxed{n_1+2}\boxed{\cdots}\boxed{n_1+n_2}\,,$$

$$T_{\lambda(3)} = \boxed{n_1+n_2+1}\boxed{n_1+n_2+3}\boxed{\cdots}\boxed{n_1+n_2+2n_3-1}\,,$$

$$T_{\lambda(4)} = \boxed{n_1+n_2+2}\boxed{n_1+n_2+4}\boxed{\cdots}\boxed{n_1+n_2+2n_3}\,,$$

and the corresponding highest weight vector

$$b = (y_1^+)^{n_1}(y_1^-)^{n_2} z_1^+\ z_1^- \cdots z_1^+\ z_1^-.$$

It is easily checked that b does not vanish in $M_{1,1}(F)$.

Let also

$$T'_{\lambda(1)} = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{n_1}\,,\quad T'_{\lambda(2)} = \boxed{n_1+1}\boxed{n_1+2}\boxed{\cdots}\boxed{n_1+n_2}\,,$$

$$T'_{\lambda(3)} = \boxed{n_1+n_2+2}\boxed{n_1+n_2+4}\boxed{\cdots}\boxed{n_1+n_2+2n_3}\,,$$

$$T'_{\lambda(4)} = \boxed{n_1+n_2+1}\boxed{n_1+n_2+3}\boxed{\cdots}\boxed{n_1+n_2+2n_3-1}\,,$$

and

$$b' = (y_1^+)^{n_1}(y_1^-)^{n_2} z_1^-\ z_1^+ \cdots z_1^-\ z_1^+$$

the corresponding highest weight vector which is not a polynomial identity of $M_{1,1}(F)$.

We claim that b and b' are linearly independent modulo $\mathrm{Id}^*(M_{1,1}(F))$. In fact, if $\alpha b + \alpha' b' \equiv 0$ (mod $\mathrm{Id}^*(M_{1,1}(F))$), for some $\alpha, \alpha' \in F$, by making the evaluation $y_1^+ = e_{11} + e_{22}$, $y_1^- = e_{11} - e_{22}$, $z_1^+ = e_{21}$ and $z_1^- = e_{12}$ we get $\alpha = \alpha' = 0$. Hence we deduce that

$$m_{\langle\lambda\rangle} \geq 2.$$

Let now $T''_{\lambda(1)}$, $T''_{\lambda(2)}$, $T''_{\lambda(3)}$, $T''_{\lambda(4)}$ be any tableaux where $\lambda(1) = (n_1)$, $\lambda(2) = (n_2)$, $\lambda(3) = \lambda(4) = (n_3)$ and f the corresponding highest weight vector. According to Proposition 1.7 (see [13, Proposition 0.1]) we may consider that these tableaux are filled in a standard way. We consider f to be a non-zero polynomial modulo the identities of $M_{1,1}(F)$.

Due to the identities $[y^+, x] \equiv 0$, $[y_1^-, y_2^-] \equiv 0$ and $y^- z + z y^- \equiv 0$, without loss of generality, we may assume that $T''_{\lambda(1)}$ is filled with the integers $1, \ldots, n_1$ and $T''_{\lambda(2)}$ with the integers $n_1+1, \ldots, n_1+n_2$.

Moreover, since $z_1^+ z_2^+ \equiv 0$ and $z_1^- z_2^- \equiv 0$, the remaining integers $n_1+n_2+1, \ldots, n_1+n_2+2n_3$ can be inserted in $T''_{\lambda(3)}$ and $T''_{\lambda(4)}$ only in two different ways: either n_1+n_2+2l lie in $T''_{\lambda(3)}$ and n_1+n_2+2l-1 lie in $T''_{\lambda(4)}$, for $1 \le l \le n_3$ or n_1+n_2+2l-1 lie in $T''_{\lambda(3)}$ and n_1+n_2+2l lie in $T''_{\lambda(4)}$, for $1 \le l \le n_3$. Hence we cover either one of the above highest weight vectors and this proves that $m_{\langle\lambda\rangle} = 2$. □

With a similar argument, one can also prove the following lemma.

Lemma 4.11. *If $\langle\lambda\rangle = ((n_1),(n_2),(n_3),(n_4))$, with $n_3 = n_4+1$ or $n_4 = n_3+1$, then $m_{\langle\lambda\rangle} = 1$ in (4.2).*

We are now in a position to present the decomposition of the $(n_1,\ldots,n_4)$-th cocharacter of $M_{1,1}(F)$.

Theorem 4.3.1. *Let*

$$\chi_{n_1,\ldots,n_4}(M_{1,1}(F)) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}$$

be the $(n_1,\ldots,n_4)$-th cocharacter of $M_{1,1}(F)$. Then $m_{\langle\lambda\rangle} = 1$ if either

1) $\langle\lambda\rangle = ((n_1),(n_2),\emptyset,\emptyset)$, $n_1+n_2>0$ or

2) $\langle\lambda\rangle = ((n_1),(n_2),(n_3+1),(n_3))$ or

3) $\langle\lambda\rangle = ((n_1),(n_2),(n_3),(n_3+1))$.

Also, $m_{\langle\lambda\rangle} = 2$ if $\langle\lambda\rangle = ((n_1),(n_2),(n_3),(n_3))$, $n_3>0$. In all other cases $m_{\langle\lambda\rangle} = 0$.

Proof. Since $\dim(M_{1,1}(F))_0^+ = \dim(M_{1,1}(F))_0^- = \dim(M_{1,1}(F))_1^+ = \dim(M_{1,1}(F))_1^- = 1$ any polynomial alternating on two symmetric or skew variables of the same homogeneous degree vanishes in $M_{1,1}(F)$ (see Proposition 1.1). This says that, if f is a highest weight vector, $f \notin \mathrm{Id}^*(M_{1,1}(F))$, then the corresponding tableaux cannot have more than one row. The conclusion of the theorem now follows from Lemmas 4.9, 4.10 and 4.11. □

We conclude this section by computing the $*$-codimensions sequence of $M_{1,1}(F)$.
We recall that, if A is a $*$-algebra, then by (1.3),

$$c_n^*(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\ldots,n_4} c^*_{n_1,\ldots,n_4}(A)$$

where $c^*_{n_1,\ldots,n_4}(A) = \dim_F P^*_{n_1,\ldots,n_4}(A)$. Since $\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}$ is the $(S_{n_1}\times\cdots\times S_{n_4})$-character of $P^*_{n_1,\ldots,n_4}(A)$, then $c^*_{n_1,\ldots,n_4}(A) = \deg\chi_{n_1,\ldots,n_4}(A)$. But

$$\deg\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\deg\chi_{\lambda(1)}\cdots\deg\chi_{\lambda(4)}$$

and $\deg \chi_{\lambda(i)} = 1$, for $1 \leq i \leq 4$. Hence, by Theorem 2.1.2 we get

$$\begin{aligned} c_n^*(M_{1,1}(F)) &= \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\ldots,n_4} \deg \chi_{n_1,\ldots,n_4}(M_{1,1}(F)) \\ &= \sum_{n_1+n_2=n} \binom{n}{n_1,n_2,0,0} + 2 \sum_{\substack{n_1+n_2+2n_3=n \\ n_3>0}} \binom{n}{n_1,n_2,n_3,n_3} + 2 \sum_{\substack{n_1+n_2+2n_3-1=n \\ n_3>0}} \binom{n}{n_1,n_2,n_3,n_3-1} \\ &= 2^n + 2\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{i!\ i!} 2^{n-2i}\ n(n-1)\cdots(n-2i+1) + 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{(i-1)!\ i!} 2^{n-2i+1} n(n-1)\cdots(n-2i+2), \end{aligned}$$

where $\lfloor\ \rfloor$ and $\lceil\ \rceil$ stand for the floor and ceiling functions, respectively. Moreover it can be checked that

$$c_n^*(M_{1,1}(F)) \approx \alpha n^k 4^n,$$

for some $\alpha, k > 0$.

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