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Shannon extensions of regular local rings.
Lefschetz properties for Gorenstein graded algebras associated to Apery Sets.

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Contents

Introduction

In this Ph.D. thesis we discuss several different results in the field of commutative algebra.

Commutative algebra is the branch of abstract algebra born to study the rings occurring in algebraic number theory and algebraic geometry.

The subject, first known as *ideal theory*, began with Richard Dedekind's work on ideals in 1879, itself based on the earlier work of Ernst Kummer and Leopold Kronecker. The concept of ideal was introduced to extend the well known factorization properties of the integers to the rings of algebraic integers, originally called *number rings*. Such rings are the object of study of the algebraic number theory, they are Dedekind rings and constitute an important class of commutative rings. In the same context, considerations related to modular arithmetic have led to the notion of a valuation ring. While the restriction of algebraic field extensions to subrings has led to the notions of integral extensions and integrally closed domains.

Later, David Hilbert introduced the term *ring* to generalize the earlier term *number ring*. Hilbert in the first years of 20th century introduced a more abstract approach and his contribution is considered the basis of the study of commutative rings occurring in a geometric context, such as the rings corresponding to algebraic varieties. He strongly influenced Emmy Noether, who recast many earlier results in terms of an ascending chain condition, now known as the Noetherian condition.

Another important milestone was the work of Hilbert's student Emanuel Lasker, who introduced primary ideals and proved the first version of the Lasker–Noether theorem which states that in a Noetherian ring every ideal has a unique primary decomposition and widely generalize the fundamental theorem of arithmetic.

The main figure responsible for the birth of commutative algebra as a mature subject was Wolfgang Krull, who introduced in the years 30's and 40s', the fundamental notions of localization and completion of a ring, as well as that of regular local rings. He established the concept of the Krull dimension of a ring, first for Noetherian rings before moving on to expand his theory to cover general valuation rings and Krull rings. To this day, Krull's principal ideal theorem is considered one of the single most important foundational theorems in commutative algebra. These results paved the way for a modern introduction of commutative algebra into algebraic geometry, an idea which would revolutionize the latter subject.

The notion of localization of a ring (in particular the localization with respect to a prime ideal, the localization consisting in inverting a single element and the total quotient ring) is one of the main differences between commutative algebra and the theory of non-commutative rings. It leads to an important class of commutative rings, the rings that have only one maximal ideal, called *local rings*. The set of the prime ideals of a commutative ring is naturally equipped with a topology, the

Zariski topology. All these notions are widely used in algebraic geometry and are the basic technical tools for the definition of scheme theory, a generalization of algebraic geometry introduced by Alexander Grothendieck in 1960. Many other notions of commutative algebra are counterparts of geometrical notions occurring in algebraic geometry. This is the case of Krull dimension, primary decomposition, regular rings, Cohen–Macaulay rings, Gorenstein rings and many others.

Much of the modern development of commutative algebra emphasizes modules and homological methods. Both ideals of a ring R and R-algebras are special cases of R-modules, so module theory encompasses both ideal theory and the theory of ring extensions. Homological algebra is the branch of mathematics that studies homology in a general algebraic setting. Its origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and theory of modules and syzygies at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert. The development of homological algebra was closely related with the emergence of category theory. It consists in the study of homological functors and the intricate algebraic structures that they describe like chain complexes, which can be studied both through their homology and cohomology. Homological algebra extracts information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. Some very important results in commutative algebra, like the unique factorization property for regular local rings have been proved using homological methods while a non-homological proof is not known.

In parallel with the development of schemes theory, module theory and homological algebra, from the 1960's many authors further studied the branch of commutative algebra called *multiplicative ideal theory* and now more generally called *commutative ring theory*. This subject, based on the initial work of Wolfgang Krull and Heinz Prüfer, is seen as a generalization of the study of rings arising from algebraic number theory. The main topics studied are factorization theory, generalization of Dedekind domains in a non-Noetherian context like Krull domains and Prüfer domains, rings of integer valued polynomials and star operations. A great contribution to this branch of algebra has been given by Robert Gilmer and many of his students in over forty years of work.

The common thread of the topics of this thesis is their relation with singularities of algebraic varieties.

In algebraic geometry, the problem of resolution of singularities asks whether every algebraic variety V has a resolution, that is a non-singular variety W with a proper birational map $W \to V$. For varieties over fields of characteristic 0 this was proved in the formidable work of Hironaka in 1964 [28]. Many years later, in 2017 again Hironaka published online a work proving the existence of a resolution of singularities for varieties over fields of characteristic p.

The main tools used by authors like Zariski [41], Abhyankhar [2] and Hironaka in the 50's were normalization and blow up of the varieties. The notion of blowing up is to replace a variety with singular points with a larger variety in which the different directions through the singular points are now distinct. For instance, the curve $x^2 - y^2 + x^3 = 0$ takes two distinct paths through the origin. By blowing up the origin in the plane, is possible to replace the origin with a projective line. Then, considering the curve in the blown-up space, the two distinct paths of the curve in the origin now intersect this projective line in two distinct points and the curve no longer intersects itself.

In general repeatedly blowing up the singular points of a curve will eventually resolve the singularities.

The content of this thesis is the following: in the first two chapters we focus on the structure and the ideal-theoretic properties of the Shannon extensions of regular local rings. A Shannon extension is a local integral domain obtained as infinite union of iterated local monoidal (or quadratic) transforms of a regular local ring R of dimension at least 2 (for the definitions and all the details about this construction see the introduction of Chapter 1).

This process of iterating local monoidal transforms of rings of the same dimension corresponds to the geometric notion of following a non singular closed point through repeated blow-ups.

While this concept were mainly studied from the 50's until the 70's by Abhyankhar [1] and Shannon [38] for their geometric interpretation, in recent times Heinzer, Loper, Olberding, Schoutens, Toeniskoetter ([25], [26]) and other authors studied the ring theoretic structure of Shannon extensions often "forgetting" about the geometric origin of such concepts. The tools used in these works are those of multiplicative ideal theory such as pullbacks, GCD domains and complete integral closure of rings.

In Chapter 1, following the work [21], we continue the study of quadratic Shannon extensions looking in particular at their classification. In order to do this, we also use some resolution of singularities in the more algebraic language of projective models rather than projective schemes.

In Chapter 2, we move to study a more general and wide class of rings formed by the monoidal Shannon extensions, introduced (not with their actual name) by Shannon in [38], but almost never studied after his classical paper.

In Chapter 3, we studied a different topic related to the theory of semigroup rings. Semigroup rings are, geometrically speaking, the local rings associated to monomial curves. They are called in this way because they can be studied using their semigroup of values, which is a numerical semigroup (a cofinite submonoid of the natural numbers). The values of the elements of a semigroup ring are given by the rank one discrete valuation induced by the integral closure of the ring (we give all the definitions needed on this topic in Chapter 3).

This natural correspondence with numerical semigroups allows to understand algebraic properties of such rings, and therefore geometric properties of the associated curves (such as the singularity of some points). Originally they were studied by authors such as Kunz and Herzog ([52], [51]) in the 70's and many researches and generalizations are still being done on this subject. Later also the study of numerical semigroups seen as pure algebraic structures started and became consistent.

An useful object used to understand properties of a geometric ring R (a ring is called *geometric* if it is the local ring of an algebraic variety or a completion of such a local ring) is the associated graded ring with respect to an ideal I of R, which, geometrically, is the coordinate ring of the tangent cone along the subvariety defined by I.

Properties like Gorensteinness and Complete Intersection of an associated graded ring can be investigated looking at quotients of such ring by a maximal regular sequence. This fundamental fact allows to transfer many properties from Artinian rings to rings of which they are quotients. Hence the study of Artinian graded algebras over a field have become very important since the last years of 1900. It is easier to deal with an Artinian graded algebra since it is possible to use combinatorial methods that fit with their natural lattice structure.

Important properties studied for Artinian graded algebras are the Lefschetz properties [50] which are motivated by the Hard Lefschetz Theorem on the cohomology rings of smooth irreducible complex projective varieties. Here we give results on the Weak Lefschetz properties for Artinian graded algebras associated to numerical semigroups.

For more technical and detailed introductions on each of the cited topics we refer to the preliminary introduction of the chapters of this thesis.

Chapter 1

Directed union of local quadratic transform of regular local rings

1.1 Introduction

We start giving the definition of local quadratic transform of a regular local ring.

Definition 1.1.1. Let R be a regular local ring with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_d)R$, where $d = \dim R$ is the Krull dimension of R. Choose $i \in \{1, \ldots, n\}$, and consider the overring $R[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}]$ of R. Choose any maximal ideal $\mathfrak{m}_1$ of $R[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}]$ that contains $\mathfrak{m}$. Then the ring

$$R_1 := R\left[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\right]_{\mathfrak{m}_1}$$

is a *local quadratic transform* of R.

It is a standard fact that R_1 is again a regular local ring and $\dim R_1 \leq n$, cf. [35, Corollary 38.2]. Iterating the process we obtain a sequence

$$R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$$

of regular local overrings of R such that for each i, R_{i+1} is a local quadratic transform of R_i. The sequence of positive integers $\{\dim R_i\}_{i \in \mathbb{N}}$ stabilizes, and $\dim R_i = \dim R_{i+1}$ for all sufficiently large i. If $\dim R_i = 1$, then necessarily $R_i = R_{i+1}$, while if $\dim R_i \geq 2$, then $R_i \subsetneq R_{i+1}$.

In general is possible to define local quadratic transforms of Noetherian local domains that need not be regular local rings along a fixed valuation overring of R.

Definition 1.1.2. Let $(R, \mathfrak{m})$ be a Noetherian local domain and let $(V, \mathfrak{m}_V)$ be a valuation domain birationally dominating R. Then $\mathfrak{m}V = xV$ for some $x \in \mathfrak{m}$. The ring

$$R_1 = R\left[\frac{\mathfrak{m}}{x}\right]_{\mathfrak{m}_V \cap R\left[\frac{\mathfrak{m}}{x}\right]}$$

is called a *local quadratic transform* (LQT) of R along V.

The ring R_1 is a Noetherian local domain that dominates R with maximal ideal $\mathfrak{m}_1 = \mathfrak{m}_V \cap R_1$. Since V birationally dominates R_1, we may iterate this process to obtain an infinite sequence $\{R_n\}_{n \geq 0}$ of LQTs of $R_0 = R$ along V. If $R_n = V$ for some n, then V is a DVR and the sequence stabilizes with $R_m = V$ for all $m \geq n$.

This fact happens if and only if V is a *prime divisor* of R, that is a valuation overring birationally dominating R such that $\mathrm{trdeg}(V/\mathfrak{m}_V, R/m)$ is equal to $\dim R - 1$.

Hence, in any case in which V is not a prime divisor of R, $\{R_n\}$ is an infinite strictly ascending sequence of Noetherian local domains.

Assume that R is an RLR with $\dim R \geq 2$ and V is minimal as a valuation overring of R. Then $\dim R_1 = \dim R$, and the process may be continued by defining R_2 to be the LQT of R_1 along V. Continuing the procedure yields an infinite strictly ascending sequence $\{R_n\}_{n\in\mathbb{N}}$ of RLRs all dominated by V.

The process of iterating local quadratic transforms of ring with the same Krull dimension is the algebraic expression of the geometric idea of following a closed point through a sequence of iterated blow-ups of a nonsingular point of an algebraic variety, with each blow up occurring at a closed point in the fiber of the previous blow-up. This geometric process is used in the works about resolution of singularities for curves on surfaces (see, for example, [3] and [9, Sections 3.4 and 3.5]), and in factorization of birational morphisms between nonsingular surfaces ([1, Theorem 3] and [41, Lemma, p. 538]). These applications depend on properties of iterated sequences of local quadratic transforms of a two-dimensional regular local ring. For a two-dimensional regular local ring R, Abhyankar [1, Lemma 12] shows that the limit of this process of iterating local quadratic transforms $R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ is a valuation ring that birationally dominates R; i.e., the infinite union $\mathcal{V} = \bigcup_{i=0}^{\infty} R_i$ is a valuation ring with the same quotient field as R and the maximal ideal of $\mathcal{V}$ contains the maximal ideal of R.

Later, David Shannon [38, Examples 4.7 and 4.17] shows with two examples that the union

$$S = \bigcup_{i=0}^{\infty} R_i$$

of an iterated sequence of local quadratic transforms of a regular local ring of Krull dimension > 2 need not to be a valuation ring. We will briefly discuss these two examples in next section.

Recently, W. Heinzer, K. A. Loper, B. Olberding, H. Schoutens and M. Toeniskoetter in [25], and some of the same authors in[26] studied from the ring-theoretic point of view the structure of such rings S and how this structure is related to properties of the sequence $\{R_i\}_{i=0}^{\infty}$. We call S a *quadratic Shannon extension* of R. In general, a quadratic Shannon extension is not necessarily a valuation ring nor a Noetherian ring, although it is always an intersection of two such rings (see Theorem 1.2.5).

The class of quadratic Shannon extensions separates into two cases, the archimedean ones and non-archimedean. A quadratic Shannon extension S is non-archimedean if there is an element x in the maximal ideal of S such that $\bigcap_{i>0} x^i S \neq 0$. This class of quadratic Shannon extensions is analyzed in detail in [25] and [26].

In this chapter of the thesis, following the article *"Directed unions of local quadratic transforms of a regular local ring and pullbacks"*[21], we use techniques from multiplicative ideal theory to classify non-archimedean quadratic Shannon extensions as the pullbacks of valuation rings of rational rank one with respect to a homomorphism from a regular local ring onto its residue field.

In Section 1.2, we first recall the most important results from the papers [25] and [26] and then we discuss some results from [21] about the multiplicity sequence of a sequence of local quadratic transform along a rank one valuation ring. The concept of multiplicity sequence (see Definition 1.2.12) will be also crucial for the pullback classification of non-archimedean quadratic Shannon extensions.

In Section 1.3, we present this classification with several variations in Lemma 1.3.6 and Theorems 1.3.8 and 1.4.1. The pullback description leads in Theorem 1.3.7 to an existence results for quadratic Shannon extensions contained in a localization of the base ring R at a any given nonmaximal prime ideal P. We define such rings as *Shannon extensions along the prime ideal* P.

As another application, in Theorem 1.4.5 of Section 1.4, we use pullbacks to characterize the quadratic Shannon extensions S of regular local rings R such that R is essentially finitely generated over a field of characteristic 0 and S has a principal maximal ideal.

The fact that non-archimedean quadratic Shannon extensions occur as pullbacks is also useful because of the extensive literature on transfer properties between the rings in a pullback square. In Section 1.5, we use the pullback classification and some structural results for archimedean quadratic Shannon extensions obtained in [25] to show in Theorem 1.5.2 that a quadratic Shannon extension is a GCD domain if and only if it is a valuation domain (in the same theorem will be also proved that it is a coherent ring if and only if it is a valuation ring).

In general, our notation for the first two chapters is as in Matsumura [33]. Thus a local ring need not be Noetherian. An element x in the maximal ideal $\mathfrak{m}$ of a regular local ring R is said to be a *regular parameter* if $x \notin \mathfrak{m}^2$. It then follows that the residue class ring R/xR is again a regular local ring. We call an extension ring B of an integral domain A an *overring of* A if B is a subring of the quotient field of A. If, in addition, A and B are local and the inclusion map $A \hookrightarrow B$ is a local homomorphism (i.e. the maximal ideal of A is contained in the maximal ideal of B), we say that B *birationally dominates* A. We use UFD as an abbreviation for unique factorization domain, RLR as abbreviation for regular local ring and DVR as an abbreviation for rank 1 discrete valuation ring. If P is a prime ideal of a ring A, we denote by $\kappa(P)$ the residue field A_P/PA_P of A_P. In our terminology, any ring of fractions over an integral domain D is said a *localization* of D even when it is not a local ring.

1.2 Preliminaries on Quadratic Shannon extensions

Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R \geq 2$ and let F denote the quotient field of R. David Shannon's work in [38] on sequences of quadratic and monoidal transforms of regular local rings motivates our terminology in Definition 1.2.1.

Definition 1.2.1. Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R \geq 2$. With $R = R_0$, let $\{R_n, \mathfrak{m}_n\}$ be an infinite sequence of RLRs, such that $\dim R_n \geq 2$ and R_{n+1} is an LQT of R_n for each n. Then the ring $S = \bigcup_{n \geq 0} R_n$ is called a *quadratic Shannon extension.*

In [25] and [26], the authors call S simply a Shannon extension of R. We have made a distinction here since in Chapter 2 we will deal with monoidal transforms and their infinite unions. Since $\dim R_n \geq 2$, we have $R_n \subsetneq R_{n+1}$ for each positive integer n and $\bigcup_n R_n$ is an infinite ascending union.

A quadratic Shannon extension S is always an integrally closed local domain with maximal ideal $\mathfrak{m}_S = \bigcup_n \mathfrak{m}_n$.

If $\dim R = 2$, then the quadratic Shannon extensions of R are precisely the valuation rings that birationally dominate R and are minimal as a valuation overring of R [1, Lemma 12]. If $\dim R > 2$, then, examples due to Shannon [38] show that there are quadratic Shannon extensions that are not valuation rings.

As mentioned in the introduction, the structure of Shannon extensions naturally separates into those that are archimedean and those that are non-archimedean. We recall the concept of archimedean domain in the following definition.

Definition 1.2.2. An integral domain A is *archimedean* if $\bigcap_{n>0} a^n A = 0$ for each nonunit $a \in A$.

An integral domain A with $\dim A \leq 1$ is archimedean. We recall here two Shannon's examples in order to give to the reader some ideas about what kind of rings we will deal with.

Example 1.2.3. [38, Examples 4.7 and 4.17]
Let $\dim R = 3$ and $\mathfrak{m} = (x, y, z)R$.

1. Define for $n \geq 1$,
$$R_n = R_{n-1}\left[\frac{y}{x^n}, \frac{z}{x^n}\right]_{(x, \frac{y}{x^n}, \frac{z}{x^n})}$$
and $S = \bigcup_{n\geq 0}^{\infty} R_n$.

   This ring S is a non archimedean domain of dimension 3 with principal maximal ideal $\mathfrak{m}_S = xS$. The unique prime of height two is the ideal $Q = \bigcap_{n\geq 0}^{\infty} x^n S$. Since $\frac{y}{z}, \frac{z}{y} \notin S$, S is not a valuation ring.

2. Let V be a rank 1 valuation overring of R such that $v(z) > v(x) + v(y)$ and V birationally dominates R.

   Then, the Shannon extension S obtained along V is an archimedean domain of dimension 2 and $\mathfrak{m}_S = \mathfrak{m}_S^2$. Hence it is not a valuation ring, since a valuation ring V with $\dim V \geq 2$ would be non-archimedean.

It happens, if $\dim R > 2$, that there are valuations rings V that birationally dominate R with V minimal as a valuation overring of R, which are not a Shannon extension of R. Indeed, if V has rank > 2, then V is not a quadratic Shannon extension of R (see [17, Proposition 7]).

Now we recall some of the results from [25] and [26] with special emphasis on non-archimedean quadratic Shannon extensions.

Theorems 1.2.4 and 1.2.5 record properties of a quadratic Shannon extension.

Theorem 1.2.4. [25, Theorems 3.3, 3.5 and 3.8]
Let $(S, \mathfrak{m}_S)$ be a quadratic Shannon extension of a regular local ring R. Then:

1. *The maximal ideal of S, $\mathfrak{m}_S$ is either principal or idempotent.*

2. *S is Noetherian if and only if it is a DVR.*

3. *Any non maximal prime ideal P of S is such that $S_P = (R_n)_{P\cap R_n}$ for $n >> 0$.*

4. *For any $n >> 0$, set $R_{n+1} = R_n\left[\frac{\mathfrak{m}_n}{x_n}\right]_{\mathfrak{m}_{n+1}}$. Then $x_n S$ is an $\mathfrak{m}_S$-primary ideal.*

There exists a collection of rank one discrete valuation rings associated to each quadratic Shannon extension. Let $S = \bigcup_{i\geq 0} R_i$ be a quadratic Shannon extension of $R = R_0$ and for each i, let V_i be the DVR defined by the *order function* ord_{R_i}, where for $x \in R_i$,

$$\operatorname{ord}_{R_i}(x) = \sup\{n \mid x \in \mathfrak{m}_i^n\}$$

and ord_{R_i} is extended to the quotient field of R_i by defining $\operatorname{ord}_{R_i}(x/y) = \operatorname{ord}_{R_i}(x) - \operatorname{ord}_{R_i}(y)$ for all $x, y \in R_i$ with $y \neq 0$. The family $\{V_i\}_{i=0}^{\infty}$ determines the set

$$V \;=\; \bigcup_{n\geq 0} \bigcap_{i\geq n} V_i = \{a \in F \mid \operatorname{ord}_{R_i}(a) \geq 0 \text{ for } i \gg 0\}.$$

The set V consists of the elements in F that are in all but finitely many of the V_i. In [25, Corollary 5.3], is proved that V is a valuation domain that birationally dominates S. For this reason V is called the *boundary valuation ring* of the Shannon extension S. The valuation associated to the boundary valuation ring of a quadratic Shannon has rank at most 2 [26, Theorem 6.4 and Corollary 8.6].

Theorem 1.2.5. [25, Theorems 4.1, 5.4 and 8.1] *Let $(S, \mathfrak{m}_S)$ be a quadratic Shannon extension of a regular local ring R. Let T be the intersection of all the DVR overrings of R that properly contain S, and let V be the boundary valuation ring of S. Then:*

1. *S is a valuation domain if and only if either dim$(S) = 1$ or dim$(S) = 2$ and the value group of V is $\mathbb{Z} \oplus G$ with $G \leq \mathbb{Q}$.*

2. *$S = V \cap T$.*

3. *There exists $x \in \mathfrak{m}_S$ such that xS is $\mathfrak{m}_S$-primary, and $T = S[1/x]$ for any such x. It follows that the units of T are precisely the ratios of $\mathfrak{m}_S$-primary elements of S and* $\dim T = \dim S - 1$.

4. *T is a localization of R_i for $i \gg 0$. In particular, T is a Noetherian regular UFD.*

5. *T is the unique minimal proper Noetherian overring of S.*

In light of item 5 of Theorem 1.2.5, the ring T is called the *Noetherian hull* of S.

The following function that we define, called w can be used to stabilize whether a quadratic Shannon extension S is non-archimedean and it is also related to the Boundary valuation ring of S

Definition 1.2.6. Let $S = \bigcup_{i\geq 0} R_i$ be a quadratic Shannon extension of a regular local ring R and let F be its quotient field.

Fix $x \in S$ such that xS is primary for the maximal ideal of S, and define

$$w : \; F \to \mathbb{R} \,\cup\, \{-\infty, \; +\infty\}$$

by defining $w(0) = +\infty$, and for each $q \in F^{\times}$,

$$w(q) \;=\; \lim_{n\to\infty} \frac{\operatorname{ord}_n(q)}{\operatorname{ord}_n(x)}.$$

The function w satisfies all the properties of a valuation, except for the fact that can assume infinite value on nonzero elements.

Next Theorem 1.2.7 shows that, in a non-archimedean quadratic Shannon extension S, there is a prime ideal Q such that S/Q is a rational rank one valuation ring and Q is also a prime ideal of the Noetherian hull T of S. In the next section this fact is going to be the basis for the classification of non-archimedean quadratic Shannon extensions as pullbacks.

Theorem 1.2.7. *Let $S = \bigcup_{n\geq 0} R_n$ be a quadratic Shannon extension of a regular local ring R with quotient field F, and let x be an element of S that is primary for the maximal ideal $\mathfrak{m}_S$ of S (see Theorem 1.2.5). Assume that* $\dim S \geq 2$. *Let $Q = \bigcap_{n\geq 1} x^n S$, and let $T = S[1/x]$ be the Noetherian hull of S. Then the following are equivalent:*

1. *S is non-archimedean.*
2. *$T = (Q :_F Q)$.*
3. *Q is a nonzero prime ideal of S.*
4. *Every nonmaximal prime ideal of S is contained in Q.*
5. *T is a (regular) local ring.*
6. *$\sum_{n=0}^{\infty} w(\mathfrak{m}_n) = \infty$, where w is as in Definition 1.2.6 and $\mathfrak{m}_n$ is the maximal ideal of R_n for each $n \geq 0$.*

Moreover if (1)–(6) hold for S and Q, then $T = S_Q$, $Q = QS_Q$ is a common ideal of S and T, and S/Q is a rational rank 1 valuation domain on the residue field T/Q of T. In particular, Q is the unique maximal ideal of T.

Proof. The equivalence of items 1 through 5 can be found in [26, Theorem 8.3]. That statement 1 is equivalent to 6 follows from [26, Theorem 6.1]. To prove the moreover statement, define $Q_\infty = \{a \in S \mid w(a) = +\infty\}$, where w is as in Definition 1.2.6. By [26, Theorem 8.1], Q_∞ is a prime ideal of S and T, and by [26, Remark 8.2], Q_∞ is the unique prime ideal of S of dimension 1. Since also item 4 implies every nonmaximal prime ideal of S is contained in Q, it follows that $Q = Q_\infty$.
By item 5, $T = S[1/x]$ is a local ring. Since xS is $\mathfrak{m}_S$-primary, we have that $T = S_Q$. Since Q is an ideal of T, we conclude that $QS_Q = Q$ and Q is the unique maximal ideal of T. By [26, Corollary 8.4], S/Q is a valuation domain, and by [26, Theorem 8.5], S/Q has rational rank 1. □

Remark 1.2.8. If statements (1) – (6) hold for S, then Theorem 1.2.7 and [15, Theorem 2.3] imply that any principal ideal of S that is primary for $\mathfrak{m}_S$ is comparable to every other ideal of S with respect to set inclusion. Conversely, if a Shannon extension S has a principal ideal that is primary for $\mathfrak{m}_S$ and is comparable to every other ideal of S, then by [15, Theorem 2.3], S satisfies statement 3 of Theorem 1.2.7, and hence S decomposes as in the statement of Theorem 1.2.7.

Remark 1.2.9. Let S be a non archimedean Shannon extension. A consequence of Theorem 1.2.7 and of [26, Theorem 8.5] is that the function w of Definition 1.2.6, restricted to the elements of the quotient field of S/Q, corresponds to the valuation associated to the valuation ring S/Q.

We can further separate the case where S is archimedean to whether or not S is completely integrally closed. We recall the definition and result.

Definition 1.2.10. Let A be an integral domain. An element x in the field of fractions of A is called *almost integral* over A if $A[x]$ is contained in a principal fractional ideal of A. The *complete integral closure* A^* of A is the ring of all the almost integral elements over A. The ring A is called *completely integrally closed* if $A^* = A$.

For a Noetherian domain, an element of the field of fractions is almost integral if and only if it is integral.

Let W be the rank 1 valuation overring of the Boundary valuation ring V of S. It is possible to characterize the complete integral closure S^* of S:

Theorem 1.2.11. [25, Theorems 6.1, 6.2, 6.9] *Let S be a quadratic Shannon extension. Then:*

1. *When S is non archimedean the complete integral closure S^* of S is equal to the Noetherian hull T.*
2. *When S is archimedean, the complete integral closure of S is*

$$S^* = (\mathfrak{m}_S :_{Q(R)} \mathfrak{m}_S) = W \cap T.$$

It follows that S is completely integrally closed if and only if V has rank 1. Moreover, in this case the function w as in Definition 1.2.6 is a rank 1 nondiscrete valuation and its valuation ring is W.

We introduce now the concept of multiplicity sequence as in [21].

Definition 1.2.12. Let R be a Noetherian local domain, let $\mathcal{V}$ be a rank 1 valuation ring dominating R with corresponding valuation ν, and let $\{(R_i, \mathfrak{m}_i)\}_{i=0}^{\infty}$ be the infinite sequence of LQTs along $\mathcal{V}$ (if $R_n = R_{n+1}$ for some integer n, then $R_n = \mathcal{V}$ is a DVR and $R_n = R_m$ for all $m \geq n$, in this case we consider the sequence to be eventually constant). Then the sequence $\{\nu(\mathfrak{m}_i)\}_{i=0}^{\infty}$ is the *multiplicity sequence of* $(R, \mathcal{V})$; see [18, Section 5]. We say the multiplicity sequence is *divergent* if $\sum_{i \geq 0} \nu(\mathfrak{m}_i) = \infty$.

Next Proposition generalizes the following fact, proved in [18, Proposition 23]: If R is a regular local ring, V is a rank 1 valuation ring birationally dominating R, and the multiplicity sequence of $(R, \mathcal{V})$ is divergent, then $\mathcal{V}$ is a quadratic Shannon extension of R. Furthermore, in [24, Proposition 7.3], is proved that in such case $\mathcal{V}$ has rational rank 1. Here we give the same result when R is any Noetherian local domain.

Proposition 1.2.13. *Let $(R, \mathfrak{m})$ be a Noetherian local domain, let $\mathcal{V}$ be a rank 1 valuation ring that birationally dominates R, and let $\{R_i\}_{i=0}^{\infty}$ be the infinite sequence of LQTs of R along $\mathcal{V}$. If the multiplicity sequence of $(R, \mathcal{V})$ is divergent, then $\mathcal{V} = \bigcup_{n \geq 0} R_n$. Thus if $\mathcal{V}$ is a DVR, then $\mathcal{V} = \bigcup_{n \geq 0} R_n$.*

Proof. Let ν be a valuation for $\mathcal{V}$ and let y be a nonzero element in $\mathcal{V}$. Suppose we have an expression $y = a_n/b_n$, where $a_n, b_n \in R_n$. Since $R_n \subseteq \mathcal{V}$, it follows that $\nu(b_n) \geq 0$. If $\nu(b_n) = 0$, then since $\mathcal{V}$ dominates R_n, we have $1/b_n \in R_n$ and $y \in R_n$.

Assume otherwise, that is, $\nu(b_n) > 0$. Then $b_n \in \mathfrak{m}_n$, and since $\nu(a_n) \geq \nu(b_n)$, also $a_n \in \mathfrak{m}_n$. Let $x_n \in \mathfrak{m}_n$ be such that $x_n R_{n+1} = \mathfrak{m}_n R_{n+1}$. Then $a_n, b_n \in x_n R_{n+1}$, so the elements $a_{n+1} = a_n/x$ and $b_{n+1} = b_n/x$ are in R_{n+1}. Thus we have the expression $y = a_{n+1}/b_{n+1}$, where $\nu(b_{n+1}) = \nu(b_n) - \nu(\mathfrak{m}_n)$.

Consider an expression $y = a_0/b_0$, where $a_0, b_0 \in R_0$. Then we iterate this process to obtain a sequence of expressions $\{a_n/b_n\}$ of y, with $a_n, b_n \in R_n$, where this process halts at some $n \geq 0$ if $\nu(b_n) = 0$, implying $y \in R_n$. Assume by way of contradiction that this sequence is infinite. For $N \geq 0$, it follows that $\nu(b_0) = \nu(b_N) + \sum_{n=0}^{N-1} \nu(\mathfrak{m}_n)$. Then $\nu(b_0) \geq \sum_{n=0}^{N} \nu(\mathfrak{m}_n)$ for any $N \geq 0$, so $\nu(b_0) \geq \sum_{n=0}^{\infty} \nu(\mathfrak{m}_n) = \infty$, which contradicts $\nu(b_0) < \infty$. This shows that the sequence $\{a_n/b_n\}$ is finite and hence $y \in \bigcup_n R_n$. $\square$

Examples of pairs $(R, \mathcal{V})$ with divergent multiplicity sequence such that $\mathcal{V}$ is not a DVR are given in [24, Examples 7.11 and 7.12].

The divergence of the multiplicity sequence in Proposition 1.2.13 is a sufficient condition for $\mathcal{V} = \bigcup_{i \geq 0} R_i$, but not a necessary condition. Indeed, in Example 1.2.14, we present two pairs $(R, \mathcal{V})$ with convergent multiplicity sequence but in the first case the union $\bigcup_{i \geq 0} R_i$ is equal to $\mathcal{V}$ while in the second case is properly contained.

Example 1.2.14. Let x, y, z be indeterminates over a field k. We first construct a rational rank 1 valuation ring V' on the field $k(x, y)$. We do this by describing an infinite sequence $\{(R'_n, \mathfrak{m}'_n)\}_{n \geq 0}$ of local quadratic transforms of $R'_0 = k[x, y]_{(x,y)}$. To indicate properties of the sequence, we define a rational valued function v on specific generators of the $\mathfrak{m}'_n$. The function v is to be additive on products. We set $v(x) = v(y) = 1$. This indicates that y/x is a unit in every valuation ring birationally dominating R'_1.

Step 1. Let R'_1 have maximal ideal $\mathfrak{m}'_1 = (x_1, y_1)R_1$, where $x_1 = x,\ y_1 = (y/x) - 1$. Define $v(y_1) = 1/2$.

Step 2. The local quadratic transform R'_2 of R'_1 has maximal ideal $\mathfrak{m}'_2$ generated by $x_2 = x_1/y_1,\ y_2 = y_1$. We have $v(x_2) = 1/2$, $v(y_2) = 1/2$.

Step 3. Define $y_3 = (y_2/x_2) - 1$ and assign $v(y_3) = 1/4$. Then $x_3 = x_2$, $v(x_3) = 1/2$.

Step 4. The local quadratic transform R'_4 of R'_3 has maximal ideal $\mathfrak{m}'_4$ generated by $x_4 = x_3/y_3,\ y_4 = y_3$. Then $v(x_4) = v(y_4) = 1/4$.

Continuing this 2-step process yields an infinite directed union $(R'_n, \mathfrak{m}'_n)$ of local quadratic transforms of 2-dimensional RLRs. Let $V' = \bigcup_{n \geq 0} R'_n$. Then V' is a valuation ring by an Abhyankhar's well known result [1, Lemma 12]. Let v' be a valuation associated to V' such that $v'(x) = 1$. Then $v'(y) = 1$ and v' takes the same rational values on the generators of $\mathfrak{m}'_n$ as defined by v. Since there are infinitely many translations as described in Steps $2n+1$ for each integer $n \geq 0$, it follows that V' has rational rank 1. For this fact see [24, Remark 5.1(4)].

The multiplicity values of $\{R'_n, \mathfrak{m}'_n)\}$ are $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \ldots$, the sum of which converges to 3.

Define $V = V'(\frac{z}{x^2y^2})$. V is the localization of the polynomial ring $V'[\frac{z}{x^2y^2}]$ at the prime ideal $\mathfrak{m}_{V'}V'[\frac{z}{x^2y^2}]$. One sometimes refers to V as a Gaussian or trivial or Nagata extension of V' to a valuation ring on the simple transcendental field extension generated by $\frac{z}{x^2y^2}$ over $k(x, y)$. It follows that V has the same value group as V' and the residue field of V is a simple transcendental extension of the residue field of V' that is generated by the image of $\frac{z}{x^2y^2}$ in $V/\mathfrak{m}_V$.

Let v denote the associated valuation to V such that $v(x) = 1$. It follows that $v(y) = 1$ and $v(z) = v(x^2y^2) = 4$. Let $R_0 = k[x, y, z]_{(x,y,z)}$. Then R_0 is birationally dominated by V. Let $\{(R_n, \mathfrak{m}_n)\}_{n\geq 0}$ be the sequence of local quadratic transforms of R_0 along V.

We describe the first few steps:

Step 1. R_1 has maximal ideal $\mathfrak{m}_1 = (x_1, y_1, z_1)R_1$, where $x_1 = x$, $y_1 = (y/x) - 1$, and $z_1 = z/x$. Also $v(y_1) = 1/2$.

Step 2. The local quadratic transform R_2 of R_1 along V has maximal ideal $\mathfrak{m}_2$ generated by $x_2 = x_1/y_1$, $y_2 = y_1$ and $z_2 = z_1/y_1$. We have $v(x_2) = 1/2$, $v(y_2) = 1/2$ and $v(z_2) = 4 - 3/2 > 3/2$.

Step 3. The local quadratic transform R_3 of R_2 along V has maximal ideal $\mathfrak{m}_3$ generated by $y_3 = (y_2/x_2) - 1$, $x_3 = x_2$ and z_3. We have $v(y_3) = 1/4$, $v(x_3) = 1/2$ and $v(z_3) > 1/2$.

Step 4. The local quadratic transform R_4 of R_3 along V has maximal ideal $\mathfrak{m}_4$ generated by $x_4 = x_3/y_3$, $y_4 = y_3$ and $z_4 = z_3/y_3$.

The multiplicity values of the sequence $\{(R_n, \mathfrak{m}_n)\}_{n\geq 0}$ along V are the same as that for $\{R'_n, \mathfrak{m}'_n)\}$, namely $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \ldots$. Let $S = \bigcup_{n\geq 0} R_n$. Since S is birationally dominated by the rank 1 valuation ring V, it follows that S is an archimedean Shannon extension. Since we never divide in the z-direction, we have $S \subseteq R_{zR}$, and S is not a valuation ring, since it is an archimedean subring of a DVR.

1.3 The relation of Shannon extensions to pullbacks

Following the paper [21], we begin this section recalling the definition of pullback.

Definition 1.3.1. Let $\alpha : A \to C$ be an extension of rings, and let B be a subring of C. The subring $D = \alpha^{-1}(B)$ of A is the *pullback* of B along $\alpha : A \to C$.

$$\begin{tikzcd} D \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ A \arrow[r, "\alpha"] & C \end{tikzcd}$$

Alternatively, D is the fiber product $A \times_C B$ of α and the inclusion map $\iota : B \to C$; see, for example, [31, page 43].

The pullback construction is often used as source of examples in non-Noetherian commutative ring theory and it is a generalization of the classical "$D + M$" construction (see [14]). In the case in which A, B, C, D are domains, α is a surjection and C is the quotient field of B, we say the diagram above is of type $\square^*$ (see [13]). For a diagram of type $\square^*$, the kernel of α is a maximal ideal of A that is a common ideal of A and D. The quotient field C of B can be identified with the residue field of this maximal ideal. If A is local with $\dim A \geq 1$ and $\dim B \geq 1$, then $A = D_M$ is a localization of D and D is non-archimedean.

For an extensive study about pullbacks in ring theory see for example [10, 11, 13]. In a more general setting in which the ring C is not a field, the diagram is simply said of type $\square$ (see [29]).

Sometimes the pullback construction can be also used as a classification tool. A simple example is given by the observation that a local domain D has a principal maximal ideal if and only if D occurs in a pullback diagram of type $\square^*$, where B is a DVR [30, Exercise 1.5, p. 7]. A second example is given by the fact that for nonnegative integers $k < n$, a ring D is a valuation domain of rank n if and only if D occurs in pullback diagram of type $\square^*$, where A is a valuation ring of rank $n-k$ and B is a valuation ring of rank k; see [10, Theorem 2.4].

A third example of classification via pullbacks of the form $\square^*$ is given by the classification of local rings of global dimension 2 by Greenberg [20, Corollary 3.7] and Vasoncelos [39]: A local ring D has global dimension 2 if and only if D satisfies one of the following:

(a) D is a regular local ring of Krull dimension 2,

(b) D is a valuation ring of global dimension 2, or

(c) D has countably many principal prime ideals and D occurs in a pullback diagram of type $\square^*$, where A is a valuation ring of global dimension 1 or 2 and B is a regular local ring of global dimension 2.

We use the pullback construction in this and the next section of this chapter to classify among the overrings of a regular local ring R those that are non-archimedean quadratic Shannon extensions of R. We prove in Theorem 1.3.8 that these are precisely the overrings of R that occur in pullback diagrams of type $\square^*$, where A is a localization of an iterated quadratic transform R_i of R at a prime ideal P and and B is a rank 1 valuation overring of R_i/P having a divergent multiplicity sequence. Thus a non-archimedean quadratic Shannon extension is determined by a rank 1 valuation ring and a regular local ring.

Next Theorem, using the characterization obtained in Theorem 1.2.7, shows that a non-archimedean quadratic Shannon extension S is a pullback of a rank one valuation ring with respect to the residue map from the Noetherian hull of S to its residue field.

Theorem 1.3.2. *Let S be a non-archimedean quadratic Shannon extension. Then there is a prime ideal Q of S and a rational rank 1 valuation ring $\mathcal{V}$ of $\kappa(Q)$ such that S_Q is the Noetherian hull of S and S is the pullback of $\mathcal{V}$ along the residue map $\alpha : S_Q \to \kappa(Q)$, as in the following diagram:*

$$\begin{tikzcd} S = \alpha^{-1}(\mathcal{V}) \arrow[r, two heads] \arrow[d, hook] & \mathcal{V} \arrow[d, hook] \\ S_Q \arrow[r, two heads, "\alpha"] & \kappa(Q) \end{tikzcd}$$

Proof. Theorem 1.2.7 implies that there is a prime ideal Q of S such that S_Q is the Noetherian hull of S, $Q = QS_Q$ and S/Q is a rational rank 1 valuation ring. Theorem 1.3.2 follows from these observations. $\square$

We want now to prove the converse of this assertion, which will be given in Theorems 1.3.8 and 1.4.1.

Let S be a non-archimedean quadratic Shannon extension. The ideal Q of Theorem 1.3.2 is a non maximal prime ideal and therefore by Theorem 1.2.4, there exists a positive integer n such that the Noetherian hull of S is $S_Q = (R_n)_{Q \cap R_n}$. By replacing, without loss of generality R with R_n as starting ring of our sequence, we can say that $S \subseteq R_P$ where $P = Q \cap R$ is a prime ideal of R. This fact motivates the following definition, which uses the terminology of Granja and Sanchez-Giralda [19, Definition 3 and Remark 4].

Definition 1.3.3. Let R be a Noetherian local domain and let $\{R_n\}_{n\geq 0}$ be an infinite sequence of local quadratic transforms of $R = R_0$. For a prime ideal P of R, we say the quadratic sequence $\{R_n\}$ is along R_P if $\bigcup_{n\geq 0} R_n \subseteq R_P$.

For a nonzero, nonmaximal prime ideal P of a Noetherian local domain $(R, \mathfrak{m})$, there is a one-to-one correspondence between sequences $\{R_n\}$ of LQTs of $R = R_0$ along R_P and sequences $\{\overline{R_n}\}$ of LQTs of $\overline{R_0} = R/P$.

Proposition 1.3.4. *Let R be a Noetherian local domain and let P be a nonzero nonmaximal prime ideal of R. Then there is a one-to-one correspondence between:*

1. *Infinite sequences $\{R_n\}_{n\geq 0}$ of LQTs of $R_0 = R$ along R_P.*
2. *Infinite sequences $\{\overline{R_n}\}_{n\geq 0}$ of LQTs of $\overline{R_0} = R/P$.*

Given such a sequence $\{R_n\}_{n\geq 0}$, the corresponding sequence is $\{R_n/(PR_P \cap R_n)\}$. Denote $S = \bigcup_{n\geq 0} R_n$ and $\overline{S} = \bigcup_{n\geq 0} \overline{R_n}$, and let $\widetilde{S}$ be the pullback of $\overline{S}$ with respect to the quotient map $R_P \to \kappa(P)$ as in the following diagram:

$$\begin{tikzcd} \widetilde{S} = \alpha^{-1}(\overline{S}) \arrow[r, two heads] \arrow[d, hook] & \overline{S} \arrow[d, hook] \\ R_P \arrow[r, two heads, "\alpha"] & \kappa(P) \end{tikzcd}$$

Then $\widetilde{S} = S + PR_P$ and $\widetilde{S}$ is non-archimedean.

Proof. The correspondence follows from [22, Corollary II.7.15, p. 165]. The fact that $\widetilde{S} = S + PR_P$ is a consequence of the fact that $\widetilde{S}$ is a pullback of $\overline{S}$ and R_P. That $\widetilde{S}$ is non-archimedean is a consequence of the observation that for each $x \in \mathfrak{m}_{\widetilde{S}} \setminus PR_P$, the fact that $PR_P \subseteq \widetilde{S} \subseteq R_P$ implies $PR_P \subseteq x^k\widetilde{S}$ for all $k > 0$. $\square$

Next lemma gives a sufficient condition for the union $S = \bigcup_{n\geq 0} R_n$ to be equal to the pullback $\widetilde{S}$. This condition involves property of the multiplicity sequence of $(\overline{R}, \overline{S})$ (see Definition 1.2.12) and the argument of the proof proceeds along the proof of Proposition 1.2.13.

Lemma 1.3.5. *Assume notation as in Proposition 1.3.4. If $\overline{S}$ is a rank 1 valuation ring and the multiplicity sequence of $(\overline{R}, \overline{S})$ is divergent, then $S = \tilde{S}$.*

Proof. Let ν be a valuation for $\overline{S}$ and assume that ν takes values in $\mathbb{R}$. Let $f \in \tilde{S}$. We claim that $f \in S$. Since $\tilde{S} = S + PR_P$, we may assume $f \in PR_P$. Write $f = \frac{g_0}{h_0}$, where $g_0 \in P$ and $h_0 \in R \setminus P$.

Suppose we have an expression of the form $f = \frac{g_n}{h_n}$, where $g_n \in PR_P \cap R_n$ and $h_n \in R_n \setminus PR_P$. Write $\mathfrak{m}_n R_{n+1} = xR_{n+1}$ for some $x \in \mathfrak{m}_n$. Since $PR_P \cap R_n \subseteq \mathfrak{m}_n$, it follows that $g_n = xg_{n+1}$ for $g_{n+1} = \frac{g_n}{x} \in R_{n+1}$. Denote the image of $h \in R_n$ in $\overline{R_n}$ by $\overline{h}$. Since $h_n \in R_n \setminus PR_P$, we have that $\overline{h_n} \neq 0$ and $\nu(\overline{h_n})$ is a finite nonnegative real number. If $\nu(\overline{h_n}) > 0$, then $h_n \in \mathfrak{m}_n$, so $h_n = xh_{n+1}$ for $h_{n+1} = \frac{h_n}{x} \in R_{n+1}$. Thus we have written $f = \frac{g_{n+1}}{h_{n+1}}$, where $g_{n+1} \in PR_P \cap R_{n+1}$ and $h_{n+1} \in R_{n+1} \setminus PR_P$, such that $\nu(\overline{h_{n+1}}) = \nu(\overline{h_n}) - \nu(\overline{\mathfrak{m}_n})$.

Since $\sum_{n\geq 0} \nu(\overline{\mathfrak{m}_n}) = \infty$ and $\nu(\overline{h_0})$ is finite, this process must halt with $f = \frac{g_n}{h_n}$ as before such that $\nu(\overline{h_n}) = 0$. Since $\nu(\overline{h_n}) = 0$, $\overline{h_n}$ is a unit in $\overline{R_n}$, so h_n is a unit in R_n, and thus $f \in R_n$. $\square$

In the case R is a regular local ring, the divergence of the multiplicity sequence of $(\overline{R}, \overline{S})$ can be shown to be exactly equivalent to the condition $S = \tilde{S}$. Moreover these two conditions are equivalent to have the Noetherian hull of the Shannon extension S equal to R_P.

Lemma 1.3.6. *Let P be a nonzero nonmaximal prime ideal of a regular local ring R. Let $\{R_n\}_{n\geq 0}$ be a sequence of LQTs of $R_0 = R$ along R_P and let $\{\overline{R_n}\}$ be the induced sequence of LQTs of $\overline{R_0} = R/P$ as in Proposition 1.3.4. Denote $S = \bigcup_{n\geq 0} R_n$ and $\overline{S} = \bigcup_{n\geq 0} \overline{R_n}$. Then the following are equivalent:*

1. *S is the pullback of $\overline{S}$ along the surjective map $R_P \to \kappa(P)$.*

2. *The Noetherian hull of S is R_P.*

3. *$\overline{S}$ is a rank 1 valuation ring and the multiplicity sequence of $(\overline{R}, \overline{S})$ is divergent.*

If these conditions hold, then $\overline{S}$ has rational rank 1.

Proof. (1) $\implies$ (2): As a pullback, the quadratic Shannon extension S is non-archimedean (see the proof of Proposition 1.3.4). Let $x \in S$ be such that xS is $\mathfrak{m}_S$-primary (see Theorem 1.2.5). By Theorem 1.2.7, the ideal $Q = \bigcap_{n\geq 0} x^n S$ is a nonzero prime ideal of S, every nonmaximal prime ideal of S is contained in Q and $T = S_Q$. Assumption (1) implies that PR_P is a nonzero ideal of both S and R_P. Hence R_P is almost integral over S. We have $S \subseteq S_Q = T \subseteq R_P$, and S_Q is an RLR and therefore completely integrally closed. It follows that $S_Q = R_P$ is the Noetherian hull of S.
(2) $\implies$ (3): Since the Noetherian hull R_P of S is local, Theorem 1.2.7 implies that S is non-archimedean and $PR_P \subseteq S$. By Theorem 1.3.2, $\overline{S} = S/PR_P$ is a rational rank 1 valuation ring. The valuation ν associated to $\overline{S}$ is equal to the valuation w of Remark 1.2.9. Hence, by item 6 of Theorem 1.2.7, we have

$$\sum_{n=0}^{\infty} \nu(\overline{\mathfrak{m}_n}) = \sum_{n=0}^{\infty} w(\mathfrak{m}_n) = \infty.$$

(3) $\implies$ (1): This is proved in Lemma 1.3.5. ☐

We observe that the proof of Lemma 1.3.6 shows that the multiplicity sequence of $(R/P, S/PR_P)$ is given by $\{w(\mathfrak{m}_i)\}$, where w is as in Definition 1.2.6. A direct consequence of Lemma 1.3.6 is the existence of a Shannon extension of R along P for every nonzero nonmaximal prime ideal of R. This fact was already proved in a different way in [32, Lemma 1.21.1].

Theorem 1.3.7 (Existence of Shannon Extensions)**.** *Let P be a nonzero nonmaximal prime ideal of a regular local ring R.*

1. *There exists a non-archimedean quadratic Shannon extension of R with R_P as its Noetherian hull.*

2. *If there exists an archimedean quadratic Shannon extension of R contained in R_P, then $\dim R/P \geq 2$.*

Proof. To prove item 1, we use a result of Chevalley that every Noetherian local domain is birationally dominated by a DVR [8]. Let V be a DVR birationally dominating R/P. We apply Lemma 1.3.6 with this R and P. Let $\{\overline{R_n}\}$ be the sequence of LQTs of $\overline{R_0} = R/P$ along V. Let S be the union of the corresponding sequence of LQTs of R given by Proposition 1.3.4. Proposition 1.2.13 implies that $\overline{S} = V$ and Lemma 1.3.6 implies that $S = \widetilde{S}$ is a non-archimedean Shannon extension with R_P as its Noetherian hull.

For item 2, if $\dim R/P = 1$, then $\dim R_P = \dim R - 1$ since an RLR is catenary. If S is an archimedean Shannon extension of R, then $\dim S \leq \dim R - 1$ by [25, Lemma 3.4 and Corollary 3.6]. Therefore R_P does not contain the Noetherian hull of an archimedean Shannon extension of R if $\dim R/P = 1$. $\square$

It is still unknown if there always exists archimedean quadratic Shannon extensions of R contained in R_P for any nonzero nonmaximal prime ideal P of R.

Now, in Theorem 1.3.8 we use Lemma 1.3.6 to characterize the overrings of a regular local ring R that are Shannon extensions of R with Noetherian hull R_P, where P is a nonzero nonmaximal prime ideal of R. Note that by Theorem 1.2.7 such a Shannon extension is necessarily non-archimedean.

Theorem 1.3.8 (Shannon Extensions with Specified Local Noetherian Hull)**.** *Let P be a nonzero nonmaximal prime ideal of a regular local ring R. The quadratic Shannon extensions of R with Noetherian hull R_P are precisely the rings S such that S is a pullback along the residue map $\alpha : R_P \to \kappa(P)$ of a rational rank 1 valuation ring $\mathcal{V}$ birationally dominating R/P whose multiplicity sequence is divergent.*

$$\begin{tikzcd} S = \alpha^{-1}(\mathcal{V}) \arrow[r, two heads] \arrow[d, hook] & \mathcal{V} \arrow[d, hook] \\ R_P \arrow[r, two heads, "\alpha"] & \kappa(P) \end{tikzcd}$$

Proof. If S is a quadratic Shannon extension with Noetherian hull R_P, then by Lemma 1.3.6, S is a pullback along the map $R_P \to \kappa(P)$ of a rational rank 1 valuation ring birationally dominating R/P whose multiplicity sequence is divergent.

Conversely, let S be such a pullback. Let $\{\overline{R_n}\}_{n\geq 0}$ denote the sequence of LQTs of $\overline{R_0} = R/P$ along $\mathcal{V}$ and let $\{R_n\}_{n\geq 0}$ denote the induced sequence of LQTs of $R_0 = R$ as in Proposition 1.3.4. Then Lemma 1.3.6 implies that $S = \bigcup_{n\geq 0} R_n$, so S is a quadratic Shannon extension. $\square$

We describe in Corollary 1.3.9 the quadratic Shannon extensions of R along the prime ideals P of R such that $\dim R/P = 1$.

Corollary 1.3.9. *Let P be a prime ideal of the regular local ring R with* $\dim R/P = 1$. *Then:*

1. *The quadratic Shannon extensions of R with Noetherian hull R_P are precisely the pullbacks along the residue map $R_P \to \kappa(P)$ of the finitely many DVR overrings $\mathcal{V}$ of R/P.*

2. *If R/P is a DVR, then $R + PR_P$ is the unique quadratic Shannon extension of R with Noetherian hull R_P.*

Proof. The Krull-Akizuki Theorem [33, Theorem 11.7] implies that R/P has finitely many valuation overrings, each of which is a DVR. By Theorem 1.3.8 there is a one-to-one correspondence between these DVRs and the Shannon extensions of R with Noetherian hull R_P. This proves item 1. If R/P is a DVR, then by item 1, the pullback $R + PR_P$ of R/P along the map $R_P \to \kappa(P)$ is the unique quadratic Shannon extension of R with Noetherian hull R_P. This verifies item 2. □

1.4 Classification of non-archimedean Shannon extensions

The first classification of non-archimedean quadratic Shannon extensions S that we give in Theorem 1.4.1 is in function of a given regular local subring R of S. Indeed, here a prime ideal of an iterated quadratic transform of R is needed for the description of the overring S as a pullback. This first result follows easily from Theorem 1.3.8.

Later, in Theorem 1.4.5 we give a characterization of certain non-archimedean quadratic Shannon extensions with principal maximal ideal that occur in an algebraic function field of characteristic 0. In this case, we are able to characterize such rings in terms of pullbacks without the explicit requirement of a regular local subring of S. This classification allows us to give an additional source of examples of non-archimedean quadratic Shannon extensions in Example 1.4.7.

Theorem 1.4.1 (Classification of non-archimedean Shannon extensions)**.** *Let R be a regular local ring with $\dim R \geq 2$, and let S be an overring of R. Then S is a non-archimedean quadratic Shannon extension of R if and only if there is a ring $\mathcal{V}$, a nonnegative integer i and a prime ideal P of an iterated local quadratic transform R_i of R, such that*

(a) *$\mathcal{V}$ is a rational rank 1 valuation ring of $\kappa(P)$ that contains the image of R_i/P in $\kappa(P)$ and has divergent multiplicity sequence over this image, and*

(b) *S is a pullback of $\mathcal{V}$ along the residue map $\alpha : (R_i)_P \to \kappa(P)$.*

$$\begin{array}{ccc} S = \alpha^{-1}(\mathcal{V}) & \twoheadrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ (R_i)_P & \xrightarrow{\alpha} & \kappa(P) \end{array}$$

Proof. Suppose $S = \bigcup_i R_i$ is a non-archimedean quadratic Shannon extension of R. By Theorem 1.2.7, the Noetherian hull T of S is a local ring, and by Theorem 1.2.5 there is $i > 0$ and a prime ideal P of R_i such that $T = (R_i)_P$. Since S is a non-archimedean quadratic Shannon extension of R_i, Theorem 1.3.8 implies there is a valuation ring $\mathcal{V}$ such that (a) and (b) hold for i, P, S and $\mathcal{V}$.

Conversely, suppose there is a ring $\mathcal{V}$, a nonnegative integer i and a prime ideal P of R_i that satisfy (a) and (b). By Theorem 1.3.8, S is a quadratic Shannon extension of R_i with Noetherian hull $(R_i)_P$. Thus S is a quadratic Shannon extension of R and it is non-archimedean by Theorem 1.2.7, since its Noetherian hull is local. □

In contrast to Theorem 1.4.1, the pullback description in Theorem 1.4.5 is without reference to a specific regular local subring of S. Indeed, in the proof we construct one regular local subring using resolution of singularities. We frame our proof in terms of projective models.

Definition 1.4.2. Let F be a field and let k be a subfield of F. Let $t_0 = 1$ and assume that $t_1, \ldots, t_n$ are nonzero elements of F such that $F = k(t_1, \ldots, t_n)$. For each $i \in \{0, 1, \ldots, n\}$, define $D_i = k[t_0/t_i, \ldots, t_n/t_i]$. The *projective model* of F/k with respect to $t_0, \ldots, t_n$ is the collection of local rings given by

$$X = \{(D_i)_P : i \in \{0, 1, \ldots, n\},\ P \in \text{Spec }(D_i)\}.$$

For more background on projective models, see [4, Sections 1.6 - 1.8] and [42, Chapter VI, §17].

If k has characteristic 0, then by resolution of singularities (see for example [9, Theorem 6.38, p. 100]) there is a projective model Y of F/k such that every regular local ring in X is in Y, every local ring in Y is a regular local ring, and every local ring in X is dominated by a (necessarily regular) local ring in Y.

We use the following terminology: by a *valuation ring of* F/k we mean a valuation ring V with quotient field F such that k is a subring of V. We also recall that a local ring is an *essentially finitely generated* k-algebra if it is a localization of a finitely generated k-algebra. Moreover, we state the well known Dimension Formula, since we are going to apply it in order to prove Theorem 1.4.5.

Definition 1.4.3. [33, Theorem 15.5, p. 118] Let A be a Noetherian integral domain and let B an extension ring of A which is an integral domain. We say that the Dimension Formula holds between A and B if for every prime ideal P of B, calling $Q = P \cap A$, we have

$$\text{ht } Q + \text{tr.deg}_{\kappa(Q)}\kappa(P) = \text{ht } P + \text{tr.deg}_A B$$

where $\text{tr.deg}_A B$ is the transcendence degree of the quotient field of B over the quotient field of A.

A ring A is *universally catenary* if A is Noetherian and every finitely generated A-algebra is catenary (all the chains of primes between two prime ideals have the same lenght).

Theorem 1.4.4. [33, Theorem 15.6, p. 119] *A Noetherian ring A is universally catenary if and only if the Dimension Formula holds between A/P and B for every prime ideal P of A and every finitely generated extension ring B of A/P.*

When A is an essentially finitely generated k-algebra and F is its quotient field, by Theorem 1.4.4, the Dimension Formula holds between k and A and hence

$$\dim A + \text{tr.deg}_k A/\mathfrak{m}_A = \text{tr.deg}_k F.$$

We prove now Theorem 1.4.5.

Theorem 1.4.5. *Let S be a local domain containing as a subring a field k of characteristic* 0. *Assume that* $\dim S \geq 2$ *and that the quotient field F of S is a finitely generated extension of k. Then the following are equivalent:*

(1) *S has a principal maximal ideal and S is a quadratic Shannon extension of a regular local ring R that is essentially finitely generated over k.*

(2) *There is a regular local overring A of S and a DVR $\mathcal{V}$ of $(A/\mathfrak{m}_A)/k$ such that* *(a)* $\operatorname{tr.deg}_k A/\mathfrak{m}_A + \dim A = \operatorname{tr.deg}_k F$*, and*

(b) *S is the pullback of $\mathcal{V}$ along the residue map $\alpha : A \to A/\mathfrak{m}_A$.*

$$\begin{tikzcd} S = \alpha^{-1}(\mathcal{V}) \arrow[r, two heads] \arrow[d, hook] & \mathcal{V} \arrow[d, hook] \\ A \arrow[r, two heads, "\alpha"] & A/\mathfrak{m}_A \end{tikzcd}$$

Proof. (1) $\Longrightarrow$ (2): Let $x \in S$ be such that $\mathfrak{m}_S = xS$. By Theorem 1.2.5, $S[1/x]$ is the Noetherian hull of S and $S[1/x]$ is a regular ring. Since $\dim S > 1$, the ideal $P = \bigcap_{k>0} x^k S$ is a nonzero prime ideal of S [30, Exercise 1.5, p. 7]. Hence S is non-archimedean. By Theorem 1.2.7, S_P is the Noetherian hull of S and hence $S_P = S[1/x]$. Let $A = S_P$ and $\mathcal{V} = S/P$. By Theorem 1.3.2, S is a pullback of the DVR $\mathcal{V}$ with respect to the map $A \to A/\mathfrak{m}_A$. By assumption, S is a quadratic Shannon extension of a regular local ring R that is essentially finitely generated over k. For sufficiently large i, we have $A = S_P = (R_i)_{P \cap R_i}$ by Theorem 1.2.4(3). Since R_i is essentially finitely generated over R, and R is essentially finitely generated over k, we have that A is essentially finitely generated over k. By the Dimension Formula (Theorem 1.4.4),

$$\operatorname{tr.deg}_k A/\mathfrak{m}_A + \dim A = \operatorname{tr.deg}_k F.$$

This completes the proof that item 1 implies item 2.

(2) $\Longrightarrow$ (1): Let $P = \mathfrak{m}_A$. By item 2b, P is a prime ideal of S, $A = S_P$, $P = PS_P$ and $\mathcal{V} = S/P$. Let $x \in \mathfrak{m}_S$ be such that the image of x in the DVR S/P generates the maximal ideal. Since $P = PS_P$, we have $P \subseteq xS$. Consequently, $\mathfrak{m}_S = xS$, and so S has a principal maximal ideal.

To prove that S is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over k, it suffices by Theorem 1.3.8 to prove:

(i) There is a subring R of S that is a regular local ring essentially finitely generated over k.

(ii) $A = S_P$ is a localization of R at the prime ideal $P \cap R$.

(iii) $\mathcal{V}$ is a valuation overring of $(R + P)/P$ with divergent multiplicity sequence.

Since F is a finitely generated field extension of k and A (as a localization of S) has quotient field F, there is a finitely generated k-subalgebra D of A such that the quotient field of D is F. By item 2a, A/P has finite transcendence degree over k. Let $a_1, \ldots, a_n$ be elements of A whose images in A/P form a transcendence basis for A/P over k. Replacing D with $D[a_1, \ldots, a_n]$, and defining $p = P \cap D$, we may assume that A/P is algebraic over $\kappa(p) = D_p/pD_p$. In fact, since the normalization of an affine k-domain is again an affine k-domain, we may assume also that D is an integrally closed finitely generated k-subalgebra of A with quotient field F. Since D is a finitely generated k-algebra, D is universally catenary. Again by the Dimension Formula we have

$$\dim D_p + \operatorname{tr.deg}_k \kappa(p) = \operatorname{tr.deg}_k F.$$

Therefore, item 2a implies

$$\dim D_p + \operatorname{tr.deg}_k \kappa(p) = \dim A + \operatorname{tr.deg}_k A/P.$$

Since A/P is algebraic over $\kappa(p)$, we conclude that $\dim D_p = \dim A$.

The normal ring A birationally dominates the excellent normal ring D_p, so A is essentially finitely generated over D_p [23, Theorem 1]. Therefore A is essentially finitely generated over k.

Since A is essentially finitely generated over k, the local ring A is in a projective model X of F/k. As discussed before the theorem, resolution of singularities implies that there exists a projective model Y of F/k such that every regular local ring in X is in Y, every local ring in Y is a regular local ring, and every local ring in X is dominated by a local ring in Y.

Since A is a regular local ring in X, A is a local ring in the projective model Y. Let $x_0, \ldots, x_n \in F$ be nonzero elements such that with $D_i := k[x_0/x_i, \ldots, x_n/x_i]$ for each $i \in \{0, 1, \ldots, n\}$, we have

$$Y = \bigcup_{i=0}^{n} \{(D_i)_Q : Q \in \text{Spec } (D_i)\}.$$

Since S has quotient field F, we may assume that $x_0, \ldots, x_n \in S$. Since A is in Y, there is $i \in \{0, 1, \ldots, n\}$ such that $A = (D_i)_{P \cap D_i}$.

By item 2b, $\mathcal{V} = S/P$ is a valuation ring with quotient field A/P. For $a \in A$, let $\overline{a}$ denote the image of a in the field A/P. Since S/P is a valuation ring of A/P, there exists $j \in \{0, 1, \ldots, n\}$ such that

$$(\{\overline{x_k/x_i}\}_{k=0}^{n})(S/P) \;=\; (\overline{x_j/x_i})(S/P). \tag{1.1}$$

Notice that $x_i/x_i = 1 \notin P$. Hence at least one of the $x_k/x_i \notin P$, and Equation 1.1 implies $x_j/x_i \notin P$. Since $A = S_P$ and $P = PS_P$, every fractional ideal of S contained in A is comparable to P with respect to set inclusion. Therefore $P \subsetneq (x_j/x_i)S$. This and Equation 1.1 imply that

$$(x_0/x_i, \ldots, x_n/x_i)S = (x_j/x_i)S. \tag{1.2}$$

Multiplying both sides of Equation 1.2 by x_i/x_j we obtain

$$D_j \;=\; k[x_0/x_j, \ldots, x_n/x_j] \;\subseteq\; S.$$

Let $R = (D_j)_{\mathfrak{m}_S \cap D_j}$. Since Y is a nonsingular model, R is a regular local ring with $R \subseteq S \subseteq A$.

We observe next that $A = R_{P \cap R}$. Since $R \subseteq A$, we have that A dominates the local ring $A' := R_{P \cap R}$. The local ring A' is a member of the projective model Y, and every valuation ring dominating the local ring A in Y dominates also the local ring A' in Y. Since Y is a projective model of F/k, the Valuative Criterion for Properness [22, Theorem II.4.7, p. 101] implies no two distinct local rings in Y are dominated by the same valuation ring. Therefore, $A = A'$, so that $A = R_{P \cap R}$.

Finally, observe that since $\mathcal{V} = S/P$ is a DVR overring of $(R + P)/P$, the multiplicity sequence of S/P over $(R + P)/P$ is divergent. By Theorem 1.3.8, S is a quadratic Shannon extension of R with Noetherian hull $A = R_{P \cap R}$. By Theorem 1.2.7, S is non-archimeean, so the proof is complete. ☐

As an application of Theorem 1.4.5, we describe, for a finitely generated field extension F/k of characteristic 0, the valuation rings with principal maximal ideal that arise as quadratic Shannon extensions of regular local rings that are essentially finitely generated over k, Recall that a valuation ring V of F/k is a *divisorial* valuation ring if

$$\text{tr.deg}_k\, V/\mathfrak{m}_V = \text{tr.deg}_k\, F - 1.$$

Such a valuation ring is necessarily a DVR (apply, e.g., [1, Theorem 1]).

Corollary 1.4.6. *Let F/k be a finitely generated field extension where k has characteristic 0, and let S be a valuation ring of F/k with principal maximal ideal.*

1. *Suppose rank $S = 1$. Then there is a sequence $\{R_i\}$ (possibly finite) of LQTs of a regular local ring R essentially finitely type over k such that $S = \bigcup_i R_i$. This sequence is finite if and only if S is a divisorial valuation ring.*
2. *Suppose rank $S > 1$. Then S is a quadratic Shannon extension of a regular local ring essentially finitely generated over k if and only if S has rank 2 and S is contained in a divisorial valuation ring of F/k.*

Proof. For item 1, assume rank $S = 1$. Hence S is a DVR. By resolution of singularities, there is a nonsingular projective model X of F/k with function field F. Let R be the regular local ring in X that is dominated by S. Let $\{R_i\}$ be the sequence of LQTs of R along S. If $\{R_i\}$ is finite, then $\dim R_i = 1$ for some i, so that R_i is a DVR. Since S is a DVR between R_i and its quotient field, we have $R_i = S$. Otherwise, if $\{R_i\}$ is infinite, then Proposition 1.2.13 implies $S = \bigcup_i R_i$ since S is a DVR. That the sequence is finite if and only if S is a divisorial valuation ring follows from [1, Proposition 4].

For item 2, suppose rank $S > 1$. Assume first that S is a Shannon extension of a regular local ring essentially finitely generated over k. By Theorem 1.2.5(1), $\dim S = 2$. By Theorem 1.4.5, S is a contained in a regular local ring $A \subseteq F$ such that $A/\mathfrak{m}_A$ is the quotient field of a proper homomorphic image of S and

$$\text{tr.deg}_k\, A/\mathfrak{m}_A + \dim A = \text{trdeg}_k\, F. \tag{1.3}$$

We claim A is a divisorial valuation ring of F/k. Since $A/\mathfrak{m}_A$ is the quotient field of a proper homomorphic image of S, it follows that

$$\text{tr.deg}_k\, A/\mathfrak{m}_A < \text{trdeg}_k F. \tag{1.4}$$

From equations 1.3 and 1.4 we conclude that $\dim A \geq 1$. As an overring of the valuation ring S, A is also a valuation ring. Since A is a regular local ring that is not a field, it follows that A is a DVR. Thus $\dim A = 1$ and equation 1.3 implies that

$$\text{tr.deg}_k\, A/\mathfrak{m}_A = \text{trdeg}_k\, F - 1,$$

which proves that A is a divisorial valuation ring.

Finally, suppose rank $S = 2$ and S is contained in a divisorial valuation ring A of F/k. Since S is a valuation ring of rank 2 with principal maximal ideal it follows that $\mathfrak{m}_A \subseteq S$ and $S/\mathfrak{m}_A$ is DVR. Since A is a divisorial valuation ring, we have

$$\text{tr.deg}_k\, A/\mathfrak{m}_A + \dim A = \text{trdeg}_k\, F.$$

As a DVR, A is a regular local ring, so Theorem 1.4.5 implies S is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over k. □

Example 1.4.7. Let k be a field of characteristic 0, let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be algebraically independent over k, and let

$$A = k(x_1, \ldots, x_n)[y_1, \ldots, y_m]_{(y_1, \ldots, y_m)}.$$

Let $\alpha : A \to k(x_1, \ldots, x_n)$ be the canonical residue map. We show that there is a one-to-one correspondence between the DVRs of $k(x_1, \ldots, x_n)/k$ and the quadratic Shannon extensions S of regular local rings that are essentially finitely generated over k, have Noetherian hull A, and have a principal maximal ideal.

For every DVR V of $k(x_1, \ldots, x_n)/k$, the ring $S = \alpha^{-1}(V)$ is by Theorem 1.4.5 a quadratic Shannon extension of a regular local ring that is essentially finitely generated over k. As in the proof that statement 2 implies statement 1 of Theorem 1.4.5, the Noetherian hull of S is A.

Conversely, suppose S is a k-subalgebra of F with principal maximal ideal such that S is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over k and S has Noetherian hull A. As in the proof that item 1 implies item 2 of Theorem 1.4.5, there is a DVR V of $k(x_1, \ldots, x_n)/k$ such that $S = \alpha^{-1}(V)$.

Theorem 1.4.5 concerns quadratic Shannon extensions of regular local rings that are essentially finitely generated over k. In Example 1.4.8 we describe a quadratic Shannon extension of a regular local ring R in a function field for which R is not essentially finitely generated over k.

Example 1.4.8. Let $F = k(x, y, z)$, where k is a field and x, y, z are algebraically independent over k. Let $\tau \in xk[[x]]$ be a formal power series in x such that x and τ are algebraically independent over k. Set $y = \tau$ and define $V = k[[x]] \cap k(x, y)$. Then V is a DVR on the field $k(x, y)$ with maximal ideal xV and residue field $V/xV = k$. Let $V(z) = V[z]_{xV[z]}$. Then $V(z)$ is a DVR on the field F with residue field $k(z)$, and $V(z)$ is not essentially finitely generated over k. Let $R = V[z]_{(x,z)V[z]}$. Notice that R is a 2-dim RLR. The pullback diagram of type $\square^*$

$$\begin{tikzcd}
S = \alpha^{-1}(k[z]_{zk[z]}) \arrow[r, two heads] \arrow[d, hook] & k[z]_{zk[z]} \arrow[d, hook] \\
V(z) \arrow[r, two heads, "\alpha"] & k(z)
\end{tikzcd}$$

defines a rank 2 valuation domain S on F that is by Theorem 1.4.1 a quadratic Shannon extension of R. For each positive integer n, define $R_n = R[\frac{x}{z^n}]_{(z, \frac{x}{z^n})R[\frac{x}{z^n}]}$. Then $S = \bigcup_{n \geq 1} R_n$.

1.5 Quadratic Shannon extensions and GCD domains

As an application of the pullback description of non-archimedean quadratic Shannon extensions given in Sections 1.3 and 1.4, we show in Theorem 1.5.2 that a quadratic Shannon extension S is a coherent domain, a GCD domain or a finite conductor domain if and only if S is a valuation domain. We extend this fact to all the quadratic Shannon extensions S, by applying structural results for archimedean quadratic Shannon extensions from [25].

Definition 1.5.1. An integral domain D is:

1. a *finite conductor domain* if for elements a, b in the field of fractions of D, the D-module $aD \cap bD$ is finitely generated. ([34])

2. a *coherent* domain if every finitely generated ideal is finitely presented.

3. a *GCD domain* if for all $a, b \in D$, $aD \cap bD$ is a principal ideal of D [14, page 76 and Theorem 16.2, p.174].

Chase [7, Theorem 2.2] proves that an integral domain D is coherent if and only if the intersection of two finitely generated ideals of D is finitely generated. Thus a coherent domain is a finite conductor domain. It is clear from the definitions that also a GCD domain is a finite conductor domain.

Examples of GCD domains and finite conductor domains that are not coherent are given by Glaz in [16, Example 4.4 and Example 5.2] and by Olberding and Saydam in [36, Prop. 3.7]. Every Noetherian integral domain is coherent, and a Noetherian domain D is a GCD domain if and only if it is a UFD. Noetherian domains that are not UFDs are examples of coherent domains that are not GCD domains.

Theorem 1.5.2. *Let S be a quadratic Shannon extension of a regular local ring. The following are equivalent:*

1. *S is coherent.*

2. *S is a GCD domain.*

3. *S is a finite conductor domain.*

4. *S is a valuation domain.*

Proof. It is true in general that if S is a valuation domain, then S satisfies each of the first 3 items. As noted above, if S is coherent or a GCD domain, then S is a finite conductor domain. To complete the proof of Theorem 1.5.2, it suffices to show that if S is not a valuation domain, then S is not a finite conductor domain. Specifically, we assume S is not a valuation domain and we consider three cases. In the non archimedean case we use the pullback description of S to show that if S is a finite conductor domain, then it is a valuation domain. In the other two cases, we find a pair of principal fractional ideals of S whose intersection is not finitely generated.

Case 1: S is non-archimedean. By Theorem 1.3.2, there is a prime ideal Q of S and a rational rank 1 valuation ring $\mathcal{V} = S/Q$ of $\kappa(Q)$ such that S_Q is the Noetherian hull of S and S is the pullback of $\mathcal{V}$ along the residue map $\alpha : S_Q \to \kappa(Q)$.

$$\begin{tikzcd} S = \alpha^{-1}(\mathcal{V}) \arrow[r, two heads] \arrow[d, hook] & \mathcal{V} \arrow[d, hook] \\ S_Q \arrow[r, two heads, "\alpha"] & \kappa(Q) \end{tikzcd}$$

We use the following argument given by Brewer and Rutter in [6, Prop. 2] to show that if S is a finite conductor domain, then $T = S_Q$ is a valuation domain: let $a, b \in S$ be nonzero. It suffices to prove that either $aS \subseteq bS$ of $bS \subseteq aS$. Now

$aS \cap bS \supseteq aQ \cap bQ$, and $aQ \cap bQ$ is a nonzero ideal in S_Q. Since S is a finite conductor domain, $aS \cap bS$ is a finitely generated ideal of S. Nakayama's lemma implies that $aS \cap bS \neq aQ \cap bQ$. Choose $y \in (aS \cap bS) \setminus aQ \cap bQ$. Write $y = (d_1 + p_1)a = (d_1 + p_2)b$ with $d_1, d_2 \in S$ and $p_1, p_2 \in Q$. One of the elements d_1 and d_2 is not in Q, say $d_1 \notin Q$. Since $d_1 + p_1 \notin Q$, $d_1 + p_1$ is a unit in S_Q. Therefore

$$a \ = (\ d_1 + p_1)^{-1}(d_2 + p_2)b \ \in \ bS_Q \quad \text{and} \quad aS_Q \ \subseteq \ bS_Q$$

It follows that S_Q is a valuation domain. Since S/Q and S_Q are valuation rings and S/Q has quotient field S_Q/Q, it follows that S is a valuation domain by [10, Theorem 2.4(1)].

Case 2: S is archimedean, but not completely integrally closed. By Theorem 1.2.5, $\dim S \geq 2$. We claim that $\mathfrak{m}_S$ is not finitely generated as an ideal of S. Since $\dim S > 1$, if $\mathfrak{m}_S$ is a principal ideal, then $\bigcap_i \mathfrak{m}_S^i$ is a nonzero prime ideal of S, a contradiction to the assumption that S is archimedean. Thus $\mathfrak{m}_S$ is not principal. By [25, Proposition 3.5], this implies $\mathfrak{m}_S^2 = \mathfrak{m}_S$. From Nakayama's Lemma it follows that $\mathfrak{m}_S$ is not finitely generated. Since S is not completely integrally closed, there is an almost integral element θ over S that is not in S. By [25, Corollary 6.6], $\mathfrak{m}_S = \theta^{-1}S \cap S$.

Case 3: S is archimedean and completely integrally closed. By Theorem 1.2.5, $\dim S \geq 2$. By Theorems 1.2.5 and 1.2.11, $S = T \cap W$, where W is the rank 1 nondiscrete valuation ring with associated valuation $w(-)$ as in Definition 1.2.6 and T is a UFD that is a localization of S. Since $\sum_{n \geq 0} w(\mathfrak{m}_n) < \infty$ by Theorem 1.2.7, and since $\mathfrak{m}_n S$ is principal and generated by a unit of T for $n \gg 0$, the w-values of units of T generate a non-discrete subgroup of $\mathbb{R}$.

Since S is archimedean, Theorem 1.2.7 implies T is a non-local UFD. Therefore there exist elements $f, g \in S$ that have no common factors in T. As in Case 1, we consider $I = fS \cap gS$. Since $S = T \cap W$, it follows that

$$\begin{aligned} I &= (fT \cap gT) \cap (fW \cap gW) \\ &= fT \cap gT \cap \{a \in W \mid w(a) \geq \max\{w(f), w(g)\}\}. \end{aligned}$$

Assume without loss of generality that $w(f) \geq w(g)$.

For $a \in I$, write $a = (\frac{a}{f})f$ in S and consider $w(a)$. Since $\frac{a}{f}$ is divisible by g in T, it is a non-unit in T, and thus it is a non-unit in S. Since W dominates S, it follows that $w(\frac{a}{f}) > 0$ and thus $w(a) > w(f)$.

We claim that $\mathfrak{m}_S I = I$. Since the w-values of the units of T generate a non-discrete subgroup of $\mathbb{R}$, for any $\epsilon > 0$, there exists an unit x in T with $0 < w(x) < \epsilon$. Then for $a \in I$ and for some x with $0 < w(x) < w(a) - w(f)$, we have $\frac{a}{x} \in I$ and thus $a \in \mathfrak{m}_S I$. Since $\mathfrak{m}_S I = I$ and $I \neq (0)$, Nakayama's Lemma implies that I is not finitely generated.

In every case, we have constructed a pair of principal fractional ideals of S whose intersection is not finitely generated. We conclude that if S is not a valuation domain, then S is not a finite conductor domain. $\square$

Chapter 2

Local monoidal transforms and GCD domains

2.1 Introduction

A prime ideal $\mathfrak{p}$ of a regular local ring R is said a regular prime if $R/\mathfrak{p}$ is again a regular local ring (i.e. $\mathfrak{p}$ is generated by regular parameters). Let $(R, \mathfrak{m})$ be a regular local ring and let $\mathfrak{p}$ be a regular prime ideal of R with ht $\mathfrak{p} \geq 2$. Let $(R_1, \mathfrak{m}_1)$ be a local ring on the blowup Proj $R[\mathfrak{p}t]$ of $\mathfrak{p}$ such that R_1 dominates R, that is, $\mathfrak{m}_1 \cap R = \mathfrak{m}$. Let $x \in \mathfrak{p} \setminus \mathfrak{p}^2$ be such that R_1 is a localization of $R[\frac{\mathfrak{p}}{x}]$ at a maximal ideal $\mathfrak{m}_1$. The ring

$$R_1 = R\left[\frac{\mathfrak{p}}{x}\right]_{\mathfrak{m}_1}$$

is called a *local monoidal transform* of R.

In his paper [38], David Shannon discussed several properties of sequences of local monoidal transforms focusing on the parallelism between the algebraic point of view of regular local rings and its global geometric interpretation of projective models.

A useful tool for the investigation on local monoidal transforms is the canonical map Spec $R_1 \to$ Spec R sending a prime ideal Q of R_1 to its contraction $Q \cap R$. One property of interest is the biregularity of such map at one prime ideal of R_1.

Definition 2.1.1. Given an extension of integral domains $A \hookrightarrow B$, we say that the map Spec $B \to$ Spec A is biregular at a prime ideal $Q \subseteq B$ if $B_Q = A_{Q \cap A}$.

We record in Proposition 2.1.2 well known properties of R_1 and of the map Spec $R_1 \to$ Spec R. The references for them are [38, Section 2] and [2, Section 1].

Proposition 2.1.2. *Assume notation as above. Then:*

1. *R_1 is a regular local ring.*

2. *$\mathfrak{p}R_1 = xR_1$ is a height one prime ideal of R_1.*

3. *$xR_1 \cap R = \mathfrak{p}$ and $(R_1)_{xR_1}$ is the order valuation ring defined by the powers of $\mathfrak{p}$.*

4. *$R_1[1/x]$ is a localization of R. Indeed $R_1[1/x] = R[1/x]$.*

5. *The map* Spec $R_1 \to$ Spec R *is biregular at every prime ideal* $\mathfrak{q}$ *of* R_1 *such that* $xR_1 \nsubseteq \mathfrak{q}$. *In particular, if* $\mathfrak{q}$ *is a height one prime of* R_1 *other than* xR_1, *then* $\mathfrak{q} \cap R$ *is a height one prime of* R, *and* $R_{\mathfrak{q}\cap R} = (R_1)_{\mathfrak{q}}$.

The properties of R_1 given by Proposition 2.1.2 are often used inductively to study sequences of local monoidal transform.

Definition 2.1.3. Let $\{(R_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ be an infinite directed sequence of local monoidal transforms of an RLR R, that is, $R_{n+1} \in$ Proj $R_n[\mathfrak{p}_n t]$, where $\mathfrak{p}_n$ is a regular prime ideal of the RLR R_n and the inclusion map $R_n \hookrightarrow R_{n+1}$ is a local map (i.e. $\mathfrak{m}_n \subseteq \mathfrak{m}_{n+1}$).

Since we are interested in studying the ideal theoretic properties of an infinite union of such rings, we assume all the ring $(R_n, \mathfrak{m}_n)$ to have the same dimension $d \geq 3$. The infinite directed union $S = \bigcup_{n\in\mathbb{N}} R_n$ is, like in the "quadratic case", a local integrally closed overring of R with maximal ideal $\mathfrak{m}_S = \bigcup_{n\in\mathbb{N}} \mathfrak{m}_n$. We call S a *monoidal Shannon extension* of R.

In [25], the authors call an infinite directed union of local quadratic transform of an RLR R a Shannon extension of R.

As said in Chapter 1, here we distinguish such class of rings, that we call quadratic Shannon extensions, from the very larger class of rings of monoidal Shannon extensions.

While there are many valuation overrings that birationally dominates R which are not quadratic Shannon extension (for instance the unique sequence of local quadratic transform of R along a valuation overring of rank at least 3 yields to a Shannon extension which is not a valuation ring [25, Theorem 8.1]), under certain hypothesis, easily fulfilled by RLRs arising in a geometric context, every valuation overring birationally dominating R is a monoidal Shannon extension.

This is one of the most important result of Shannon's paper [38] and we state it for completeness.

Theorem 2.1.4. *Let* $(R, \mathfrak{m})$ *be an excellent regular local ring of dimension greater than one such that one of the following conditions hold:*

1. *The residue field* $R/\mathfrak{m}$ *has characteristic zero (in any dimension of* R*).*

2. R *is equicharacteristic of dimension at most* 3.

Then every valuation ring V *that birationally dominates* R *is a union of local monoidal transform of* R.

To prove it Shannon used the notion of a *strongly principalizable* ring extension. Given an extension of rings $R \subseteq V$, we say that it is principalizable if, for every ideal $I \subseteq R$, IR_t is principal in some iterated monoidal transform R_t of R birationally dominated by V. An extension $R \subseteq V$ is strongly principalizable if, for any iterated local monoidal transform R_n of R birationally dominated by V, the extension $R_n \subseteq V$ is principalizable.

Shannon proved that if V is a valuation overring that birationally dominates R and the extension $R \subseteq V$ is strongly principalizable, then V is a directed union of local monoidal transform of R.

It was previously known that an extension $R \subseteq V$ is often strongly principalizable. In particular for an excellent regular local ring R it was proved by Hironaka in [28] when the residue field of R has characteristic zero and by Abhyankar in [2] in the equicharacteristic case if the dimension of R is at most 3.

Hence, in a ring theoretic language, assuming "sufficiently good" hypothesis on R we can say that every valuation overring that birationally dominates R is a monoidal Shannon extension of R. The converse is clearly not true, as Shannon itself mentioned providing the examples 1.2.3. Such examples are quadratic Shannon extensions but, since a local quadratic transform can be factorized in a finite number of local monoidal transforms, it turns out that a quadratic Shannon extension is also a monoidal Shannon extension.

An interesting problem is now to characterize with ideal-theoretic tools the monoidal Shannon extension which are not quadratic extensions. It seems to be an hard and wide open problem. In the monoidal setting, we lost good properties of quadratic transform like the uniqueness of a sequence along a valuation overring. Indeed, given a valuation ring V birationally dominating a RLR R, we may have some possible different "directions" of transforming along V.

In this chapter we provide examples of classes of monoidal non-quadratic Shannon extensions and we study their properties, focusing sometimes on the case in which R has dimension 3.

We prove here many of the results holding in this case in the more general context where $\dim R_n = d$ and ht $\mathfrak{p}_n = d-1$ for every n.

The most general setting for monoidal transform is when ht $\mathfrak{p}_n = 2$ for every n. Such transforms are called by Shannon *elementary monoidal transform* since any monoidal transform can be factorized in elementary monoidal tranforms (see [38, Remark 2.5]).

Elementary monoidal transforms represent minimal possible birational extensions of RLRs. Indeed, if R_1 is an elementary monoidal transform of R and there exists one regular local ring R' such that $R \subseteq R' \subseteq R_1$ and both inclusion are birational, then either $R = R'$ or $R_1 = R'$ [38, Proposition 3.7].

Hence we can define different classes of monoidal Shannon extensions in the following way: call $\mathcal{M}_i(R)$ the set of the monoidal Shannon extensions of R such that ht $\mathfrak{p}_n \geq i$ for every n where $2 \leq i \leq d = \dim R$.

What said until now implies that there is an inclusion

$$\mathcal{M}_i(R) \subseteq \mathcal{M}_{i-1}(R).$$

With this notation, the set of quadratic Shannon extensions is $\mathcal{M}_d(R)$ and the set of the all possible monoidal Shannon extensions is $\mathcal{M}_2(R)$.

In the study of monoidal Shannon extensions we are going to focus on the GCD property of such rings (see Definition 1.5.1). In the last section of Chapter 1 we showed that a quadratic Shannon extension of a regular local ring is a GCD domain if and only if it is a valuation ring. In the monoidal case we find many Shannon extensions which are GCD domain but not valuation domain.

In order to prove this, in Section 2.2 we give some criteria that make an integral domain a GCD domain.

In Section 2.3, we apply the results of Section 2.2 to a specific example of a monoidal non quadratic Shannon extension.

In Section 2.4, after some general results, we characterize a monoidal Shannon extension $S = \bigcup_{n \in \mathbb{N}} R_n \in \mathcal{M}_{d-1}(R)$ in function of the behavior of the prime ideals

$\mathfrak{p}_n$. We prove that an infinite chain of such primes has as a union a prime ideal of S and the localization of S at this prime ideal is a quadratic Shannon extension of a regular local ring of dimension $d-1$.

In Section 2.5, we describe a specific Noetherian overring of S which can be built under some hypothesis on the chains of primes realizing S.

We apply the previous results in Section 2.6 to give other examples of classes of monoidal Shannon extensions and study their basic properties.

In Section 2.7 we extend what proved in Sections 2.2 and 2.3 to characterize the GCD domains among a class of monoidal Shannon extensions.

Many results of this chapter have been obtained in collaboration with W. Heinzer, B. Olberding and M. Toeniskoetter.

2.2 GCD domains

We recall that an integral domain D is a GCD domain if for all $a, b \in D$, $aD \cap bD$ is a principal ideal of D. (See Definition 1.5.1)

The motivation for the results in this section is the following: in Theorem 1.5.2 is proved that a quadratic Shannon extension of a regular local ring is a GCD domain if and only if it is a valuation domain.

In Section 2.3, and later in Section 2.7, we will show that for monoidal Shannon extensions the situation is more subtle. We construct monoidal Shannon extensions that are GCD domains but not valuation domains. The construction gives a monoidal Shannon extension S such that there exist nonassociate prime elements $x, z \in S$ such that S_{xS}, S_{zS} and $S[1/xz]$ are GCD domains. These properties, along with the results we prove in this section, imply then that S is a GCD domain.

Let x be a nonzero prime element of an integral domain D. We prove in Theorem 2.2.5 that D is a GCD domain if and only if D_{xD} and $D[1/x]$ are GCD domains. Corollary 2.2.6 generalizes this fact to finitely many nonassociate prime elements and it will imply that the monoidal Shannon extension S in Construction 2.3.1 is a GCD domain.

The survey paper of Dan Anderson [5] and the book of Robert Gilmer [14] are good references for GCD domains.

All the following results are needed to prove the main theorem of this section, namely Theorem 2.2.5. In this section we will put ourselves in a general context, working with an integral domain D instead of speaking of Shannon extensions.

Lemma 2.2.1. *Let xD be a nonzero principal prime ideal of an integral domain D. Then $D = D[1/x] \cap D_{xD}$, and $yD = yD[1/x] \cap yD_{xD}$ for every y in the field of fractions of D.*

Proof. Clearly $D \subseteq D[1/x] \cap D_{xD}$. If $y \in D[1/x]$, then $x^n y \in D$ for some integer $n \geq 0$. If $y \notin D$, then $n \geq 1$, and we can choose n minimal such that $x^n y \in D$. Then $x^n y = a \in D \setminus xD$. If also $y \in D_{xD}$, then $y = b/c$, where $b \in D$ and $c \in D \setminus xD$. Then

$$y = \frac{a}{x^n} = \frac{b}{c} \quad \Longrightarrow \quad ac = x^n b \in xD.$$

This contradicts the fact that xD is prime and a and c are not in xD. We conclude that $D = D[1/x] \cap D_{xD}$. Since multiplication by y distributes over the intersection, the second equality also holds. $\square$

Proposition 2.2.2. *Let x be a nonzero prime element of an integral domain D with quotient field F, and let $P = \bigcap_{n\in\mathbb{N}} x^n D$. Then:*

1. *P is a prime ideal, $xP = P$, every prime ideal of D properly contained in xD is contained in P, and D_{xD}/PD_{xD} is a DVR.*
2. *Let I be an ideal of D such that $I \nsubseteq PD[1/x] = P$, and $ID[1/x] \cap D = I$. If $ID[1/x]$ is a principal ideal of $D[1/x]$, then I is a principal ideal of D.*
3. *Let J be a proper principal ideal in $D[1/x]$ such that $J \nsubseteq P$. Then $J \cap D$ is a principal ideal in D and there exists $b \in D \setminus xD$ such that $J \cap D = bD$*
4. *Let I be an ideal of D such that $ID[1/x]$ is a finitely generated ideal of $D[1/x]$. Then $(D :_F I)D_{xD} = (D_{xD} :_F ID_{xD})$.*

Proof. The assertions in item 1 are well known, see [30, Exercise 1.5, p. 7].

The assumption in item 2 that $I = ID[1/x] \cap D$ is equivalent to the statement that x is a regular element on D/I. Every element in $ID[1/x]$ has the form a/x^n, where $a \in I$ for some $n \geq 0$. Let $y \in I$ be such that $yD[1/x] = ID[1/x]$. Since $I \nsubseteq P$, $y \notin xD$. It follows that x is regular on D/yD, for if $a \in D$ and $ax \in yD$, then $ax = yb$, and $y \notin xD$ implies $b \in xD$. Therefore

$$yD \;=\; yD[1/x] \,\cap\, D \;=\; ID[1/x] \,\cap D \;=\; I.$$

For item 3, first notice that P is a common ideal of D and $D[1/x]$. Then, since $J \nsubseteq P$ and J is a proper ideal of $D[1/x]$, $J \cap D \nsubseteq xD$. Let $a \in D$ be such that $aD[1/x] = J$. Then $a \notin P$, and for some integer $n \geq 0$, we have $a/x^n = b \in D \setminus xD$. It follows that $bD[1/x] = J$, and $bD = J \cap D$.

For item 4, let I be an ideal of D such that $ID[1/x]$ is a finitely generated ideal of $D[1/x]$. By Lemma 2.2.1, we have $D = D_{xD} \cap D[1/x]$. Thus

$$(D :_F I) = (D_{xD} :_F ID_{xD}) \cap (D[1/x] :_F ID[1/x]).$$

Since localization commutes with finite intersections, this implies

$$(*) \qquad (D :_F I)D_{xD} = (D_{xD} :_F ID_{xD}) \cap (D[1/x] :_F ID[1/x])D_{xD}.$$

By item 1, every prime ideal of D properly contained in xD is contained in P, so $D[1/x]D_{xD} = D_P$. Therefore,

$$\begin{aligned}(D[1/x] :_F ID[1/x])D_{xD} &= (D[1/x] :_F ID[1/x])D[1/x]D_{xD}\\ &= (D[1/x] :_F ID[1/x])D_P\\ &= (D_P :_F ID_P),\end{aligned}$$

where the last equality follows from the fact that $ID[1/x]$ is a finitely generated ideal of $D[1/x]$ and D_P is a localization of $D[1/x]$. Therefore, by $(*)$,

$$(D :_F I)D_{xD} = (D_{xD} :_F ID_{xD}) \cap (D_P :_F ID_P).$$

Since $D_{xD} \subseteq D_P$, this implies

$$(D :_F I)D_{xD} = (D_{xD} :_F ID_{xD}).$$

□

Remark 2.2.3. Assume notation as in Proposition 2.2.2.

(1) If $P \neq 0$, then P is not finitely generated as an ideal of D. It may happen that $PD[1/x]$ is a principal ideal of $D[1/x]$. Thus the assumption in Proposition 2.2.2(2) that $I \nsubseteq P$ is necessary.

(2) If $y \in P$ and $y \neq 0$, then $yD[1/x] \subseteq D$, and $I = yD[1/x]$ is a common ideal of D and $D[1/x]$. The ideal I is principal as an ideal of $D[1/x]$, but is not finitely generated as an ideal of D.

(3) Assume that D_{xD} is a valuation domain. Then D_P is a valuation overring of D_{xD}. It may happen that P is contained in more than one maximal ideal of D. For instance, let S be the monoidal Shannon extension of Construction 2.3.1, and let $D = S_{xS} \cap S_{zS}$. Then xD and zD are distinct maximal ideals of D and $\bigcap_{n\in\mathbb{N}} x^n D = P = \bigcap_{n\in\mathbb{N}} z^n D$.

We recall now that a domain D is a *finite conductor* if the intersection of any two principal ideal of D is finitely generated.

Lemma 2.2.4. *If D is a local finite conductor domain with principal maximal ideal M, then D is a valuation domain.*

Proof. Let $x \in D$ such that $M = xD$, and let $P = \bigcap_{i>0} x^i D$. By Proposition 2.2.2(1), P is a prime ideal of D. Then $D_P = D[1/x]$ and $PD[1/x] = P$. Thus we have a commutative diagram that describes D as a pullback:

$$\begin{tikzcd} D \arrow[r, two heads] \arrow[d, hook] & D/P \arrow[d, hook] \\ D_P \arrow[r, two heads] & D_P/P \end{tikzcd} \tag{2.1}$$

Since D is a finite conductor domain, it follows from [12, Proposition 4.3] that D_P is a valuation domain. Gabelli and Houston mentioned in [12] that the proof proceeds as in the proof given by Brewer and Rutter in [6, Prop. 2]. This ideal-theoretic proof has been described while proving case 1 of Theorem 1.5.2. By Proposition 2.2.2(1), D/P is a DVR. Since D/P and D_P are valuation rings and D/P has quotient field D_P/P, it follows that D is a valuation domain [10, Theorem 2.4(1)]. $\square$

Theorem 2.2.5. *Let xD be a nonzero prime ideal of an integral domain D. Then D is a GCD domain if and only if D_{xD} and $D[1/x]$ are GCD domains.*

Proof. Since localizations commute with finite intersections, every localization of a GCD domain is a GCD domain. Hence if D is a GCD domain, then D_{xD} and $D[1/x]$ are GCD domains.

Conversely, assume that D_{xD} and $D[1/x]$ are GCD domains. Since every GCD domain is a finite conductor domain, Lemma 2.2.4 implies that D_{xD} is a valuation domain. Let F denote the quotient field of D. Let a, b be nonzero elements in D, and let $I = aD \cap bD$. It suffices to prove that I is principal.

Let $J = (a, b)D$. Since J is finitely generated and D_{xD} is a valuation domain, we have by Proposition 2.2.2(4)

$$JJ^{-1}D_{xD} = JD_{xD}(D_{xD} : JD_{xD}) = D_{xD}.$$

Thus $JJ^{-1} \nsubseteq xD$. Choose $q \in J^{-1}$ such that $qJ = (qa, qb)D \nsubseteq xD$. Either $qa \notin xD$ or $qb \notin xD$, and $qI = qaD \cap qbD$ is principal if and only if I is principal. Replacing a, b with qa, qb and relabeling if necessary, we may assume $a \notin xD$. We may also assume that $b \neq 0$. Then $J^{-1} = \frac{1}{a}D \cap \frac{1}{b}D$, and $abJ^{-1} = I$. To prove that I is principal it suffices to show that $K := aJ^{-1}$ is principal. Now

$$K = aJ^{-1} = \ D \ \cap \ \tfrac{a}{b}D \ = \ \{c \in D \mid cb \ \in \ aD\} = aD :_D b.$$

Notice that if $xy \in K$ with $y \in D$, then $y \in K$. To see this, assume that $xyb = az$, where $z \in D$. Then $a \notin xD$ and, since xD a prime ideal, $z \in xD$. Hence $yb \in aD$. From this, it follows $K = KD[1/x] \cap D$.

Denote $P = \bigcap_{i>0} x^i D$. Then P is a prime ideal by Proposition 2.2.2. Since $a \notin xD$, it follows that $a \notin P$, so $b \in aD_P$ and hence $KD_P = D_P$. Therefore $K \nsubseteq P$. If $K = D$, then K is principal and I is principal.
Assume that K is a proper ideal of D. Then $KD[1/x] \cap D = K$ implies that $KD[1/x]$ is a proper ideal of $D[1/x]$.

Since $D[1/x]$ is a GCD domain, $KD[1/x]$ is a principal ideal of $D[1/x]$. Proposition 2.2.2(2,3) implies that $KD[1/x] \cap D = yD$, where $y \in D \setminus xD$.

Since $K = KD[1/x] \cap D$, it follows $K = yD$ is principal. □

Corollary 2.2.6. *Let $x = x_1 \cdots x_r$ be an element of an integral domain D, where $x_1, \ldots, x_r$ are nonassociate prime elements of D. If $D_{x_i D}$ is a GCD domain for $1 \leq i \leq r$ and $D[\frac{1}{x}]$ is a GCD domain, then D is a GCD domain.*

Proof. We proceed by induction on r, where the $r = 1$ case is given by Theorem 2.2.5. Assume the hypotheses of the corollary. Since $x_r \notin x_i D$ for $1 \leq i < r$, it follows that

$$D[\tfrac{1}{x}]_{x_r D[\frac{1}{x}]} = D[\tfrac{1}{x_1 \cdots x_{r-1}}]_{x_r D[\frac{1}{x_1 \cdots x_{r-1}}]}.$$

Since $D[\frac{1}{x}] = D[\frac{1}{x_1 \cdots x_{r-1}}][\frac{1}{x_r}]$, we have $D[\frac{1}{x_1 \cdots x_{r-1}}]$ is a GCD domain by Theorem 2.2.5. We conclude by the inductive hypothesis that D is a GCD domain. □

The next corollary gives a criterion, based of properties of $D[1/x]$ and D_{xD}, to see whether D is a Bezout domain. We recall that a Bezout domain is an integral domain such that any finitely generated ideal is principal and a Prüfer domain is a domain A such that A_P is a valuation domain for every prime ideal P of A. It is a well known fact that a Prüfer domain is a GCD domain if and only if it is a Bezout domain. This fact can be seen as a generalization in a non-Noetherian context of the also well known fact that a Dedekind domain is a PID if and only if it is a UFD.

Corollary 2.2.7. *Let xD be a nonzero principal maximal ideal of an integral domain D. If $D[1/x]$ is a Bezout domain and D_{xD} is a finite conductor domain, then D is a Bezout domain.*

Proof. Lemma 2.2.4 implies that D_{xD} is a valuation domain. Since xD is a maximal ideal and $D[1/x]$ is a Bezout domain, $D_{\mathfrak{q}}$ is a valuation domain for every prime ideal $\mathfrak{q}$ of D. Therefore D is a Prüfer domain. By Theorem 2.2.5, D is a GCD domain. Therefore D is a Bezout domain. □

2.3 A GCD monoidal Shannon extension

In this section we give a first example of a monoidal Shannon extension of a 3-dimensional RLR which is neither a quadratic Shannon extension nor a valuation domain. We discuss some properties of this ring, including the property of being a GCD domain.

Construction 2.3.1. Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring with maximal ideal $\mathfrak{m} = (x, y, z)R$. Define

$$R_1 = R\left[\frac{y}{x}\right]_{(x, \frac{y}{x}, z)R[\frac{y}{x}]},$$

and

$$R_2 = R_1\left[\frac{y}{xz}\right]_{(x, \frac{y}{xz}, z)R_1[\frac{y}{xz}]}.$$

Thus R_1 is the local monoidal transform of R obtained by blowing up the prime ideal $(x, y)R$, dividing by x, and localizing at the maximal ideal generated by $x, y/x$ and z; and R_2 is the local monoidal transform of R_1 obtained by blowing up the prime ideal $(\frac{y}{x}, z)R_1$, dividing by z, and localizing at the maximal ideal generated by x, $\frac{y}{xz}$ and z.

Define R_{2n+1} and R_{2n+2} inductively so that R_{2n+1} is the local monoidal transform of R_{2n} obtained by blowing up the prime ideal $(x, \frac{y}{x^n z^n}))R_{2n}$, dividing by x, and localizing at the maximal ideal generated by x, $\frac{y}{x^{n+1}z^n}$ and z; and R_{2n+2} is the local monoidal transform of R_{2n+1} obtained by blowing up the prime ideal $(\frac{y}{x^{n+1}z^n}, z)R_{2n+1}$, dividing by z, and localizing at the maximal ideal generated by x, $\frac{y}{x^{n+1}z^{n+1}}$ and z. Call $S = \bigcup_{n\in\mathbb{N}} R_n$.

We record properties of S and of the sequence $\{R_n\}$ in Theorem 2.3.2. Among other things, we prove that this ring is not Noetherian and admits a unique minimal Noetherian overring T. We call T the Noetherian hull of S like in the quadratic case.

Theorem 2.3.2. *Assume notation as in Construction 2.3.1. Let $S = \bigcup_{n\in\mathbb{N}} R_n$, and let $\mathfrak{p} = y\mathcal{V} \cap S$ where $\mathcal{V} = R_{yR}$. Then:*

1. *The maximal ideal of $S = \bigcup_{n\in\mathbb{N}} R_n$ is $\mathfrak{m}_S = (x, z)S$ and*

$$\mathfrak{p} = \bigcap_{n\in\mathbb{N}} x^n S = \bigcap_{n\in\mathbb{N}} z^n S$$

   is a non finitely generated prime ideal of S.

2. *The principal ideals xS and zS are nonmaximal prime ideals of S of height 2.*

3. *$S/\mathfrak{p}$ is a 2-dimensional RLR that is isomorphic to R/yR. This isomorphism defines 1-to-1 correspondence of the prime ideals of S of height 2 containing $\mathfrak{p}$ with the prime ideals of R of height 2, containing y.*

4. *The localizations S_{xS} and S_{zS} are rank 2 valuation domains, and the map* Spec $S \to$ Spec R *is not biregular at these two prime ideals.*

5. *Let $\mathfrak{q}$ be a prime ideal of S of height 1. If $\mathfrak{q} \neq \mathfrak{p}$, then $\mathfrak{q}$ is a principal ideal generated by a prime element $f \in R$ such that $f \notin (x, y)R \cup (y, z)R$. It follows that $R_{fR} = S_{\mathfrak{q}}$.*

6. *The map* Spec $S \to$ Spec R *is biregular on the height 2 primes of* S *containing* y *other than* xS *and* zS*, and is biregular on all the height 1 primes of* S.

7. $S^* = S[\frac{1}{xz}] = T$ *is a 2-dimensional regular Noetherian UFD that is the complete integral closure of* S *and the unique minimal Noetherian overring of* S.

8. $S = T \cap V_1 \cap V_2$*, where* $V_1 = S_{xS}$ *and* $V_2 = S_{zS}$ *are the valuation rings of item 2.*

9. S *is a GCD domain.*

Proof. By definition $\frac{y}{x^i z^j} \in \mathfrak{p}$ for all $i, j \in \mathbb{N}$. Moreover, all the elements of this form are necessary to generate $\mathfrak{p}$. Thus $\mathfrak{p}$ is non finitely generated. It is also clear that $\mathfrak{p} = \bigcap_{n \in \mathbb{N}} x^n S$ and $\mathfrak{p} = \bigcap_{n \in \mathbb{N}} z^n S$. Hence $\mathfrak{m}_n \subseteq (x, z)S$ for every $n \in \mathbb{N}$. This proves item 1.

For item 2 observe that, since x and z are prime elements in R_n for every $n \in \mathbb{N}$, then they are prime elements in S. Moreover:

$$(S_{xS})_{\mathfrak{p}} = (S_{zS})_{\mathfrak{p}} = S_{\mathfrak{p}} = \mathcal{V}.$$

By [10, Theorem 2.4], it follows that xS and zS are prime ideals of height 2 and S_{xS} and S_{zS} are rank 2 valuation domains. The DVR $\mathcal{V}$ is the rank 1 valuation overring of both S_{xS} and S_{zS}.

For item 3, just notice that

$$\frac{R}{yR} = \frac{R_1}{(\mathfrak{p} \cap R_1)} = \cdots = \frac{R_n}{(\mathfrak{p} \cap R_n)} = \cdots = \frac{S}{\mathfrak{p}}.$$

Therefore $S/\mathfrak{p}$ is a 2-dimensional RLR that is isomorphic to R/yR.

Since $xS \cap R = (x, y)R$ and $zS \cap R = (y, z)R$ and $R_{(x,y)R}$ and $R_{(y,z)R}$ are 2-dimensional RLRs, the map Spec $S \to$ Spec R is not biregular at xS and zS. This proves item 4.

For item 5, let $\mathfrak{q}$ be a height 1 prime ideal of S with $\mathfrak{q} \neq \mathfrak{p}$. Then $\mathfrak{q} \nsubseteq xS \cup zS$. By repeated applications of Proposition 2.1.2(6), the map Spec $S \to$ Spec R is biregular at $\mathfrak{q}$, that is $S_{\mathfrak{q}} = R_{\mathfrak{q} \cap R}$. Since $\dim S_{\mathfrak{q}} = 1$, $\mathfrak{q} \cap R = fR$, where f is a prime element of R with $f \notin (x, y)R \cup (y, z)R$. Repeated applications of Proposition 2.1.2(5) imply that $\mathfrak{q} = fS$.

Item 6 follows from items 4 and 5.

For item 7, Proposition 2.1.2 implies that $S[\frac{1}{xz}]$ is a localization of R. Hence $S[\frac{1}{xz}]$ is a regular Noetherian UFD. Since $R/yR = S/\mathfrak{p}$, we have $\dim S[\frac{1}{xz}] = 2$. Since $\frac{y}{x^i z^j} \in S$ for all $i, j \in \mathbb{N}$, $S[\frac{1}{xz}]$ is almost integral over S.

It follows that $S^* = S[\frac{1}{xz}]$.

If A is a Noetherian overring of S, then $\frac{1}{xz}$ is almost integral and therefore integral over A. Since $xz \in S \subseteq A$, it follows that $\frac{1}{xz} \in A$, cf. [30, page 10, Theorem 15]. We conclude that $S[\frac{1}{xz}] = T$ is the Noetherian hull of S.

To prove item 8, let

$$\frac{a}{x^i z^j} \in S[\frac{1}{xz}] \cap S_{xS} \cap S_{zS},$$

where $a \in S$ and $i \geq 0$ and $j \geq 0$ are minimal for such a representation. Then $a \in S \setminus (xS \cup zS)$ and xS and zS distinct nonzero principal prime ideals implies $i = 0 = j$. Therefore $S = T \cap V_1 \cap V_2$.

Since T and S_{xS} and S_{zS} are GCD domains, Corollary 2.2.6 implies that S is a GCD domain. This proves item 9 and completes the proof of Theorem 2.3.2. $\square$

The rings $V_1 = S_{xS}$ and $V_2 = S_{zS}$ are playing the role for this monoidal Shannon extension that is played by the boundary valuation ring V in the case where S is a quadratic Shannon extension. Indeed, we recall that by Theorem 1.2.5(2) a quadratic Shannon extension is the intersection of its Boundary valuation ring and of its Noetherian hull.

Instead considering the limit point of the order valuation rings $\{V_n\}$ of the sequence $\{R_n\}$ in Construction 2.3.1, that is the ring

$$V \;=\; \bigcup_{n\geq 0} \bigcap_{i\geq n} V_i = \{a \in F \mid \mathrm{ord}_{R_i}(a) \geq 0 \text{ for } i \gg 0\},$$

we do not find the equality $S = T \cap V$. To see this, we can take for instance the element x/z which is in T and in V_n for all n but it is not in S.

It is possible to prove in another way that S is a GCD domain using its pullback representation.

Remark 2.3.3. The monoidal Shannon extension S obtained in Construction 2.3.1 is a pullback of the canonical homomorphism $S \to S/\mathfrak{p}$ with respect to the canonical injection $S \hookrightarrow T = S[\frac{1}{xz}]$ as in the following diagram

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d, hook] & S/\mathfrak{p} \cong R/yR \arrow[d, hook] \\ T \arrow[r, two heads] & T/\mathfrak{p} \end{tikzcd} \tag{2.2}$$

In the terminology of Evan Houston and John Taylor in [29], Equation 2.2 is a pullback diagram of type $\square$. This differs from a pullback of type $\square^*$ in that the integral domain $T/\mathfrak{p}$ in the lower right of the diagram is not a field.

We want to apply the next Theorem to this ring. In [29], the authors define a generalization of GCD domains. An integral domain D with an overring E is an E-*GCD domain* if $J^{-1} \cap E$ is principal for every finitely generated ideal J of D. Notice that D is a GCD domain if it is an E-GCD domain for E equal to the quotient field of D.

Theorem 2.3.4. [29, Theorem 5.11] *Consider the following diagram of type $\square$ where I is a maximal t-ideal of T and $T = (I : I)$.*

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\ T \arrow[r, two heads] & E = T/I \end{tikzcd} \tag{2.3}$$

Then, R is a GCD domain if and only if T is a GCD domain, D and E have the same quotient field, D is an E-GCD domain and the natural map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is surjective.

Corollary 2.3.5. *The ring S in Construction 2.3.1 is a GCD domain.*

Proof. Let F denote the field of fractions of S and let F' denote the field of fractions of $S/\mathfrak{p}$. Then $T = (\mathfrak{p} : \mathfrak{p}) = \{a \in F \mid a\mathfrak{p} \subseteq \mathfrak{p}\}$.

The ideal $\mathfrak{p}$ is principal in the Noetherian ring T and hence it is a maximal t-ideal.

The rings T and $S/\mathfrak{p}$ are Noetherian regular UFDs and therefore they are GCD domains. By permutability of localizations and residue class formations, $T/\mathfrak{p}$ is a localization of $S/\mathfrak{p}$ with respect to the multiplicative set generated by the image of xz in $S/\mathfrak{p}$.

We claim that $S/\mathfrak{p}$ is a $T/\mathfrak{p}$-GCD domain. Indeed, let J be an ideal of $S/\mathfrak{p}$. The fractional ideal J^{-1} is principal generated by an element of F'. Without loss of generality we may assume $J^{-1} = (1/g)S/\mathfrak{p}$ for $g \in S/\mathfrak{p}$. Hence, if $1/g \in T/\mathfrak{p}$, $J^{-1} \cap T/\mathfrak{p} = J^{-1}$ and it is principal.

Assume instead $1/g \notin T/\mathfrak{p}$ and take $f \in J^{-1} \cap T/\mathfrak{p}$, then $f = d/g$ for some $d \in gT/\mathfrak{p} \cap S/\mathfrak{p}$. Since the images of x and z in $S/\mathfrak{p}$ are prime elements and they do not divide g (g is not a unit in $T/\mathfrak{p}$), we have $gT/\mathfrak{p} \cap S/\mathfrak{p} = gS/\mathfrak{p}$ and therefore $f \in S/\mathfrak{p}$. It follows that $J^{-1} \cap T/\mathfrak{p} = S/\mathfrak{p}$ is a principal ideal of $S/\mathfrak{p}$ and thus $S/\mathfrak{p}$ is a $T/\mathfrak{p}$-GCD domain.

Finally, the multiplicative group $\mathcal{U}(T)$ of units of T maps surjectively onto the group of units $\mathcal{U}(T/\mathfrak{p})$ of $T/\mathfrak{p}$. With $\mathcal{U}(S/\mathfrak{p})$ the group of units of $S/\mathfrak{p}$, it follows that the natural map $\mathcal{U}(T) \to \mathcal{U}(T/\mathfrak{p})/\mathcal{U}(S/\mathfrak{p})$ is surjective. Now Theorem 2.3.4 implies that S is a GCD domain. □

2.4 Generalities about directed unions of local monoidal transforms

In this section we give a more systematic and detailed study of the ring structure of a monoidal Shannon extension. Our setting is the following: let $\{(R_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ be an infinite directed sequence of local monoidal transforms of a regular local ring R where $R_{n+1} = R_n[\mathfrak{p}_n/x_n]$ for every n. Assume $\dim R_n = d \geq 3$ for all n and let $S = \bigcup_{n\in\mathbb{N}} R_n$. Also let $\mathfrak{m}_S = \bigcup_{n\in\mathbb{N}} \mathfrak{m}_n$ denote the unique maximal ideal of S.

We start by looking at properties correspondent to those proved for quadratic Shannon extensions in Theorem 1.2.4. In such theorem it is proved that a quadratic extension is Noetherian if and only if is a DVR. We are going to prove that a monoidal extension is Noetherian if and only if it is a RLR of dimension less than $\dim R$.

Proposition 2.4.1. *Let S be a monoidal Shannon extension of a regular local ring R. If $\mathfrak{m}_S$ is finitely generated, then it is minimally generated by at most $d-1$ elements. Moreover, any minimal generating set of $\mathfrak{m}_S$ is a regular sequence on S and part of a regular system of parameters of R_n for $n \gg 0$.*

Proof. Take a minimal generating set for the maximal ideal $\mathfrak{m}_S$ of S, say $\mathfrak{m}_S = (x_1, \ldots, x_t)$, and take $n \geq 0$ such that $x_1, \ldots, x_t \in R_n$. The inclusion $\mathfrak{m}_n \subseteq \mathfrak{m}_S$ induces a map $\mathfrak{m}_n/\mathfrak{m}_n^2 \to \mathfrak{m}_S/\mathfrak{m}_n\mathfrak{m}_S = \mathfrak{m}_S/\mathfrak{m}_S^2$. Let $\overline{x_1}, \ldots, \overline{x_t}$ denote the images of $x_1, \ldots, x_t$ in $\mathfrak{m}_n/\mathfrak{m}_n^2$. By Nakayama's Lemma, the images of $\overline{x_1}, \ldots, \overline{x_t}$ in $\mathfrak{m}_S/\mathfrak{m}_S^2$ are linearly independent over $S/\mathfrak{m}_S$, hence $\overline{x_1}, \ldots, \overline{x_t}$ are linearly independent over the subfield $R_n/\mathfrak{m}_n$. We conclude that $x_1, \ldots, x_t$ are part of a regular system of parameters for R_n, and in particular $t \leq d$.

Moreover, $x_1, \ldots, x_t$ is a regular sequence in S. To see this, let $0 \leq k < t$ and let $ax_{k+1} \in (x_1, \ldots, x_k)S$ for some $a \in S$. Write $ax_{k+1} = \sum_{i=1}^{k} c_i x_i$ for some $c_1, \ldots, c_k \in S$. Take $n \geq 0$ such that $x_1, \ldots, x_{k+1}, c_1, \ldots, c_k, a \in R_n$, so that $ax_{k+1} \in (x_1, \ldots, x_k)R_n$. Since $x_1, \ldots, x_{k+1}$ are part of a regular system of parameters in R_n and hence a regular sequence on R_n, it follows that $a \in (x_1, \ldots, x_k)R_n \subseteq (x_1, \ldots, x_k)S$. We conclude that $x_1, \ldots, x_t$ is a regular sequence in S.

Finally, since R_{n+1} is a proper birational extension essentially of finite type of R_n, Zariski's Main Theorem implies that ht $\mathfrak{m}_n R_{n+1} < d$ for all $n \geq 0$, so since $\dim R_n = d$ for all $n \geq 0$, it follows that $\mathfrak{m}_n R_{n+1} \subsetneq \mathfrak{m}_{n+1}$ for all $n \geq 0$. If $t = d$, then $\mathfrak{m}_n = (x_1, \ldots, x_t)R_n$ for $n \gg 0$, contradicting this fact, so we conclude that $t \leq d - 1$. $\square$

Corollary 2.4.2. *Let S be a monoidal Shannon extension of a regular local ring R. If S is Noetherian, then S is a regular local ring of dimension at most* $\dim R - 1$.

Proof. Since $\mathfrak{m}_S$ is finitely generated, Proposition 2.4.1 implies that $\mathfrak{m}_S$ is minimally generated by a regular sequence on S, so S is a regular local ring. Since $\mathfrak{m}_S$ is minimally generated by at most $\dim R - 1$ elements, Krull's Altitude Theorem implies that $\dim S \leq \dim R - 1$. $\square$

Example 2.4.3. It is possible to construct a Noetherian monoidal Shannon extension of a RLR $(R, \mathfrak{m})$ under the assumption of the existence of a DVR V that birationally dominates R but it is not a prime divisor over R (we recall that a prime divisor is a valuation overring birationally dominating R such that $\operatorname{trdeg}(V/\mathfrak{m}_V, R/m)$ is equal to $\dim R - 1$.). For instance take $R = k[x, y]_{(x,y)}$ where $y \in xk[[x]]$ is a formal power series in x and y, x are algebraically independent over k. Then, the DVR $V = k[[x]] \cap k(x, y)$ has residue field k and hence is not a prime divisor over R. Consider the sequence $(R_n, \mathfrak{m}_n)$ of local quadratic transform of R along V. By Proposition 1.2.13, $V = \bigcup_{n \in \mathbb{N}} R_n$. Take some indeterminates $z_1, \ldots, z_n$ over the quotient field of R and consider the sequence of rings

$$R_n[z_1, \ldots, z_n]_{(\mathfrak{m}_n, z_1, \ldots, z_n)}.$$

Each ring is a local monoidal transform of the previous ring obtained localizing the blow up of the prime ideal $\mathfrak{p}_n = \mathfrak{m}_n$. The directed union of this sequence is the RLR $V[z_1, \ldots, z_n]_{(x, z_1, \ldots, z_n)}$.

The example obtained in Construction 2.3.1 can be seen as a prototype to study the structure of a class of monoidal Shannon extension. In that particular sequence of local monoidal transforms we see that the prime ideals $\mathfrak{p}_n$ form two different chains, which are

$$\mathfrak{p}_0 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_4 \subseteq \cdots \subseteq \bigcup_{k \geq 0}^{\infty} \mathfrak{p}_{2k} = xS$$

and

$$\mathfrak{p}_1 \subseteq \mathfrak{p}_3 \subseteq \mathfrak{p}_5 \subseteq \cdots \subseteq \bigcup_{k \geq 0}^{\infty} \mathfrak{p}_{2k+1} = zS.$$

The directed union of each chain is a prime ideal of the union ring S. We prove that this is a general fact for a monoidal Shannon extension and we explain how the structure of these rings can be understood looking at the chains formed by the prime ideals $\mathfrak{p}_n$. In most of the cases we will assume the locus ideals to form a finite number of chains.

The terminology of the following definition is inspired by the concept of "fundamental locus" of a birational transformation used by Zariski in [40].

Definition 2.4.4. We call the prime ideals $\mathfrak{p}_n$, the *locus ideals* of the sequence $\{(R_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ and the ideal $\mathcal{L} = \sum_{n\geq 0}^{\infty} \mathfrak{p}_n \subseteq S$, the locus ideal of S. Notice that $\mathcal{L}$ is a proper ideal of S.

Definition 2.4.5. Let $\{\mathfrak{p}_{n_i}\}_{i\in\mathbb{N}}$ a family of locus ideals of the sequence $\{R_n\}_{n\in\mathbb{N}}$. The family $\{\mathfrak{p}_{n_i}\}_{i\in\mathbb{N}}$ is a *chain* if $\mathfrak{p}_{n_i} \subseteq \mathfrak{p}_{n_{i+1}}$ for every i. An infinite chain is said *minimal* if for all but finitely many $m \notin \{n_i\}_{i\in\mathbb{N}}$, the ideal $\mathfrak{p}_m \nsubseteq \bigcup_{i\in\mathbb{N}} \mathfrak{p}_{n_i}$.

Theorem 2.4.6. *Let $\{\mathfrak{p}_{n_i}\}_{i\in\mathbb{N}}$ be an infinite chain of locus ideals of the sequence $\{R_n\}_{n\in\mathbb{N}}$ and for every n, take $x_n \in \mathfrak{p}_n$ such that $\mathfrak{p}_n S = x_n S$. Denote $Q = \bigcup_{i\in\mathbb{N}} \mathfrak{p}_{n_i}$. Then:*

1. *Q is a prime ideal of S and it is maximal if and only if $Q \cap R_{n_i} = \mathfrak{m}_{n_i}$, for infinitely many i.*
2. *Q is either principal or it is not finitely generated.*

Proof. For item 1, take $ab \in Q = \bigcup_{i\in\mathbb{N}} \mathfrak{p}_{n_i}$. Then there exists n_i such that $ab \in \mathfrak{p}_{n_i}$ and hence either a or b is in $\mathfrak{p}_{n_i} \subseteq Q$ and hence Q is prime.

Assume $Q \cap R_{n_i} = \mathfrak{m}_{n_i}$ for infinitely many i. Since for every n, $\mathfrak{m}_n \subseteq \mathfrak{m}_{n+1}$, we have $Q = \bigcup_{n\geq 0}^{\infty} \mathfrak{m}_n = \mathfrak{m}_S$. Conversely if Q is maximal, $Q \cap R_{n_i} = \mathfrak{m}_{n_i}$ for every i.

For item 2, since $\mathfrak{p}_{n_i} S = x_{n_i} S$ for some $x_{n_i} \in \mathfrak{p}_{n_i}$, we have that $Q = \bigcup_{i\in\mathbb{N}} \mathfrak{p}_{n_i} = \bigcup_{i\in\mathbb{N}} x_{n_i} S$ is an ascending union of principal ideals. It follows that Q it is finitely generated if and only if it is principal. $\square$

Definition 2.4.7. Let $\{\mathfrak{p}_{n_i}\}_{i\in\mathbb{N}}$ an infinite chain of locus ideals of the sequence $\{R_n\}_{n\in\mathbb{N}}$. We call the prime ideal $Q = \bigcup_{i\in\mathbb{N}} \mathfrak{p}_{n_i}$ a *chain-prime ideal.*

We apply by induction Proposition 2.1.2(4) to study the biregularity of the map $\operatorname{Spec} S \to \operatorname{Spec} R$. Next result will be an important tool for our further investigation.

Lemma 2.4.8. *Let S be a monoidal Shannon extension of R. Let P be a prime ideal of S which contains only finitely many locus ideals $\mathfrak{p}_n$ (for instance a chain-prime ideal). Then, there exists $N \geq 0$, which depends on P, such that the map* $\operatorname{Spec} S \to \operatorname{Spec} R_N$ *is biregular at P.*

Proof. Take N such that $\mathfrak{p}_n \nsubseteq P \cap R_n$ for all $n \geq N$. Applying Proposition 2.1.2(4) with an inductive argument, we get $(R_N)_{P\cap R_N} = (R_n)_{P\cap R_n}$ for all $n \geq N$ and this gives $S_P = (R_N)_{P\cap R}$. $\square$

In the introduction of this chapter we defined the set $\mathcal{M}_i(R)$ consisting of the monoidal Shannon extensions of R such that $\operatorname{ht} \mathfrak{p}_n \geq i$ for every n. Set, as before, $d = \dim R_n$ for every n. As application of Proposition 2.4.8, we can bound the dimension of the valuation rings belonging to each set $\mathcal{M}_i(R)$.

Proposition 2.4.9. *Let $S \in \mathcal{M}_{d-i}(R)$ be a monoidal Shannon extension of R and let Q be a prime ideal of S which contains infinitely many locus ideal $\mathfrak{p}_n$. then,*

$$\operatorname{ht} Q \geq \dim S - i.$$

Proof. Assume there is an ascending chain of prime ideals of S,

$$Q \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_i \subsetneq \mathfrak{m}_S,$$

of lenght $i+1$ between Q and $\mathfrak{m}_S$. Since all the inclusions are strict, we can find a sufficiently large n such that

$$\mathfrak{p}_n \subseteq Q \cap R_n \subsetneq Q_1 \cap R_n \subsetneq \ldots \subsetneq Q_i \cap R_n \subsetneq \mathfrak{m}_S \cap R_n = \mathfrak{m}_n.$$

But this contradicts the assumption of $S \in \mathcal{M}_{d-i}(R)$, since in this case ht $\mathfrak{p}_n \geq d-i$. $\square$

Theorem 2.4.10. *Let $V \in \mathcal{M}_{d-i}(R)$ be a valuation ring which is a monoidal Shannon extension of R. Then,*

$$\dim V \leq i+2$$

Proof. First assume $\dim V = 1$. In this case, by Theorem 1.2.5(1) $V \in \mathcal{M}_d(R)$ is a quadratic Shannon extension.

Thus assume $\dim V \geq 2$ and call Q the height two prime ideal of V. The ring V_Q is a two dimensional valuation domain and therefore it is not a localization of R_n for any n. Hence by Proposition 2.4.8, Q must contain infinitely many locus ideals $\mathfrak{p}_n$. By Proposition 2.4.9, $2 = \text{ht } Q \geq \dim V - i$, and hence $\dim V \leq i+2$. $\square$

This Theorem generalizes the fact that a valuation ring which is a quadratic Shannon extensions has dimension at most two (Theorem 1.2.5(1)). We can use an application of it to show that the inclusion between the sets of monoidal extensions $\mathcal{M}_i(R) \subseteq \mathcal{M}_{i-1}(R)$ is proper for every $d = \dim R$ and every $i = 3, \ldots, d$.

Theorem 2.4.11. *Let R be a regular local ring of dimension d with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_d)$. For every $d \geq 3$ and for every $i = 2, \ldots, d-1$, there exists a discrete valuation ring $V \in \mathcal{M}_{d-i}(R) \setminus \mathcal{M}_{d-i+1}(R)$.*

Proof. We recall that the rank of a valuation overring of R is at most d. By Theorem 2.4.10, any valuation ring V of rank i is not in $\mathcal{M}_{d-i+3}(R)$. We want to prove by induction on d that there is a discrete valuation ring of rank i, $V \in \mathcal{M}_{d-i+2}(R)$. For $d = 3$, a discrete valuation ring of rank 3 is a monoidal non-quadratic Shannon extension and therefore is in $\mathcal{M}_2(R)$. Indeed, if R satisfies the assumption of Theorem 2.1.4, this is clear, otherwise we refer to an explicit construction made in Example 2.6.6.

Hence, we assume the statement true for d and we prove it for $e = d+1$. Let R a regular local ring of dimension e and let $R' = R_{(x_1,\ldots,x_{e-1})}$. Such ring R' is a regular local ring of dimension d, and hence we can find a discrete valuation ring V' of rank $j-1$ which is in $\mathcal{M}_{d-j+3}(R')$. Write $V' = \bigcup_{i=0}^{\infty} R'_n$ where $R'_n = R'_{n-1}[\frac{\mathfrak{q}_n}{y_n}]_{\mathfrak{m}'_n}$. Let V be the discrete rank j valuation overring of R occurring in the pullback diagram

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d, hook] & \dfrac{R}{(x_1, \ldots, x_{e-1})} \arrow[d, hook] \\ V' \arrow[r, two heads] & \kappa(\mathfrak{m}_{V'}). \end{tikzcd}$$

Consider the ring S obtained in a sequence of local monoidal transform of R in the following way: in the even steps $0, 2, 4, \ldots, 2k$ we blow up the ideals $\mathfrak{q}_k \cap R_{2k}$ dividing by y_k and localizing at $((\mathfrak{m}_{V'} \cap R_{2k}[\frac{\mathfrak{q}_k \cap R_{2k}}{y_k}]) + x_e R_{2k}[\frac{\mathfrak{q}_k \cap R_{2k}}{y_k}])$.

In the odd steps $1, 3, 5, \ldots, 2k+1$ we blow up the maximal ideal of R_{2k+1} dividing by x_e and localizing at $((\mathfrak{m}_{V'} \cap R_{2k+1}[\mathfrak{m}_{2k+1}/x_e]) + x_e R_{2k+1}[\mathfrak{m}_{2k+1}/x_e])$.

The direct union S of this local monoidal transforms is equal to V. Indeed, by construction $\frac{S}{\mathfrak{m}_{V'} \cap S} = \frac{R}{(x_1, \ldots, x_{e-1})}$ and $V' = S_{(V' \cap S)}$. Moreover $\mathfrak{m}_{V'} \cap S = \bigcap_{i \geq 0} x_e^i S \subseteq tS$ for every $t \notin \mathfrak{m}_{V'} \cap S$, and therefore $\mathfrak{m}_{V'} \cap S = \mathfrak{m}_{V'}$ and this implies that S is the pullback of the previous diagram.

Since $V' \in \mathcal{M}_{d-j+3}(R')$, the ideals $\mathfrak{q}_k \cap R_{2k}$ have height at least $d-j+3 = e-j+2$. Hence all the locus ideals of the sequence along V have height at least $e-j+2$ and thus $V \in \mathcal{M}_{e-j+2}(R)$. $\square$

Choosing $i = d-1$ we get a set of monoidal Shannon extensions with some nice properties, in particular we can prove that the localization of a monoidal extension $S \in \mathcal{M}_{d-1}(R)$ at a non maximal chain-prime ideal is a quadratic Shannon extension of a RLR of dimension $d-1$. This fact allows to understand more properties of these rings. Notice that if $\dim R = 3$, then all the monoidal Shannon extensions are of this form and thus the next results well describe this case.

Proposition 2.4.12. *Let $S = \bigcup_{i \in \mathbb{N}} R_n \in \mathcal{M}_{d-1}(R)$ and let $\{\mathfrak{p}_{n_i}\}_{i \in \mathbb{N}}$ be an infinite collection of locus ideals of the sequence $\{R_n\}_{n \in \mathbb{N}}$. Denote $Q = \bigcup_{i \in \mathbb{N}} \mathfrak{p}_{n_i}$. If Q is not maximal, then Q is prime if and only if $\{\mathfrak{p}_{n_i}\}_{i \in \mathbb{N}}$ is a chain.*

Proof. When $\{\mathfrak{p}_{n_i}\}_{i \in \mathbb{N}}$ is a chain, then Q is prime by item 1 of Theorem 2.4.6. Conversely, if Q is prime and not maximal, for every $n_i >> 0$, we have $\mathfrak{p}_{n_i} = Q \cap R_{n_i}$ again by Theorem 2.4.6, because ht $\mathfrak{p}_{n_i} = d-1$ for $n_i >> 0$. Thus:

$$\mathfrak{p}_{n_i} = Q \cap R_{n_i} \subseteq Q \cap R_{n_{i+1}} = \mathfrak{p}_{n_{i+1}}.$$

This proves the thesis. $\square$

Theorem 2.4.13. *Let $S \in \mathcal{M}_{d-1}(R)$ and let $Q = \bigcup_{i \in \mathbb{N}} \mathfrak{p}_{n_i} \subsetneq \mathfrak{m}_S$ a chain-prime ideal of S. Then the chain $\{\mathfrak{p}_{n_i}\}_{i \in \mathbb{N}}$ is minimal and S_Q is a quadratic Shannon extension of a RLR of dimension $d-1$.*

Proof. Since $Q \subsetneq \mathfrak{m}_S$, then $Q \cap R_n \subsetneq \mathfrak{m}_n$ for $n >> 0$. Hence, by assumption on the height of the primes $\mathfrak{p}_n$, we have $Q \cap R_{n_i} = \mathfrak{p}_{n_i}$ for $i >> 0$. Now, since for $n < m$, $Q \cap R_n \subseteq Q \cap R_m$, then $Q \cap R_n = \mathfrak{p}_n$ if and only if $n \in \{n_i\}_{i \in \mathbb{N}}$ an this means that the chain is minimal.

In this way we proved that $Q \subsetneq \mathfrak{m}_S$ implies that the chain $\{\mathfrak{p}_{n_i}\}_{i \in \mathbb{N}}$ is minimal. Now consider the rings $R'_n = (R_n)_{(QS_Q \cap R_n)}$. Set theoretically we have $S_Q = \bigcup_{n \geq 0}^{\infty} R'_n$ but we need to show that these rings form a sequence of local quadratic transforms of the RLR $R_{\mathfrak{p}_0}$.

For any $k \notin \{n_i\}_{i \in \mathbb{N}}$, since the chain is minimal, $x_k S = \mathfrak{p}_k S \nsubseteq Q$ and hence $x_k \notin Q$. The ring R_{k+1} is a localization of $R_k[\frac{\mathfrak{p}_k}{x_k}]$ at a maximal ideal containing $\mathfrak{p}_k$, therefore $R'_{k+1} = R'_k$.

Hence we can restrict ourselves to consider the directed union $S_Q = \bigcup_{i \in \mathbb{N}} R'_{n_i}$. For any large $i >> 0$, the contraction $Q \cap R_{n_i}$ is a prime ideal strictly contained in the maximal ideal $\mathfrak{m}_{n_i}$ and hence $Q \cap R_{n_i} = \mathfrak{p}_{n_i}$. By this fact and since $\mathfrak{p}_{n_i} R'_{n_i}$ is the maximal ideal of R'_{n_i}, for every i the ring $R'_{n_{i+1}}$ is a local quadratic transform of R'_{n_i} and this completes the proof of the theorem. $\square$

2.5 A Noetherian overring of some monoidal Shannon extensions

We recall the notion of arithmetic rank of an ideal:

Definition 2.5.1. The arithmetical rank of an ideal I of a ring A is

$$\min\{n : \sqrt{I} = \sqrt{(x_1, \dots, x_n)A}, \text{ where } x_1, \dots, x_n \in I\}.$$

The maximal ideal of a quadratic Shannon extension has arithmetical rank one (Theorem 1.2.4(3)). We show that under the assumption of arithmetical rank one for each chain-prime ideals of a monoidal Shannon extension, it is possible to build a specific Noetherian overring of S which is a UFD like the Noetherian hull of a quadratic Shannon extension (which is described in Theorem 1.2.5).

Theorem 2.5.2. *Let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal Shannon extension of R with finitely many chain-prime ideals $Q_1, \dots, Q_c$. Then the following assertions are equivalent:*

1. *There exists an uniform $N >> 0$, such that the map* $\operatorname{Spec} S \to \operatorname{Spec} R_N$ *is biregular at any prime ideal P of S such that $P \not\supseteq Q_j$ for all $j = 1, \dots, c$.*
2. *For every $i = 1, \dots, c$, Q_i has arithmetical rank one.*

Proof. Assume item 1. First we show that any element $x \in \mathfrak{m}_S$ is contained in at most finitely many height one primes of S. Since the chain-prime ideals of S are a finite number, if x is contained in infinitely many height one primes, it follows by the assumption that x is contained in infinitely many height one prime P such that $S_P = (R_N)_{P \cap R}$ for a sufficiently large N.

Hence, for $n \geq N$ such that $x \in R_n$, we would have x contained in infinitely many height one primes of R_n, but this is impossible since a Noetherian ring has the finite character.

Now, consider a chain of locus ideal. Set $\mathfrak{p}_{n_i} R_{n_i+1} = x_{n_i} R_{n_i+1}$. Hence $\mathfrak{p}_{n_i} S = x_{n_i} S$ and therefore

$$x_{n_i} S \subseteq x_{n_{i+1}} S \subseteq x_{n_{i+2}} S \subseteq \ldots \subseteq Q = \bigcup_{j \geq i} x_{n_j} S.$$

Since x_{n_i} is contained in only finitely many height one primes of S, it follows that there exists $j \geq i$ such that x_{n_j} is not contained in any height one prime of S (except possibly Q). We can also find j large enough to have x_{n_j} not contained in any other non maximal chain-prime ideal Q_1, otherwise we would have $Q = Q_1$ by minimality of the two chains (Theorem 2.4.13).

We claim $Q = \sqrt{x_{n_j} S}$. Indeed, suppose $x_{n_j} \in P$ with P a prime ideal that does not contain Q. Hence P is not a chain-prime ideal and it does not contain any chain-prime ideal, since by Proposition 2.4.9, the height of a chain-prime ideal is at least $\dim S - 1$. It follows that x_{n_j} is in the maximal ideal of S_P which, by assumption of item 1, is equal to the Noetherian ring $R_{(P \cap R)}$. Thus x_{n_j} is contained in an height one prime of S contained in P and this is a contradiction. It follows that $Q = \sqrt{x_{n_j} S}$.

Assume now item 2 and write for every $i = 1, \dots, c$, $Q_i = \sqrt{x_{n_i} S}$ where $x_{n_i} \in R_{n_i}$. Call $N = \max n_i$. For $n \geq N$, since there are finitely many chains, any locus ideal $\mathfrak{p}_n$ is such that $\sqrt{\mathfrak{p}_n S} = Q_i$ for some i. Hence if a prime ideal P of S does not contain any of the chain-prime ideals, then it does not contain any locus ideal $\mathfrak{p}_n$ for $n \geq N$. Applying Proposition 2.1.2(4) with an inductive argument, we get $(R_N)_{P\cap R_N} = (R_n)_{P\cap R_n}$ for all $n \geq N$ and this gives $S_P = (R_N)_{P\cap R_N}$. ☐

These conditions are satisfied by the ring of Construction 2.3.1 discussed in Section 2.3, since in that case the two chain-prime ideals are principal.

Theorem 2.5.3. *Let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal Shannon extension of R with finitely many chain-prime ideals $Q_1, \dots, Q_c$.*

Assume that for every i, $Q_i = \sqrt{x_i S}$ for some $x_i \in S$. Then the overring

$$T = S\left[\frac{1}{x_1 \cdots x_c}\right]$$

is a localization of R_n for a large n and then it is a Noetherian UFD.

Proof. Let $n >> 0$ such that $x_1, \dots, x_c \in R_n$. We show that T is flat over R. By Richman's criterion [37, Theorem 2], T is flat over R if and only if for every maximal ideal M of T, $T_M = (R_n)_{M\cap R_n}$. Maximal ideals of T naturally correspond to prime ideals of S not containing $x_1, \dots, x_c$, and by flatness of T over S, $T_M = S_P$ where $P = M \cap S$. By Theorem 2.5.2, all the primes of S which does not contain any chain-prime ideal are biregular over R_N, for $N >> 0$. Thus we get that $S_P = (R_N)_{P\cap R_N} = (R_N)_{M\cap R_N}$ and this implies the flatness of T over R_N.

Now, by a theorem of Heinzer and Roitman [27, Theorem 2.5], since R_n is a Noetherian UFD, T is a localization of R_n and it is again a Noetherian UFD. ☐

2.6 Two classes of monoidal Shannon extensions

Now we discuss with more details two classes of monoidal Shannon extensions in $\mathcal{M}_{d-1}(R)$. The first one is formed by the monoidal Shannon extensions with principal maximal ideal, the second one by the monoidal Shannon extensions with only one chain-prime ideal P properly contained in $\mathfrak{m}_S$ and having an element $x \notin P$ which is a regular parameter in all the rings R_n. The setting of this section is the same of the previous one and we assume $\dim R_n = d \geq 3$ for all n.

Proposition 2.6.1. *Let S be any monoidal Shannon extension of S such that $\mathfrak{m}_S = xS$ is principal. Then $\mathfrak{m}_S$ is a chain-prime ideal.*

Proof. Consider the collection of locus ideals $\mathcal{C} = \{\mathfrak{p}_{n_i} \mid x \in \mathfrak{p}_{n_i}\}$. This collection is infinite, since otherwise we would find an $n >> 0$ and $y \in \mathfrak{m}_n \setminus xR_n$ such that also $y \in \mathfrak{m}_S \setminus xS$. We prove that $\mathcal{C}$ is a chain, proving that, if $x \in \mathfrak{p}_{n_i}$, then $\mathfrak{p}_{n_i} R_{n_i+1} = xR_{n_i+1}$. Indeed, if $x \in \mathfrak{p}_{n_i} R_{n_i+1} = tR_{n_i+1}$, we have $t \in \mathfrak{m}_S$ and x/t is a unit in S. Hence x/t is a unit in R_{n_i+1} and the proof is complete. ☐

Now we describe the locus ideals of a quadratic Shannon extension seen as a monoidal extension.

Lemma 2.6.2. *A quadratic Shannon extension of a regular local ring R is a monoidal Shannon extension of R with only one chain-prime ideal equal to the maximal ideal $\mathfrak{m}_S$.*

Proof. Let $S = \bigcup_{n\in\mathbb{N}} R_n$ be a quadratic Shannon extension of R. Consider the first rings of the correspondent sequence of local quadratic transforms: R, $R_1 = R[\mathfrak{m}/x]_{\mathfrak{m}_1}$, $R_2 = R_1[\mathfrak{m}_1/x_1]_{\mathfrak{m}_2}$. We can obtain the ring R_2 as iterated monoidal transform of R in the following way: take a regular prime ideal $\mathfrak{p}$ which contains the regular parameter x and write $\mathfrak{m} = \mathfrak{p} + aS$ for a regular parameter $a \in \mathfrak{m}$. We have

$$R \subseteq R' = R[\mathfrak{p}/x]_{\mathfrak{m}'} \subseteq R_1 = R'[(a,x)R'/x]_{\mathfrak{m}_1} \subseteq R'_1 = R_1[\mathfrak{p}_1/x_1]_{\mathfrak{m}'_1} \subseteq R_2$$

with $\mathfrak{m}' \subseteq \mathfrak{m}_1$ and $\mathfrak{p}_1$ contains x and x_1. Going ahead iterating this process we can factor the sequence $\{(R_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ in a sequence of local monoidal transforms with the requested properties. □

The converse of Lemma 2.6.2 is not true in general. There are many monoidal non quadratic Shannon extension with only one chain-prime ideal equal to $\mathfrak{m}_S$. But instead it turns out to be true if we assume $S \in \mathcal{M}_{d-1}(R)$ and $\mathfrak{m}_S$ to be principal. To prove this we are going to use the characterization of quadratic Shannon extensions as pullbacks given in Theorem 1.3.8.

Theorem 2.6.3. *Let S be any monoidal Shannon extension of S such that $\mathfrak{m}_S = xS$ is principal. Let $Q = \bigcap_{j\geq 0}^{\infty} x^j S$. The following are equivalent:*

1. *S is a quadratic Shannon extension of R_n for some n.*
2. *The ring S_Q is a localization of R_n at the prime ideal $Q \cap R_n$ for some n.*
3. *There are only finitely many locus ideals $\mathfrak{p}_n$ contained in Q.*

Proof. First consider the case in which $Q = (0)$ and therefore S is a DVR by [30](Exercise 1.5). Now, since S birationally dominates R, S is a quadratic Shannon extension by [21](Proposition 3.4). Clearly (2) and (3) are true in this case.

Suppose now $Q \neq (0)$. By [10](Theorem 2.4), S occurs in the pullback diagram

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d, hook] & S/Q \arrow[d, hook] \\ S_Q \arrow[r, two heads] & \kappa(Q) \end{tikzcd}$$

where S/Q is a DVR. As a consequence of Theorem 1.3.8, (1) and (2) are equivalent. Indeed, if S is a quadratic Shannon extension, then it is non archimedean and S_Q is its Noetherian hull. Hence (2) follows by Theorem 1.2.5(4). Conversely, if S_Q is equal to $(R_n)_{Q\cap R_n}$, Theorem 1.3.8 applies directly to say that S is a quadratic Shannon extension of R_n since a DVR has divergent multiplicity sequence with respect to any of its regular local subrings (See Definition 1.2.12).

We observe that (3) implies (2) by Proposition 2.4.8. To conclude we need to show that (1) implies (3). By Lemma 2.6.2, the locus ideals $\mathfrak{p}_n$ form eventually a unique chain whose union is $\mathfrak{m}_S$. Since $Q \subsetneq \mathfrak{m}_S$ only finitely many of them can be contained in Q. □

Corollary 2.6.4. *Let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal non-quadratic Shannon extension of S such that $\mathfrak{m}_S = xS$ is principal. Let $Q = \bigcap_{j\geq 0}^{\infty} x^j S$. Then Q is a chain-prime ideal.*

Proof. By Theorem 2.6.3 the ideal Q contains infinitely many locus ideals $\mathfrak{p}_n$. Since $S \in \mathcal{M}_{d-1}(R)$, for any of such ideals and $n >> 0$, $\mathfrak{p}_n = Q \cap R_n$. Hence the infinitely many locus ideals contained in Q form a chain. By Proposition 2.4.12, the union of such chain is a prime ideal of S, which is of height at least $\dim(S) - 1$ by Proposition 2.4.9. Hence such union is equal to Q and Q is a chain-prime ideal. □

We describe now the monoidal non-quadratic Shannon extension with principal maximal ideal. Let $x \in S$ such that $\mathfrak{m}_S = xS$ nd let $Q = \bigcap_{j\geq 0}^{\infty} x^j S$. Notice that $Q \neq 0$ since S cannot be a DVR, because a DVR would be a quadratic extension. Hence S is the pullback of the diagram

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d, hook] & S/Q \arrow[d, hook] \\ S_Q \arrow[r, two heads] & \kappa(Q) \end{tikzcd}$$

where S/Q is a DVR ([10](Theorem 2.4)).

Theorem 2.6.5. (Description of monoidal non-quadratic Shannon extension in $\mathcal{M}_{d-1}(R)$ with principal maximal ideal) *Let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal, but not quadratic Shannon extension of S such that $\mathfrak{m}_S = xS$ is principal and let $Q = \bigcap_{j\geq 0}^{\infty} x^j S$. Then:*

1. *S has exactly two chain-prime ideals Q and $\mathfrak{m}_S$.*
2. *S_Q is a quadratic Shannon extension of a RLR of dimension $d-1$.*
3. *The following are equivalent:*
   (i) S is a valuation domain.
   (ii) S is a GCD domain.
   (iii) S_Q is a GCD domain.
   (iv) S_Q is a valuation domain.
4. *The Noetherian hull T of S exists and it is equal to the Noetherian hull of S_Q.*
5. *Let V' be the boundary valuation ring of S_Q and let V the valuation ring defined by the pullback*

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d, hook] & S/Q \arrow[d, hook] \\ V' \arrow[r, two heads] & \kappa(\mathfrak{m}_{V'}). \end{tikzcd}$$

   Then $S = V \cap T$.

6. *The complete integral closure of S is equal to the complete integral closure of S_Q.*

Proof. For item 1, Proposition 2.6.1 implies that $\mathfrak{m}_S$ is a chain-prime ideal.

By Corollary 2.6.4, Q is a nonzero chain-prime ideal. Moreover, it is the unique prime ideal of S of height $\dim(S) - 1$. Take another chain-prime ideal $Q_i = \bigcup_{i\in\mathbb{N}} \mathfrak{p}_{n_i}$ of S. Theorem 2.4.13(2) implies that the height of Q_i is at least $\dim(S) - 1$, hence either $Q_i = Q$ or $Q_i = \mathfrak{m}_S$.

By Corollary 2.6.4, the ideal Q is a chain-prime ideal. Hence item 2 follows from Theorem 2.4.13.

We prove now item 3. The implication (i) $\Longrightarrow$ (ii) is clear. Then (ii) implies (iii) since a localization of a GCD domain is again a GCD domain. The implication (iii) $\Longrightarrow$ (iv) is given by Theorem 1.5.2. Finally (iv) $\Longrightarrow$ (i) follows by [10](Theorem 2.4) since S/Q is a DVR.

For item 4, let T' be the Noetherian hull of S_Q. If T exists, $T \subseteq T'$. Take a Noetherian overring A of S. For any $y \in Q$, the element $y/x^n \in S \subseteq A$ for every $n \geq 0$. But A is Noetherian and therefore there is $n \geq 1$ such that $y/x^n \in (y, y/x, \ldots, y/x^{n-1})A$. Hence, multiplying x^n, we get $y = y(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x)$ for some $a_i \in A$. It follows that $1 = x(\sum_{i=1}^{n} a_i x^{i-1})$ and x is a unit in A. Hence $S_Q = S[1/x] \subseteq A$ and $T' \subseteq A$. Thus the Noetherian hull of S exists and coincides with T'.

We prove now item 5. Notice that V is well defined since $S/Q \subseteq \kappa(Q) \subseteq \kappa(\mathfrak{m}_{V'})$ and $S \subseteq V$, since $\mathfrak{m}_{V'} \cap S = Q$.

Hence $S \subseteq V \cap T$. By Theorem 1.2.5(2), $V' \cap T = S_Q$. Take $a/b \in V \cap T$ with $a, b \in S$ and $b \neq 0$. When b is a unit, $a/b \in S$, hence consider the case in which $b \in \mathfrak{m}_S$. Since $V \cap T \subseteq V' \cap T = S_Q$ we can assume $b = x^n u \in \mathfrak{m}_S \setminus Q$ with u a unit of S. The maximal ideal of V is xV and the value group of V is isomorphic to $G \bigoplus \mathbb{Z}$ where G is the value group of V'. Assuming $a/b \in V \cap T$, we must have the values $v(a) \geq v(b) = (0_G, n)$. Hence $a \in x^n V \cap S = x^n S$ and therefore $a/b \in S$.

Finally we prove item 6. The element $1/x$ is almost integral over S, hence we have $S_Q = S[1/x] \subseteq S^* \subseteq (S_Q)^*$. Assume first S_Q non archimedean. In this case, by Theorem 1.2.11(1), $(S_Q)^* = T = S_Q[1/y]$ where $y \in QS_Q = Q \subseteq S$ is such that $QS_Q = \sqrt{yS_Q}$. Hence $(S_Q)^* = S[1/y]$ and $1/y$ is almost integral over S. It follows that $(S_Q)^* = S^*$. Consider now the case in which S_Q is archimedean. Its complete integral closure $(S_Q)^*$ is equal to

$$(QS_Q :_F QS_Q) = (Q :_F Q)$$

where F is the quotient field of R (see Theorem 1.2.11(2)). Since $(S_Q)^* = (Q :_F Q)$ is a fractional ideal of S that contains S^*, it follows that $(S_Q)^* = S^*$. $\square$

Example 2.6.6. Let $\mathfrak{m} = (x, y, z)R$ the maximal ideal of R and let $s_k := \sum_{j=1}^{k} j$. We start blowing up the ideal $\mathfrak{p}_0 = (x, y)R$ and dividing by x and then blowing up $\mathfrak{p}_1 = (y/x, z)R_1$ and dividing by y/x.

We iterate this process defining two chain of locus ideals $\{\mathfrak{p}_{2k}\}_{k \in \mathbb{N}}$ and $\{\mathfrak{p}_{2k+1}\}_{k \in \mathbb{N}}$ where $\mathfrak{p}_{2k} = (x, y/x^k)$ and $\mathfrak{p}_{2k+1} = (y/x^k, (zx^{s_k})/y^k)$, assuming hence that $\mathfrak{p}_{2k}S = xS$ and $\mathfrak{p}_{2k+1}S = \frac{y}{x^k}S$ for all k.

In this way we obtain the sequence of rings $\{(R_n, \mathfrak{m}_n)\}_{n \in \mathbb{N}}$ where for $k \geq 0$ the maximal ideals are

$$\mathfrak{m}_{2k} = (x, \frac{y}{x^k}, \frac{zx^{s_k}}{y^k})$$

and

$$\mathfrak{m}_{2k+1} = (x, \frac{y}{x^{k+1}}, \frac{zx^{s_k}}{y^k}).$$

Let $S = \bigcup_{n \in \mathbb{N}} R_n$ be the monoidal Shannon extension of R obtained as direct union of this sequence of rings. The maximal ideal $\mathfrak{m}_S$ of S is the principal ideal xS.

Moreover we can note that for every $n \in \mathbb{N}$ the elements $\frac{y}{x^n} \in S$ and $\frac{z}{y^n} = \frac{zx^{s_{n+1}}}{y^{n+1}} \frac{y}{x^{s_{n+1}}} \in S$. Thus we have $z \in Q := \bigcap_{n \geq 0} y^n$ and $y \in P := \bigcap_{n \geq 0} x^n$.

We claim that S is a discrete valuation ring of rank 3. Indeed, since S is a local domain with principal maximal ideal, the ideal P is prime and S is a valuation ring if and only if his localization S_P is a valuation ring. The ideal P is generated by the family of elements $\{\frac{y}{x^n}\}_{n\in\mathbb{N}}$ and hence, since x is a unit in S_P we have $PS_P = yS_P$. By the same fact S_P is a valuation ring if and only if $(S_P)_{QS_P}$ is a valuation ring, but $(S_P)_{QS_P} = R_{zR}$ is a DVR and hence S is a valuation ring.

The union of the ideals of the first chain $\{\mathfrak{p}_{2k}\}_{k\in\mathbb{N}}$ is the maximal ideal $\mathfrak{m}_S$ and hence this chain is not minimal. Instead the chain $\{\mathfrak{p}_{2k+1}\}_{k\in\mathbb{N}}$ is minimal and has as union the ideal P.

In this case the Noetherian hull of S is the overring $S_Q = S[\frac{1}{x\frac{y}{x}}] = S[1/y]$ defined in Theorem 2.5.3.

Now we turn to consider a second class of monoidal Shannon extensions. Assume $\dim R = d$ and let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal Shannon extension of a RLR R with only one chain-prime ideal $P \subsetneq \mathfrak{m}_S$. Assume there exists an element $x \in \mathfrak{m}_S \setminus P$ such that, for every $n >> 0$, x is a regular parameter in R_n.

Hence the ring $R'_n := R_n/xR_n$ is well defined and it is a RLR with maximal ideal $\mathfrak{p}_n/xR_n$. Moreover $\overline{R_n}$ is a local quadratic transform of $\overline{R_{n-1}}$.

Since $S \in \mathcal{M}_{d-1}(R)$, Theorem 2.4.13 implies that S_P is a quadratic Shannon extension of a regular local ring of dimension $d-1$, and Proposition 2.4.9 implies that $\dim(S) = \dim(S_P) + 1$.

A motivation for the study of this rings is the following example.

Example 2.6.7. Let $R = k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$ be a localized polynomial ring over a field and take for S the same assumptions and notations of the paragraph above. It follows that

$$S = \bigcup_{n\in\mathbb{N}} R_n = \bigcup_{n\in\mathbb{N}} \overline{R_n}[x]_{(x,\mathfrak{p}_n)} = (\bigcup_{n\in\mathbb{N}} \overline{R_n})[x]_{(x,P)} = \overline{S}[x]_{(\mathfrak{m}_{\overline{S}},x)}$$

where $\overline{S}$ is a quadratic Shannon extension of the $d-1$-dimensional RLR R/xR. Hence, with R a localized polynomial ring S is a localized polynomial ring in one variable over a quadratic Shannon extension and many of his properties can be easily found looking at this construction.

When $d = 3$, by Abhyankhar's result ([1][Lemma 12]) $\overline{S}$ is a valuation ring, and then S, being a localized polynomial ring over a valuation ring is a GCD domain.

Proposition 2.6.8. *Take the same assumptions as above and suppose there exists $y \in P$ such that $P = \sqrt{yS}$. If S_P is a DVR, then S is a RLR of dimension 2.*

Proof. First notice that P has to be principal. Indeed, if it is non finitely generated, the quadratic Shannon extension S_P would have an idempotent maximal ideal and hence it would not be a DVR. Moreover, S has to be Noetherian. Indeed, if there is a non finitely generated ideal I, I would be contained in a non-finitely generated prime ideal of S. But, by Theorem 2.5.2, the map Spec $S \to$ Spec R is biregular at any prime ideal different from P and therefore all the prime ideals of S are finitely generated. Hence S is a Noetherian local ring of dimension 2 with maximal ideal generated by two non associated prime elements and thus it is a RLR. $\square$

Proposition 2.6.9. *Take the same assumptions as above. Suppose S_P is not a DVR and there exists $y \in P$ such that $P = \sqrt{yS}$. Then:*

1. $T = S[1/y]$ is the Noetherian hull of S.

2. We have the decomposition $S = S_P \cap T$.

3. The complete integral closure S^* of S is T if S_P is non archimedean and is $T \cap S_P^*$ if S_P is archimedean. Hence S is completely integrally closed if and only if S_P is completely integrally closed.

Proof. We prove item 1. By Theorem 2.5.3 follows that T is a Noetherian UFD. Let A be a Noetherian overring of S and let M be a maximal ideal of A. Take V a DVR that dominates A_M, hence, if $M \cap S = \mathfrak{m}_S$, then V dominates also S. Consider the DVR $W = V \cap F'$ where F' is the quotient field of R/xR which is clearly a subfield of F =Quot(R). Therefore W dominates the quadratic Shannon extension $\overline{S} = \bigcup_{n \in \mathbb{N}} \overline{R_n}$. It follows that $\overline{S}$ is a DVR and thus P is an height one principal ideal and S_P is a DVR. This is a contradiction and hence $M \cap S \subsetneq \mathfrak{m}_S$. For the same reason V cannot dominate S_P, since S_P is a quadratic Shannon extension and not a DVR. Hence $M \cap S$ is a prime ideal that does not contain P. It follows that $y \notin M \cap S$ and $T \subseteq A_M$. This holds for every maximal ideal M of A and therefore $T \subseteq A$.

We prove that

$$R_n = R_n[1/y] \cap (R_n)_{PS_P \cap R_n} = R_n[1/y] \cap (R_n)_{\mathfrak{p}_n}$$

for any $n >> 0$ such that $y \in \mathfrak{p}_n$. Take $s \in R_n[1/y] \cap (R_n)_{\mathfrak{p}_n}$. Write $s = a/y^k$ with $a \in S$ and $k \geq 0$. Since $y \in \mathfrak{p}_n$ and R_n is a UFD, y^k divides a in S and hence $s \in S$. Item 2 now follows taking the union over $n \in \mathbb{N}$ at both sides of the equality.

For item 3 assume first S_P to be non archimedean. By Theorem 2.5.3, $T = S[1/y]$ is a UFD, hence completely integrally closed. Since S_P is non archimedean, the ideal $\cap_{n\geq 0}^{\infty} y^n S$ is non zero and therefore the element $1/y$ is almost integral over S. It follows that T is the complete integral closure of S.

Assume instead S_P to be archimedean. The inclusion $S^* \subseteq T \cap S_P^*$ is clear. Take an element $s/y^n \in T \cap S_P^*$ with $s \in S$. By definition, there exists an element $b/c \in S_P$ with $b \in S$ and $c \in S \setminus P$ such that $s^m b/y^{nm} c \in S_P$ for every $m \geq 0$. But since $y \in P$, this implies $bs^m \in y^{nm} S$ for every $m \geq 0$ and therefore $s/y^m \in S^*$. From item 2 it follows that S is completely integrally closed if and only if S_P is completely integrally closed. □

In the next section we give a general theorem which characterizes when the two class of monoidal Shannon extensions introduced in this section are GCD domains.

2.7 GCD property for monoidal Shannon extensions with a finite number of chains

Let $S = \bigcup_{n \in \mathbb{N}} R_n \in \mathcal{M}_{d-1}(R)$ be a monoidal Shannon extension of a RLR R and, as usual, assume $\dim(R_n) = d \geq 3$ for every n.

Moreover, we consider as before the case in which in S there is just a finite number of chain-prime ideals. Call them $Q_1, \dots, Q_c$. We assume through this Section that the ideals Q_i have arithmetical rank one. We proved in Proposition 2.4.9 and in

Theorem(2.4.13) that, when $Q_i \subsetneq \mathfrak{m}_S$, the height of Q_i is equal to dim$S-1$ and the ring S_{Q_i} is a quadratic Shannon extension of a RLR of dimension $d-1$.

In this case, for every element x such that $\sqrt{xS} = Q_i$, there exists some n, such that $\mathfrak{p}_n R_{n+1} = xR_{n+1}$ and hence x is a regular parameter in some ring of the sequence.

The following result extends in the this monoidal setting the characterization of the GCD property given for a quadratic Shannon extension in Theorem 1.5.2.

Theorem 2.7.1. *Let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal Shannon extension of a RLR R with finite chain-prime ideals $Q_1, \dots, Q_r$. Suppose that every chain-prime ideal Q_i of S is not maximal and has arithmetical rank one. Then S is a GCD domain if and only if S_{Q_i} is a valuation domain for every i.*

The strategy to prove the "if" part of this result is the following: for every $i = 1, \dots, r$, we can consider $y_i \in S$ such that $Q_i = \sqrt{y_i S}$. The overring $T = [1/y_1 \cdots y_r]$ of S is a GCD domain by Theorem 2.5.3.

We prove by induction that for $i = 1, \dots, r-1$, the rings $S_i = S[1/y_1 \cdots y_i]$ are GCD domains. Then the final step will be to show that S_1 GCD domain implies S to be a GCD domain. To prove all those fact we are using inductively Theorem 2.7.5. For it we need some preliminary lemmas.

Lemma 2.7.2. *Let D be a domain and let I be a proper ideal of D that properly contains a principal prime ideal xD. Then I cannot have arithmetical rank one.*

Proof. Suppose $I = \sqrt{fD}$ for some element $f \in D$, hence $f \in I$. If $f \in xD$, then $\sqrt{fD} \subseteq \sqrt{xD} = xD \subsetneq I$, therefore we must have $f \notin xD$. Now we have $x \in \sqrt{fD}$ and hence there exists $k \geq 0$ such that $x^k = fd$ for some $d \in D$. But xD is prime and $f \notin xD$, hence $d \in xD$ and by cancellation of x we have $x^{k-1} = fd_1$ for some $d_1 \in D$. By induction, there exist $t \in D$ such that $1 = x^0 = ft$ and f is a unit. This contradicts the fact that I is a proper ideal. □

Lemma 2.7.3. *Let $S \in \mathcal{M}_{d-1}(R)$ be a monoidal Shannon extension in which all the chain-prime ideals Q_i are not maximal and they all have arithmetical rank one. Assume that S is not a valuation ring. Then, the maximal ideal $\mathfrak{m}_S$ of S cannot have arithmetical rank one.*

Proof. Clearly $\mathfrak{m}_S$ is not principal by Theorems 2.6.5 and 2.6.3, hence we just need to prove that there exists some prime element in S and apply Lemma 2.7.2. If there is only one chain-prime ideal $Q \subsetneq \mathfrak{m}_s$, since S is not a valuation ring, there exists an height one prime P which does not contain Q and it is principal by Proposition 2.4.8. Hence its generator is a prime element.

In the other cases, take some elements $x_1, \dots, x_r$ (with $r > 1$) such that $Q_i = \sqrt{x_i S}$. Set $y := x_1 + x_2 x_3 \cdots x_r$ and we can assume, by reordering the indexes and since x_1 is a regular parameter in some R_n, that y has order one in $\mathfrak{m}_m$ for some $m \geq 0$. Hence y is prime in R_m.

Moreover it is easy to see that $y \notin Q_i$ for every i and hence $y \notin \mathfrak{p}_n$ for every n. We show that this implies y is a prime element in R_n for $n \geq m$ and therefore y is prime in S.

Let $R_{m+1} = R_m[\frac{\mathfrak{p}_m}{t}]_{\mathfrak{m}_{m+1}}$. Since $y \notin \mathfrak{p}_m$, it follows $y \notin tR_{m+1} = \mathfrak{p}_m R_{m+1}$. Consider a prime element $f \in R_{m+1}$ such that $y \in fR_{m+1}$. By biregularity (Proposition

2.1.2(6)), $fR_{m+1} \cap R_m$ is an height one prime, but, since y is prime in R_m, we have $fR_{m+1} \cap R_m = yR_m$.

Moreover there exists $k \geq 0$ such that $t^k f \in fR_{m+1} \cap R_m = yR_m$ and hence $t^k f = yc$ for some $c \in R_m$. It follows $f = yc/t^k$, but t is prime in R_{m+1} and it does not divide y. Therefore $c/t^k \in R_{m+1}$, $f \in yR_{m+1}$ and y is prime in R_{m+1}. By induction it follows y is prime in in R_n for $n \geq m$. $\square$

Lemma 2.7.4. *Let $D = \cup_{i\geq 0}^{\infty} D_i$ be an infinite directed union of Noetherian UFDs such that every maximal ideal $\mathfrak{m}_i$ of D_i is contained in a maximal ideal of D. Assume there exists some $y \in D$ such that $Q = \sqrt{yD}$ is a prime ideal of D. Assume that either Q is maximal or D is local with maximal ideal $\mathfrak{m}_D$, $ht(Q) = dim(D) - 1$ and $\mathfrak{m}_D$ does not have arithmetical rank one. Call $T = D[1/y]$. Then:*

(1) If $a \in D \setminus Q$ and $af \in D$ for $f \in T$, then $f \in D$.

(2) Let zT be a principal proper ideal of T and assume $I := zT \cap D \nsubseteq Q$. Then I is principal in D.

(3) Assume also that D_Q is a valuation domain. Then, if I is a finitely generated ideal of D, $I^{-1}D_Q = (ID_Q)^{-1}$.

Proof. (1) Write $f = s/y^k$ and assume by way of contradiction $s \notin y^k D$. Take $n \geq 0$ such that $af = as/y^k \in D_i$ that is a UFD. Hence, since $a \notin yD$, there exists some non unit $g \in D_i$ such that $a, y \in gD_i \subseteq gD$. Thus $\sqrt{yD} = Q \subsetneq \sqrt{gD}$. Now, if the ring S is local and Q has height equal to $\dim(D) - 1$, we must have $\sqrt{gD} = \mathfrak{m}_D$. But this is impossible by the assumption on the arithmetical rank of m_D. If instead Q is maximal,$\sqrt{gD} = D$ and g is a unit in D and hence a unit in D_i. Both these contradictions imply $f = s/y^k \in D$.

(2) By multiplying a unit of T, we can assume $z \in D$. There exists an element $zs/y^k \in I \setminus Q$, hence $z \notin \bigcap_{k\geq 0}^{\infty} y^k D$. Again by multiplying z for some power of $1/y$ we find an element $u = z/y^m \in D \setminus Q$ such that $uT = zT$. Indeed if $z/y^k \in Q$ for every $k \geq 0$, we would have $z \in \bigcap_{k\geq 0}^{\infty} y^k Q \subseteq \bigcap_{k\geq 0}^{\infty} y^k D$ that is a contradiction. We prove $I = uD$. Clearly $uD \subseteq I$, hence take $h = uf \in I$ with $f \in T$. Item (1) implies $f \in D$ and therefore $I = uD$.

(3) We proceed along the argument given for item 4 of Proposition 2.2.2. First we show $D = D_Q \cap T$ (notice that when y is prime this is proved in Lemma 2.2.1). Take $f \in D_Q \cap T$. Hence there exists $a \in D \setminus Q$ such that $af \in D$. By item (1), it follows $f \in D$. The other inclusion is trivial.

Hence we can write
$$I^{-1} = (ID_Q)^{-1} \cap (IT)^{-1}$$
and since localization commutes with finite intersection, this implies

$$I^{-1}D_Q = (ID_Q)^{-1} \cap (IT)^{-1}D_Q.$$

Since D_Q is a valuation ring, every prime ideal properly contained in yD is contained in $P = \bigcap_{i=0}^{\infty} y^n D$, and thus $D[1/x]D_{xD} = D_P$. Therefore,

$$\begin{aligned} (IT)^{-1}D_Q &= (IT)^{-1}TD_Q \\ &= (IT)^{-1}D_P \\ &= (ID_P)^{-1}, \end{aligned}$$

where the last equality follows from the fact that IT is a finitely generated ideal of T and D_P is a localization of T. Therefore,

$$I^{-1}D_Q = (ID_Q)^{-1} \cap (ID_P)^{-1}.$$

Since $D_Q \subseteq D_P$, this implies $I^{-1}D_Q = (ID_Q)^{-1}$. $\square$

Theorem 2.7.5. *Let D be a domain which fulfills the same assumptions of Lemma 2.7.4. Also assume that D_Q is a valuation domain and that $T = D[1/y]$ is a GCD domain. Then D is a GCD domain.*

Proof. Take a and b nonzero elements of D. We proceed along the method used in the proof of Theorem 2.2.5. Set $J = (a, b)D$. The ideal J is finitely generated and therefore JD_Q is invertible in the valuation ring D_Q. By Lemma 2.7.4(3), we have

$$JJ^{-1}D_Q = JD_Q(JD_Q)^{-1} = D_Q,$$

and thus we can find an element $q \in J^{-1}$ such that $qJ \nsubseteq Q$.

Obviously J is principal if and only if qJ is principal. Replacing J with qJ, we may assume without loss of generality that $a \notin Q$. Again as in 2.2.5 we have that the ideal $aD \cap bD$ is principal if and only if the ideal $K := D \cap \frac{a}{b}D = aD :_D b$ is principal. We first prove that $KT \cap D = K$.

Indeed, clearly $K \subseteq KT \cap D$. Take $z \in KT \cap D$, hence $z = c/y^n$ with $c \in K$ and $zb = as/y^n$. Since $a \notin Q$, by Lemma 2.7.4(1), $s/y^n \in D$ and $z \in K$.

Now, if $K = D$, then it is principal, hence assume $K \subsetneq D$. This implies $KT \subsetneq T$. Since T is a GCD domain, the ideal KT is principal and its contraction $KT \cap D$ is not contained in Q since contains the element a. By Lemma 2.7.4(2), $KT \cap D$ is principal generated by an element of D. Hence $KT \cap D = K$ is principal and therefore $aD \cap bD$ is principal. $\square$

We can now prove Theorem 2.7.1 using by induction Theorem 2.7.5.

Proof. (of Theorem 2.7.1)

For every $i = 1, \dots, r$, take $y_i \in S$ such that $Q_i = \sqrt{y_i S}$. By Theorem 2.5.3, the overring $T = S[1/y_1 \cdots y_r]$ is a GCD domain.

Since the localizations of a GCD domain are GCD domains and a quadratic Shannon extension is a GCD domain if and only if is a valuation ring, the first implication of the theorem is clear.

Assume then S_{Q_i} to be a valuation domain for every i.

We prove by induction that for $i = 1, \dots, r-1$, the ring $S_i = S[1/y_1 \cdots y_i]$ is a GCD. Indeed, take a such i and assume S_{i+1} to be a GCD domain. Call $Q := Q_{i+1}S_i$. The prime ideals of the ring S_i are in one-to-one correspondence with the prime ideals of S except $\mathfrak{m}_S, Q_1, \dots, Q_i$. Hence $\dim(S_i) = \dim(S) - 1 = \mathrm{ht}Q_{i+1} = \mathrm{ht}(Q)$ and Q is maximal in S_i. We notice that $S_{i+1} = S_i[1/y_{i+1}]$ and $(S_i)_Q = S_{Q_i}$. Moreover we can consider S_i as the directed union of the Noetherian UFDs $R_n[1/y_1 \cdots y_i]$ for large $n \in \mathbb{N}$. It follows that S_i and its prime ideal Q fulfill the hypothesis of Lemma 2.7.4 and Theorem 2.7.5 and thus S_i is a GCD domain.

Finally, by Lemma 2.7.3, $\mathfrak{m}_S$ does not have arithmetical rank one and also S fulfills the hypothesis of Lemma 2.7.4. Further, since $S_1 = S[1/y_1]$ is a GCD domain and S_{Q_1} is a valuation ring, S is a GCD domain again by Theorem 2.7.5. $\square$

Question 2.7.6. Let S be a monoidal Shannon extension with finitely many chain-prime ideals. Then is it true that S is a GCD-domain if and only if S_{Q_i} is a valuation domain for every i also in the case in which $\mathfrak{m}_S$ is a chain-prime ideal? This is equivalent to prove that if $\mathfrak{m}_S$ is a chain-prime ideal, then S is a GCD domain if and only if it is a valuation domain. We know this to be true when S is a quadratic extension and also when $\mathfrak{m}_S$ is principal.

Chapter 3

Lefschetz Properties for Gorenstein graded algebras associated to Apery Sets

Introduction

The Lefschetz properties for standard graded Artinian K-algebras are algebraic concepts introduced by Stanley in [54], motivated by the Hard Lefschetz Theorem on the cohomology rings of smooth irreducible complex projective varieties. The notion of Poincaré duality for these cohomology rings inspired the definition of Poincaré duality for algebras which is equivalent to the Gorensteiness. Hence many results about Lefschetz properties have been proved in the Gorenstein case.
In [56], it has been shown that almost all Artinian Gorenstein algebras have the Strong Lefschetz property. But in general it is a difficult problem to know whether a given specific algebra has the Strong (or the Weak) Lefschetz property.
Using Macaulay-Matlis duality in characteristic zero it is possible to present Artinian Gorenstein algebras in the form $A = Q/\mathrm{Ann}_Q(f)$ with $f \in R = K[x_1, \ldots, x_n]$ an homogeneous polynomial and $Q = K[X_1, \ldots, X_N]$ where $X_i := \frac{\partial}{\partial x_i}$ are differential operators (e.g. [53]). In [55] and [53], using this presentation of the algebras, the autors introduced a criterion based on determinants of Higher Hessians that estabilishes whether an algebra has or not Lefschetz properties.
In [48] this criterion was used to construct explicit examples of Artinian Gorenstein algebras that do not satisfy one or both Lefschetz properties.
Even if the "Lefschetz properties problem" has a very simple formulation, it is in general open even in low codimension. Indeed, while in codimension two it is known that all the Artinian Gorenstein graded algebras have the Strong Lefschetz property, in codimension three this is not known but is conjectured to be true. The first examples of algebras without Strong or Weak Lefschtz properties appear in codimension four.
Furthermore, we do not know if there are examples of algebras without one or both the Lefschetz properties (in any given codimension) that belong to the smaller class of Complete Intersection rings. Indeed it is conjectured that all the Complete Intersection Artinian graded algebras have the SLP. For all this results and open conjectures we refer to the monography *The Lefschetz properties* [50].
In this work we study the Weak Lefschetz property (WLP) for a class of graded

Artinian Gorenstein algebras built up starting from the Apéry Set of a numerical semigroup and of which they reflect the lattice structure. Our goal is to study wheter these algebras have the WLP in codimension three and in the Complete Intersection case. In both cases we are going to have a positive answer, finding thus a class of algebras confirming both the conjectures about WLP.
The structure of this chapter is the following: in Section 3.1 we recall some definitions and known results about Lefschetz properties and we focus on the Hessians criteria proved in [53].
In Section 3.2 we prove a key theorem (3.2.2) that will be our main tool for the results of the work. It states that if a graded Artinian Gorenstein algebra

$$A \cong \frac{K[x_1, \ldots, x_n]}{I}$$

has the WLP, then also the quotient ring

$$\frac{A}{(0 :_A L)}$$

is Gorenstein and it has the WLP for any linear element $L \in A$.
In general it is known that if an Artinian graded algebra A, non necessarily Gorenstein, has the SLP and if $L \in A$ is a Lefschetz element for A (see definition in Section 3.1), then $\frac{A}{(0 :_A L)}$ has the SLP; moreover, if A has the SLP in the narrow sense (see Definition(3.1.1)), then for any linear element $L_1 \in A$, the quotient ring $\frac{A}{(0 :_A L_1)}$ has the SLP if it has the same Hilbert function of $\frac{A}{(0 :_A L)}$ ([50], 3.11 and 3.40). So we prove a similar result about WLP under the hypothesis of Gorensteiness of A in an explicit way using standard linear algebra methods without making any assumption on the nature of the linear element L.
In Section 3.3 we construct the graded Artinian algebra associated to the Apéry Set of a numerical semigroup. This is the same ring appeared in [43] and in [44] and used to prove results about the Gorensteiness and the Complete Intersection property of the associated graded ring of a semigroup ring.
In Section 3.4 and 3.5 we present the two main results of the paper. In Section 3.4 we deal with the Complete Intersection case and we recall results from [44] to estabilish when our algebras associated to Apéry Sets are Complete Intersections. Then we use an useful known criterion about WLP for Complete Intersection rings combined with Theorem(3.2.2) to get our result.
Finally, in Section 3.5 we assume the codimension to be three; in this case we are able to completely characterize the defining ideal of all the graded algebras associated to Apéry Sets in function of their socle degree and we find that any such algebra is of the form

$$A = \frac{K[y, z, w]}{I} = \frac{G}{(0 :_G z^C)}$$

with G a Complete Intersection Artinian graded ring and C a positive integer. This characterization will imply the WLP of A as an easy consequence of Theorem(3.2.2).
The reference for all Chapter 3, where not specified, is the paper *"Lefschetz Properties of Gorenstein Graded Algebras associated to the Apéry Set of a Numerical Semigroup"* [49].

3.1 Lefschetz Properties

We start recalling definitions and important results about the Lefschetz properties.

Let K be a field of characteristic zero and let $A = \bigoplus_{i\geq 0}^{D} A_i$ be a standard graded Artinian K-algebra. Since it is Artinian, A is a finite dimensional K-vector space.

Consider the polynomial ring in n variables $K[x_1, \ldots, x_n]$ with $n \geq 1$. We can always write

$$A \cong \frac{K[x_1, \ldots, x_n]}{I}$$

where $I \subseteq K[x_1, \ldots, x_n]$ is an homogeneous ideal of height n. The integer $n = \dim_K(A_1)$ is said the *codimension* of the ring A.

Definition 3.1.1. We say that:

1. A has the Weak Lefschetz property (WLP) if there is an element $L \in A_1$ such that the multiplication map $\times L : A_i \to A_{i+1}$ has maximal rank for every $i = 0, \ldots, D-1$.

2. A has the Strong Lefschetz property (SLP) if there is an element $L \in A_1$ such that the multiplication map $\times L^d : A_i \to A_{i+d}$ has maximal rank for every $i = 0, \ldots, D$ and $d = 0, \ldots, D-i$.

3. A has the Strong Lefschetz property in the narrow sense if there is an element $L \in A_1$ such that the multiplication map $\times L^{D-2i} : A_i \to A_{D-i}$ is bijective for every $i = 0, \ldots, [D/2]$.

A linear form $L \in A_1$ such that each map $\times L : A_i \to A_{i+1}$ has maximal rank is said a *Weak Lefschetz element*. If instead each map $\times L^d : A_i \to A_{i+d}$ has maximal rank, L is said a *Strong Lefschetz element*.

The ring $A = \bigoplus_{i\geq 0}^{D} A_i$ is Gorenstein if there is a perfect pairing of its homogeneous components, that is $A_i \cong A_{D-i}$ for every i.

Hence, if A is Gorenstein, it has a symmetric Hilbert function, that means:

$$\dim_K(A_i) = \dim_K(A_{D-i}), \qquad \forall i.$$

We call the integer D the *socle degree* of A.

In this work we are always dealing with Gorenstein algebras and in this case it is known that the SLP is equivalent to the Strong Lefschetz property in the narrow sense [53].

The Artinian ring $A \cong \dfrac{K[x_1, \ldots, x_n]}{I}$ is a Complete Intersection (CI) if I is minimally generated by exactly n elements (notice that in general I is generated by at least n elements). It is well known that a Complete Intersection ring is always Gorenstein. Here we state some known results and still open problems about Lefschetz properties of Gorenstein rings.
All the details about these topics can be found in ([50], 3.15, 3.48, 3.35, 3.46, 3.80).

Let $A \cong \dfrac{K[x_1, \ldots, x_n]}{I}$ be a standard graded Artinian Gorenstein K-algebra, then:

- If $n = 2$, then A has the SLP.

- If $n = 3$ and A is a CI, then A has the WLP.

- For every n, if A is a CI and I is a monomial ideal, then A has the SLP.

- It is conjectured that if A is a CI, then A has the SLP.

- If $n = 3$, it is unknown if there exist a such ring A that does not have the WLP or the SLP.

The most important tools that we need to study if a Gorenstein algebra has the Lefchetz properties are the higher Hessians.
We define the ring of differential operators $Q := K[X_1, \dots, X_n]$ where

$$X_i := \frac{\partial}{\partial x_i},$$

and we give some definitions and results taken from a classical work of Maeno and Watanabe [53] and from a recent work of Gondim and Zappalá [47].

Proposition 3.1.2. *Let A be a standard graded Artinian Gorenstein K-algebra and $k := [D/2]$ where D is the socle degree of A. Thus we have:*

1. *If D is an odd number, A has the WLP if there is an element $L \in A_1$ such that the multiplication map $\times L : A_k \to A_{k+1}$ is an isomorphism.*

2. *If D is an even number, A has the WLP if there is an element $L \in A_1$ such that the multiplication map $\times L : A_k \to A_{k+1}$ is surjective or equivalentely the multiplication map $\times L : A_{k-1} \to A_k$ is injective.*

Theorem 3.1.3. [53](Theorem 2.1) *Let A be a standard graded Artinian Gorenstein K-algebra. Then there exists a polynomial $F \in K[x_1, \dots, x_n]$ such that A is isomorphic to the quotient $Q/\mathrm{Ann}_Q(F)$.*

This shows that A is generated over K exactly by the homogeneous monomials in $K[x_1, \dots, x_n]$ that do not annihilate F when considered as differential operators.

Definition 3.1.4. Let F be a polynomial in $K[x_1, \dots, x_n]$ and $d \geq 1$ an integer. Take a K-linear basis $\mathcal{B}_d = \{\alpha_i\}_{i \geq 0}^{s}$ of A_d.
We define the d-th Hessian of F the matrix

$$Hess^d_{\mathcal{B}_d}(F) := \{(\alpha_i(X)\alpha_j(X)F(x))_{i,j=1}^{s}\}.$$

We call the $hess^d_{\mathcal{B}_d}(F)$ the determinant of this matrix. The singularity of the matrix is independent of the chosen basis and hence we can write simply $Hess^d(F)$ and $hess^d(F)$. Clearly $hess^d(F)$ is a polynomial in $K[x_1, \dots, x_n]$.

Definition 3.1.5. Let F be a polynomial in $K[x_1, \dots, x_n]$.
Taking two integers $d, t \geq 1$ and two basis of A_d and A_t, we define the mixed Hessians of the polynomial F as

$$Hess^{d,t}(F) := \{(\alpha_i(X)\beta_j(X)F(x))\}$$

where $\{\alpha_i\}_{i=1}^{s_1}$ and $\{\beta_j\}_{j=1}^{s_2}$ form respectively the basis of A_d and A_t.

Theorem 3.1.6. [47](Theorem 2.10) *Let A be a standard graded Artinian Gorenstein K-algebra and $k := [D/2]$ where D is the socle degree of A. Thus we have:*

1. *The algebra $A = Q/\mathrm{Ann}_Q(F)$ has the SLP if and only if all the Hessians $Hess^d(F)$ for $d = 1, \dots, k$, have maximal rank (hence if they have nonzero determinant). Moreover, a linear form $L = \sum a_i x_i \in A_1$ is a Strong Lefschetz element if $F(a_1, \dots, a_n) \neq 0$ and $hess^d(F)(a_1, \dots, a_n)$ is nonzero for all d.*

2. *If D is an odd number, the algebra $A = Q/Ann_Q(F)$ has the WLP if and only if the maximal Hessian $Hess^k(F)$ has nonzero determinant. Moreover, a linear form $L = \sum a_i x_i \in A_1$ is a Weak Lefschetz element if $F(a_1, \dots, a_n) \neq 0$ and $hess^k(F)(a_1, \dots, a_n)$ is nonzero.*

3. *If D is an even number, the algebra $A = Q/Ann_Q(F)$ has the WLP if and only if the mixed Hessian $Hess^{k-1,k}(F)$ has maximal rank. Moreover, a linear form $L = \sum a_i x_i \in A_1$ is a Weak Lefschetz element if $F(a_1, \dots, a_n) \neq 0$ and the matrix $Hess^{k-1,k}(F)(a_1, \dots, a_n)$ has maximal rank.*

The previous results follow from some facts that we want to recall:

Remark 3.1.7. Let $L = \sum a_i x_i \in A_1$ a linear element of A.
Then the matrix $Hess^d(F)(a_1 x_1, \dots, a_n x_n)$ is the symmetric matrix associated to the map $\times L^{D-2d} : A_d \to A_{D-d}$ and hence the first assertion of Theorem(3.1.6) follows from the fact that, in the Gorenstein case, SLP is equivalent to SLP in the narrow sense.
In particular, if the socle degree D is odd, $Hess^k(F)(a_1 x_1, \dots, a_n x_n)$ is the matrix associated to the map $\times L : A_k \to A_{k+1}$.
If instead D is even, we have that $Hess^{k-1,k}(F)(a_1 x_1, \dots, a_n x_n)$ is the matrix associated to the map $\times L : A_k \to A_{k+1}$, while the matrix ${}^t Hess^{k-1,k}(F)(a_1 x_1, \dots, a_n x_n)$ is associated to the map $\times L : A_{k-1} \to A_k$.

3.2 WLP of quotient algebras

We prove a theorem that will allow us to transfer the Weak Lefschetz Property from a Gorenstein algebra to some of its quotients. Also in this section K is a field of characteristic zero.

Lemma 3.2.1. *Let $G = \bigoplus_{i \geq 0}^{D} G_i$ be a standard graded Gorenstein Artinian K-algebra that has the WLP.
Then, it is always possible to find a Weak Lefschetz element $L = \sum_{j=1}^{n} a_j x_j \in G_1$ with $a_j \neq 0$ for all j.*

Proof. Let $k = [D/2]$ and write $G \cong Q/\mathrm{Ann}_Q(F)$ as in Theorem(3.1.3).
Let H be $Hess^k(F)$ if D is odd or a square submatrix of maximal order of $Hess^{k-1,k}(F)$ if D is even. Such matrix H exists by Theorem(3.1.6) since G has the WLP.
In Theorem(3.1.6) is also proved that L is a Weak Lefschetz element if and only if $\det(H)(a_1, \dots, a_n) \neq 0$. Since G has the WLP, $\det(H)$ is a non constant polynomial in $K[x_1, \dots, x_n]$. Hence there exist points $(a_1, \dots, a_n) \in \mathbb{A}^n_K$ such that $a_j \neq 0$ for all j and $\det(H)(a_1, \dots, a_n) \neq 0$. $\square$

Theorem 3.2.2. *Let $G = \bigoplus_{i\geq 0}^{D} G_i \cong \dfrac{K[x_1,\dots,x_n]}{I}$ be a standard graded Gorenstein Artinian K-algebra that satisfies the WLP.*
Then, for $l = 1,\dots,n$, the quotient ring

$$A = \frac{G}{(0 :_G x_l)}$$

is also a standard graded Gorenstein Artinian K-algebra. If A and G have the same codimension, then also A has the WLP.

Proof. Since G is Gorenstein, there is a perfect pairing of its homogeneous components, that is $G_d \cong G_{D-d}$ for every d. These isomorphism are obtained associating to an homogeneous element $f \in G_d$, the element $\varphi \in G_{D-d}$ such that $f\varphi = q$ where q is the generator of the socle G_D. We observe that the socle of A is the homogeneous component A_{D-1} and its unique generator (modulo I) as K-vector space is the monomial $x_l^{-1}q \in G_{D-1}$. Therefore it is easy to see that A is a standard graded Gorenstein Artinian algebra.

Since G and A have the same codimension, let $L = \sum_{j=1}^{n} a_j x_j \in G_1 = A_1$ be a Weak Lefschetz Element for G with $a_j \neq 0$ for all j (3.2.1) and let $k := [D/2]$. Call $z := x_l$, $a := a_l$ and $J = (0 :_G x_l)$. As usual we denote by F the polynomial such that $G \cong Q/\mathrm{Ann}_Q(F)$.

We are going to use the characterization of WLP given in Proposition(3.1.2). We divide the proof in two subcases:

(1) D odd:
By Proposition(3.1.2), we have that the multiplication map $\times L : G_k \to G_{k+1}$ is an isomorphism and we want to prove that the map $\times L : A_k \to A_{k+1}$ is surjective.

The ideal $J = (0 :_G x_l)$ can be seen as a K-vector subspace of G and by definition $A \cap J = (0)$. Hence $G \cong A \oplus J$ as K-vector spaces and thus we can write the elements of G in the form (a, j) with $a \in A$ and $j \in J$ and we have $(a, j) \in J$ if and only if $a = 0$.

Take $(a, j) \in G_{k+1}$ with $a \neq 0$. The map $\times L$ is an isomorphism, so we can consider its preimage $\times L^{-1}(a, j) = (a_1, j_1) \in G_k$. Showing $a_1 \neq 0$, we will have that $\times L : A_k \to A_{k+1}$ is surjective.
Assume $a_1 = 0$, then by definition $(a, j) = \times L(a_1, j_1) = \times L(0, j_1) = (0, Lj_1) \in J$, and thus $a = 0$. This is a contradiction.

(2) D even:
Now, Proposition(3.1.2) implies that the multiplication map $\times L : G_{k-1} \to G_k$ is injective and we want to prove that the map $\times L : A_{k-1} \to A_k$ is also injective (or surjective because $A_{k-1} \cong A_k$).
Since $G \cong A \oplus J$ as K-vector spaces, the map $\times L : A_{k-1} \to G_k$ is injective on his image and therefore is injective on A_k if we show $\times L(A_{k-1}) \cap J = (0)$.
By assumption we have $J = (0 :_G z)$, so we need to show $zLf \neq 0$ for all $f \in A_{k-1}$.

Let Z be the matrix associated to the multiplication map $\times zL : G_{k-1} \to G_{k+1}$. By Remark(3.1.7), the matrix associated to $\times L : G_{k-1} \to G_k$ is ${}^tH(a_1x_1,\dots,a_nx_n)$ where $H := Hess_{k-1,k}(F)$, while the matrix associated to $\times z : G_k \to G_{k+1}$ is $H(0,\dots,z,\dots,0)$ where z appear in the l-th place. Thus

$$Z = H(0,\dots,z,\dots,0)^t H(a_1x_1,\dots,a_nx_n)$$

is a square matrix of dimension $b = \dim_K(G_{k-1})$ (observe that $G_{k-1} \cong G_{k+1}$).

Now take a K-linear basis $f_1, \ldots, f_b$ of G_{k-1} formed by monomials. Since $G_{k-1} \cong A_{k-1} \oplus J_{k-1}$, we can assume this basis to be the union of a basis of A_{k-1} and a basis of J_{k-1}. We define two set of indexes: $\mathcal{I} := \{i \in \{1, 2, \ldots, b\}, f_i \notin J\}$ and $\mathcal{H} := \{i \in \{1, 2, \ldots, b\}, f_i \in J\}$.

We want to show $Z \cdot f \neq 0$ for every $f = \sum_{i=1}^{b} t_i f_i \in A_{k-1}$. Consider the i-th row of the mixed hessian H. By construction, there exists j such that $H_{ij} = z$ if and only if $zf_i \notin I$, that means $zf_i \neq 0$ in G. This is equivalent to say $f_i \notin J$ and therefore it is also equivalent to say $i \in \mathcal{I}$.

Thus, by definition of Z, we have

$$Z_{ij} := \sum_{u=1}^{b} [H(0, \ldots, z, \ldots, 0)]_{uj} [{}^t H(a_1 x_1, \ldots, a_n x_n)]_{iu}$$

and hence the entry $Z_{ii} = az^2$ if and only if $i \in \mathcal{I}$. Otherwise, if $i \in \mathcal{H}$ all the i-th row of Z is zero. Hence Z is a square matrix with $|\mathcal{H}| = \dim_K(J_{k-1})$ zero rows.

Consider the submatrix $M := \{H_{ij}\}_{i \in \mathcal{I}}$. A square submatrix of maximal order of M (that we may call $\tilde{M}$) has the element az^2 on all the entries on the principal diagonal and it has either the elements $a_j x_j z$ with $x_j \neq z$ or 0 elsewhere (depending on the entries of H). Therefore, it is easy to see that $\tilde{M}$ has nonzero determinant and hence M has maximal rank. Indeed $\det(\tilde{M}) = z^{2m} + p(x_1, \ldots, x_n)$ with $m > 1$ and $\deg(z, p) < 2m$.

Hence

$$\dim_K(\operatorname{Ker}(Z)) = \dim_K(G_{k-1}) - \dim_K(\operatorname{Im}(Z)) = b - |\mathcal{I}| = |\mathcal{H}|.$$

The last thing that we need to prove is that $\operatorname{Ker}(Z) = J_{k-1}$ because this will imply $Z \cdot f \neq 0$ for every $f \in A_{k-1}$. Since $\operatorname{Ker}(Z)$ and J_{k-1} have the same dimension, we need to show just one inclusion. But for every $t \in J$, $Lt \in J = (0 :_G z)$, hence $zLt = 0$ and thus $J_{k-1} \subseteq \operatorname{Ker}(Z)$ and this concludes the proof. $\square$

By using a linear change of coordinates on $x_1, \ldots, x_n$, we find as an easy corollary that, if G ia an Artinian standard graded Gorenstein algebra with the WLP, then any quotient $\dfrac{G}{(0 :_G f)}$ with $f \in G_1$ has the WLP. We did not prove directly this result since the construction of the matrix would have been more complicate.

Corollary 3.2.3. *Let $G = \bigoplus_{i \geq 0}^{D} G_i \cong \dfrac{K[x_1, \ldots, x_n]}{I}$ be a standard graded Gorenstein Artinian K-algebra with the WLP and let $f \in G_1$ a linear element. Then, the quotient ring*

$$A = \frac{G}{(0 :_G f)}$$

is also a standard graded Gorenstein Artinian K-algebra. If A and G have the same codimension, then also A has the WLP.

Proof. Write $f = \sum b_i x_i$ and, assuming $b_1 \neq 0$, make the linear change of coordinates $\varphi : K[x_1, \ldots, x_n] \longrightarrow K[y_1, \ldots, y_n]$ given by $y_1 := \varphi(f)$ and $y_i := \varphi(x_i)$ for $i \geq 2$. Consider the surjective homomorphism

$$K[x_1, \ldots, x_n] \xrightarrow{\varphi} K[y_1, \ldots, y_n] \twoheadrightarrow \frac{K[y_1, \ldots, y_n]}{\varphi(I)} =: G'$$

whose kernel is the ideal I.
Therefore $G \cong G'$. Moreover the ideal $(0 :_{G'} y_1)$ of G' is the image of the ideal $(0 :_G f)$ of G. Hence the result follows applying Theorem(3.2.2) to the ring $\dfrac{G'}{(0 :_{G'} y_1)}$. $\square$

3.3 Algebras associated to Apéry Sets

We want to investigate on the Lefschetz properties of a class of Artinian Gorenstein algebras obtained from numerical semigroups.

Let $S = \langle g_1 = m, g_2, \dots, g_n \rangle \subseteq \mathbb{N}$ be a numerical semigroup. Recall that in this case $\gcd(g_1, \dots, g_n) = 1$. For all the basic knowledge about numerical semigroups and semigroup rings consider as references [46] and [45].

The Apéry set of S with respect to the minimal generator of the semigroup is defined as the set

$$\mathrm{Ap}(S) := \{s \in S : s - g_1 \notin S\} = \{0 = \omega_1 < \omega_2 < \cdots < \omega_m = f + g_1\},$$

where f :=max$(\mathbb{N} \setminus S)$ is the Frobenius number of S. Note that $\mathrm{Ap}(S)$ is a finite set and $|\mathrm{Ap}(S)| = g_1 = m$.

Definition 3.3.1. Let $s \in S$. A representation of s is an n-uple $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $s = \sum_{i\geq 1}^n \lambda_i g_i$. The order of s is defined as

$$\mathrm{ord}(s) := \max\{\sum_{i\geq 1}^n \lambda_i : \lambda \text{ is a representation of } s\}.$$

A representation is said to be maximal if $\mathrm{ord}(s) = \sum_{i\geq 1}^n \lambda_i$.

Definition 3.3.2. The semigroup S is said M-pure symmetric if for each $i = 0, \dots, m$:
(1) $\omega_i + \omega_{m-i} = \omega_m$ and
(2) $\mathrm{ord}(\omega_i) + \mathrm{ord}(\omega_{m-i}) = \mathrm{ord}(\omega_m)$.

Therefore the Apéry set of a M-pure symmetric semigroup has the structure of a symmetric lattice.

Let K be a field of characteristic zero and consider the homomorphism:

$$\Phi : K[x_1, \dots, x_n] \longrightarrow K[t]$$

$$x_i \longmapsto t^{g_i}.$$

The one dimensional ring $R = K[S] := K[t^{g_1}, \dots, t^{g_n}] \cong \dfrac{K[x_1, \dots, x_n]}{ker(\Phi)}$ is called the semigroup ring associated to S.
We can associate to any representation of an element $s \in S$ a monomial in R by the correspondence

$$s = \sum_{i\geq 1}^n \lambda_i g_i \longleftrightarrow x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

induced by the preovious homomorphism. We can observe that the monomials in R that correspond to different representations of the same element s are equivalent modulo $ker(\Phi)$.

Consider now the zero dimensional ring $\overline{R} := R/x_1R$.
Notice that for $s = \sum_{i\geq 1}^{n} \lambda_i g_i \in \mathrm{Ap}(S)$, we have $\lambda_1 = 0$ and the correspondent monomial $\prod_{i=2}^{n} \overline{x_i}^{\lambda_i} \neq 0$ in $\overline{R}$. Conversely if $s \notin \mathrm{Ap}(S)$, then x_1 divides x^λ for at least one representation λ of s and hence

$$\overline{R} = \langle \overline{x^\lambda} \mid \sum_{i\geq 1}^{n} \lambda_i g_i \in \mathrm{Ap}(S) \rangle_K$$

is generated as a K-vector space by the classes modulo the ideal x_1R of the monomials x^λ for every representation λ of any element of $\mathrm{Ap}(S)$. Notice that in this way there is a one to one correspondence between the elements of $\mathrm{Ap}(S)$ and the generators of $\overline{R}$ as a K-vector space.

Recall that for a graded ring R and an homogeneous ideal $\mathfrak{m}$, the associate graded ring of R with respect to $\mathfrak{m}$ is defined as

$$\mathrm{gr}_{\mathfrak{m}}(R) := \bigoplus_{i\geq 0} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}.$$

Definition 3.3.3. Let $\overline{\mathfrak{m}}$ be the maximal homogeneous ideal of $\overline{R}$. Define

$$A = \mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{R})$$

to be the *associated graded algebra of the Apéry set of S.*

By definition of associated graded ring, we have that

$$A = \bigoplus_{i\geq 0}^{D} A_i = \langle \overline{x^\lambda} \mid \sum_{i\geq 1}^{n} \lambda_i g_i \in \mathrm{Ap}(S) \text{ and } \lambda \text{ is maximal } \rangle_K$$

is an Artinian standard graded K-algebra generated by the monomials $\overline{x^\lambda}$ associated to the maximal representations of the elements of $\mathrm{Ap}(S)$. Notice that the socle degree is $D = \mathrm{ord}(f + g_1)$ and that we can think of the homogeneous K-generators of A to have the same lattice structure of $\mathrm{Ap}(S)$. In the work *Goto Numbers of a Numerical Semigroup ring and the Gorensteiness of Associated Graded Rings* [43], Lance Bryant characterized when this kind of rings are Gorenstein.

Proposition 3.3.4. *Let S be a numerical semigroup. Then the ring A associated to* $\mathrm{Ap}(S)$ *is Gorenstein if and only if S is M-pure symmetric.*

Example 3.3.5. Consider the numerical semigroup $S = \langle 8, 10, 11, 12\rangle$.
Its Apéry set is $\mathrm{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$ and it can be easily checked that S is M-pure symmetric. The associated graded Artinian algebra is

$$A = K \oplus \langle y, z, w\rangle K \oplus \langle yz, yw, zw\rangle K \oplus \langle yzw\rangle K.$$

Since in the semigroup $22 = 11 + 11 = 10 + 12$, then in the ring A, $yw \equiv z^2$ and hence

$$A \cong \frac{K[y, z, w]}{(y^2, z^2 - yw, w^2)}.$$

In order to check when this kind of graded rings have the Lefschetz properties we want to use Theorem(3.1.6) and some consequences of it and therefore we need to find who is the polynomial F such that A is isomorphic to $Q/\mathrm{Ann}_Q(F)$ (see Theorem(3.1.3)) and compute his Hessians.

It is possible to write the Apéry set of S as $\mathrm{Ap}(S) = \bigcup_{d\geq 0}^{D} Ap_d$ where Ap_d is the set of the element of $\mathrm{Ap}(S)$ of order d. It is clear from the definition of A that the dimension of A_d as K-vector space is equal to the cardinality of Ap_d.

From now on we are going to use the previous notation x^λ for the homogeneous monomials in A instead of the heavier notation $\overline{x^\lambda}$ (or we will use the variables y, z, w if in codimension 3). We can choose as basis $\mathcal{B}_d$ of A_d a set of of monomials $\{x^{\lambda^1}, \dots, x^{\lambda^{b_d}}\}$ where the λ^j are maximal representations of the elements $\omega_j \in Ap_d$ and $b_d = \dim_K(A_d)$. For instance in Example(3.3.5) we can choose equivalently as basis for A_2 either the set $\{yz, yw, zw\}$ or the set $\{yz, z^2, zw\}$ since $yw \equiv z^2$ in such ring A.

Proposition 3.3.6. *Call Λ the set of the maximal representations of the maximal element of* $\mathrm{Ap}(S)$*, $f + g_1$.*
The graded ring A associated $\mathrm{Ap}(S)$ *is isomorphic to $Q/Ann_Q F$, where $Q = K[X_1, \dots, X_n]$ and $F = \sum_{\lambda\in\Lambda} x^\lambda$.*

Proof. We want to apply to this particular case of graded rings associated to the Apéry set of a semigroup the general proof of the existence of the polynomial F given by Maeno and Watanabe ([53], Theorem 2.1).
Identifying the algebra A with the quotient of Q by an ideal I (called the defining ideal of A), we have the exact sequence of modules $Q \to A \to 0$. That sequence induces another exact sequence

$$0 \to \mathrm{Hom}(A, K) \cong A \xrightarrow{\theta} \mathrm{Hom}(Q, K) \cong K[[x_2, \dots, x_n]].$$

Maeno and Watanabe proved that F is equal to $\theta(1) \in K[[x_2, \dots, x_n]]$ with $1 := (1, 0, \dots, 0) \in A$.
In order to use this fact we recall that the isomorphism between the ring A and $\mathrm{Hom}(A, K)$ is the application that maps $a = (a_0, \dots, a_D) \in A$ to the map $\varphi_a : A \to K$ defined by $\varphi_a(c) = \sum_{i=0}^{D} a_i c_{D-i}$ for each $c = (c_0, \dots, c_D) \in A$. We also recall that the isomorphism between $K[[x_2, \cdots, x_n]]$ and $\mathrm{Hom}(Q, K)$ is obtained identifying a formal power series f with the homomorphism which maps every monomial of Q to its correspondent numerical coefficient in the power series f.
Thus we have $1 \in A$ identified with the homomorphism φ_1 mapping $c = (c_0, \dots, c_D) \in A$ to his last component c_D. Hence we compute

$$F = \theta(1) = \sum \overline{\varphi_1}(X_2^{s_2} \cdots X_n^{s_n}) x_2^{s_2} \cdots x_n^{s_n},$$

where the sum is taken over the infinite basis of Q over K and $\overline{\varphi_1}(\alpha) := \varphi_1(\alpha + I)$ for $\alpha \in Q$.
Now we compute $\overline{\varphi_1}(X_2^{s_2} \cdots X_n^{s_n})$ for every possible values of the s_i. Let $\alpha = X_2^{s_2} \cdots X_n^{s_n}$ and $s = \sum_{i=1}^{n} s_i g_i \in S$. If $s \notin \mathrm{Ap}(S)$ or $s \in \mathrm{Ap}(S)$ but $\mathrm{ord}(s) > \sum_{i=1}^{n} s_i$, then $\alpha \in I$ and $\overline{\varphi_1}(\alpha) = 0$.
All the other monomials of the K-basis of Q are associated to a maximal representation of an element $\omega \in Ap_d$, therefore for every $\alpha = X_2^{s_2} \cdots X_n^{s_n}$ there exists a monomial X^λ such that $\alpha \equiv X^\lambda \bmod I$.

If $d < D$, the D-th component of X^λ is equal to zero and in this case $\overline{\varphi_1}(\alpha) = \overline{\varphi_1}(X^\lambda) = 0$. If instead $d = D$, it is clear that α is a maximal representation of $f + m$ and $\overline{\varphi_1}(\alpha) = \overline{\varphi_1}(X^\lambda) = 1$. Thus in the sum only survive the maximal representations of $f + g_1$ with coefficient 1. $\square$

Example 3.3.7. Let $S = \langle 16, 18, 21, 27\rangle$. We compute the Hessians to study the Lefschetz properties of the algebra A associated to the Apéry set of S.
We have that $\mathrm{Ap}(S) = \{0, 18, 21, 27, 36, 39, 42, 45, 54, 57, 60, 63, 72, 78, 81, 99\}$ and S is M-pure symmetric.

Doing the computation as in Example(3.3.5) we see that in this case

$$A \cong \frac{K[y, z, w]}{(y^5, z^3 - y^2 w, w^2, zw, y^3 z)}.$$

Hence the socle degree of A is $D = 5$. In the semigroup $99 = 4\cdot 18 + 27 = 2\cdot 18 + 3\cdot 21$ and by (3.3.6) the polynomial $F = y^4 w + y^2 z^3$. We choose as basis respectively $\{y, z, w\}$ for A_1 and $\{y^2, yz, z^2, yw\}$ for A_2.

We compute the first Hessian of F,

$$Hess^1(F) = \begin{pmatrix} y^2 w + z^3 & yz^2 & y^3 \\ yz^2 & zy^2 & 0 \\ y^3 & 0 & 0 \end{pmatrix}$$

The second Hessian is

$$Hess^2(F) = \begin{pmatrix} w & 0 & z & y \\ 0 & z & y & 0 \\ z & y & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix}$$

The hessians have both maximal rank, hence A has the SLP.

3.4 Complete Intersections Algebras

In this section we recall some results of D'Anna, Micale and Sammartano [44] that we need to characterize when the graded algebra associated to the Apéry Set of a numerical semigroup is a Complete Intersection. Let $S = \langle g_1 = m, g_2, \ldots, g_n\rangle$ be a numerical semigroup. In [44] the autors introduced two hyper-rectangles in $\mathbb{N}^{n-1}$ that contain the representations of the elements $\mathrm{Ap}(S)$ and whose properties determine in some way when the associated graded algebra $A = \bigoplus_{i\geq 0}^{D} A_i$ is a Complete Intersection. At the end of this section, combining Theorem(3.2.2) and a classical criterion for Weak Lefschetz properties of Complete Intersection algebras, we prove that any Complete Intersection algebra A associated to the Apéry Set of a numerical semigroup has the WLP.

Definition 3.4.1. For $2 \leq i \leq n$, define:
$\beta_i := \max\{h \in \mathbb{N} \mid hg_i \in \mathrm{Ap}(S) \text{ and } \mathrm{ord}(hg_i) = h\}$;
$\gamma_i := \max\{h \in \mathbb{N} \mid hg_i \in \mathrm{Ap}(S),\ \mathrm{ord}(hg_i) = h \text{ and } hg_i \text{ has a unique maximal representation}\}$.

The positive natural numbers β_i and γ_i are strongly related to the degrees of the generators of the defining ideal of A seen as quotient of the polynomial ring in $n-1$ variables.

Remark 3.4.2. For all $i = 2, \dots, n$, $\gamma_i \leq \beta_i$. But always $\gamma_2 = \beta_2$ and $\gamma_n = \beta_n$.

Proof. By definition $\gamma_i \leq \beta_i$ for every i. For the second statement assume $\gamma_2 < \beta_2$, and hence we must have that $(\gamma_2 + 1)g_2 = \sum_{j\neq 2} \lambda_j g_j$ are two different maximal representations of the same element of $\mathrm{Ap}(S)$. Hence they have the same order and therefore $(\gamma_2 + 1) = \sum_{j\neq 2} \lambda_j$, but this is impossible since $g_2 < g_3 < \ldots < g_n$. For the same reason it follows that $\gamma_n = \beta_n$. $\square$

For the proofs of all the following facts see [44](Section 2).

Proposition 3.4.3. *Let $\omega = \sum_{i=2}^{\nu} \lambda_i g_i \in \mathrm{Ap}(S)$ with $\lambda = (\lambda_2, \dots, \lambda_n)$ a maximal representation. Then $\lambda_i \leq \beta_i$ for each i. If λ is the maximum of the set of maximal representations of s with respect to the lexicographic order, then $\lambda_i \leq \gamma_i$ for each i.*

Definition 3.4.4. Define two hyper-rectangles in $\mathbb{N}^{n-1}$:
$B = \left\{ \sum_{i=2}^{n} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \beta_i \right\}$ and $\Gamma = \left\{ \sum_{i=2}^{n} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \gamma_i \right\}$

Using Proposition(3.4.3), it can be proved that

$$\mathrm{Ap}(S) \subseteq \Gamma \subseteq B$$

and moreover we can give some characterizations of when the possible equalities hold.

Proposition 3.4.5. *The following assertions are equivalent for a numerical semigroup S:*

1. $\mathrm{Ap}(S) = B$.
2. *The maximal element of* $\mathrm{Ap}(S)$, $f + g_1$ *has a unique maximal representation.*
3. *All the element of* $\mathrm{Ap}(S)$ *have a unique maximal representation.*
4. $g_1 = \prod_{i=2}^{n}(\beta_i + 1)$.
5. $D = \mathrm{ord}(f + g_1) = \sum_{i=2}^{n} \beta_i$.

Proposition 3.4.6. *The following assertions are equivalent for a numerical semigroup S:*

1. $\mathrm{Ap}(S) = \Gamma$.
2. $g_1 = \prod_{i=2}^{n}(\gamma_i + 1)$.
3. $D = \mathrm{ord}(f + g_1) = \sum_{i=2}^{n} \gamma_i$.

Consider now the ring $A = \bigoplus_{i\geq 0}^{D} A_i \cong \dfrac{K[x_2, \dots, x_n]}{I}$. associated to $\mathrm{Ap}(S)$. Such ring is a Complete Intersection if and only if I is minimally generated by $n-1$ elements. The next proposition is the key to characterize when it happens.

Proposition 3.4.7. *The defining ideal I of A always contains the ideal*

$$\widetilde{I} = (x_i^{\gamma_i+1} - \rho_i \prod_{j\neq i} x_j^{\lambda_j} \;:\; i = 2\ldots, n)$$

where $\rho_i = 0$ if $\beta_i = \gamma_i$ and $\rho_i = 1$ if $\beta_i > \gamma_i$. In the second case $(\gamma_i + 1)g_i = \sum_{j\neq i} \lambda_j g_j$ are two different maximal representation of the same element of $\mathrm{Ap}(S)$*. Furthermore $I = \widetilde{I}$ if and only if A is a Complete Intersection.*

This result is proved using the definitions of β_i and γ_i and considering the monomials of A corresponding to the representations of the elements of S. Let $s \in S$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a representation of s, the key fact is that the monomial $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in I$ if either $s \notin \mathrm{Ap}(S)$ or if λ is not maximal.

Theorem 3.4.8. *The followings assertions hold:*

1. *A is a Complete Intersection if and only if* $\mathrm{Ap}(S) = \Gamma$.
2. *A is a Complete Intersection and its defining ideal I is generated by monomials if and only if* $\mathrm{Ap}(S) = B$.

Proof. Observe that $\widetilde{I}$ is an ideal of $K[x_2, \ldots, x_n]$ generated by a regular sequence of $n-1$ elements, hence $\dfrac{K[x_2, \ldots, x_n]}{\widetilde{I}}$ is an Artinian Complete Intersection ring and therefore it has finite dimension as K-vector space.
Assertion 1 follows from the following inequality:

$$g_1 = |\mathrm{Ap}(S)| = \dim_K(\frac{K[x_2, \ldots, x_n]}{I}) \leq \dim_K(\frac{K[x_2, \ldots, x_n]}{\widetilde{I}}) = |\Gamma| = \prod_{i=2}^{n}(\gamma_i + 1)$$

applying Proposition(3.4.7) and Proposition(3.4.6).
For assertion 2 just observe that if $\mathrm{Ap}(S) = B$, then $\gamma_i = \beta_i$ for all i and use Proposition(3.4.7) and Proposition(3.4.5). $\square$

Example 3.4.9. Consider the numerical semigroup $S = \langle 15, 21, 35\rangle$.
Its Apéry Set is $\mathrm{Ap}(S) = \{0, 21, 35, 42, 56, 70, 63, 77, 91, 84, 98, 112, 119, 133, 154\}$. We can see that $84 = 21 \cdot 4 \in \mathrm{Ap}(S)$ and $105 = 21 \cdot 5 \notin \mathrm{Ap}(S)$, then $70 = 35 \cdot 2 \in \mathrm{Ap}(S)$ and $105 = 35 \cdot 3 \notin \mathrm{Ap}(S)$, hence $\beta_2 = \gamma_2 = 4$, $\beta_3 = \gamma_3 = 2$ and we can verify that

$$B = \Big\{ \sum_{i=2}^{3} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \beta_i \Big\} = \mathrm{Ap}(S).$$

The associated graded algebra is

$$A = \frac{K[y, z]}{(y^5, z^3)}.$$

and it is a monomial Complete Intersection.

Consider now the semigroup $S = \langle 8, 10, 11, 12\rangle$ as in Example(3.3.5).

In this case $\mathrm{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$ and we can see that $20 = 10\cdot 2 \notin \mathrm{Ap}(S)$, $24 = 12 \cdot 2 \notin \mathrm{Ap}(S)$, $33 = 11 \cdot 3 \in \mathrm{Ap}(S)$, $44 = 11 \cdot 4 \notin \mathrm{Ap}(S)$ and $11 \cdot 2 = 10 + 12$.
Hence $\beta_2 = \gamma_2 = 1$, $\beta_4 = \gamma_4 = 1$ but $1 = \gamma_3 < \beta_3 = 3$. Therefore

$$\Gamma = \Big\{ \sum_{i=2}^{4} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \gamma_i \Big\} = \mathrm{Ap}(S) \subsetneq B.$$

The associated graded algebra is

$$A = \frac{K[y,z,w]}{(y^2, z^2 - yw, w^2)}.$$

and it is a Complete Intersection but it is not monomial.

The next proposition is a standard result for WLP of Complete Intersection algebras that states that if there exists a minimal generator of the defining ideal having a sufficiently big degree (about the half of the socle degree), then A has the WLP. For the proof see ([50], 3.52 and 3.54).

Proposition 3.4.10. *Let* $A = \dfrac{K[x_1,\ldots,x_n]}{(f_1,\ldots,f_n)}$ *a Complete Intersection standard graded Artinian* K*-algebra.*
Call $d_i := degf_i$ *and assume that* $d_n \geq d_i \geq 2$ *for all* i*. Then, the condition*

$$d_n \geq d_1 + \ldots + d_{n-1} - n$$

implies that A *has the WLP.*

Corollary 3.4.11. *Let* $A = \bigoplus_{i\geq 0}^{D} A_i$ *be a standard graded algebra associated to the Apéry Set of a numerical semigroup. If* A *is a Complete Intersection and there exist* $\gamma_i \geq \dfrac{D-2}{2}$*, then* A *has the WLP.*

Proof. Since A is a Complete Intersection, $D = \sum_{i=2}^{n} \gamma_i$ by Proposition(3.4.6) and Theorem(3.4.8). Assume, by changing the order of the generators of the defining ideal of A, that $\gamma_n \geq \gamma_i$ for all i. Using notation of Proposition(3.4.10) we have $d_i = \gamma_i + 1$. Thus $W := d_n - (d_2 + \ldots + d_{n-1} - n + 1) = \gamma_n + 1 - (\sum_{i=2}^{n}(\gamma_i + 1) - \gamma_n - 1 - n + 1) = \gamma_n + 1 - (D + n - 1 - \gamma_n - n) = 2\gamma_n - D + 2$. By assumption $W \geq 0$ and hence by Proposition(3.4.10), A has the WLP. □

Theorem 3.4.12. *Let* $A = \bigoplus_{i\geq 0}^{D} A_i$ *be the graded algebra associated to the Apéry Set of a numerical semigroup. If* A *is a Complete Intersection, then* A *has the WLP.*

Proof. By Proposition(3.4.7), the defining ideal of A is

$$(f_2, \ldots, f_n) := (x_i^{\gamma_i+1} - \rho_i \prod_{j\neq i} x_j^{\lambda_j} \ : \ i = 2\ldots, n)$$

and we recall that $\rho_2 = \rho_n = 0$ by Remark(3.4.2).
By Corollary(3.4.11), if there exists $\gamma_i \geq t = \frac{D-2}{2}$, then A has the WLP. Hence assume $\gamma_i < t$ for every i and consider the Artinian Complete Intersection ring

$$B := \frac{K[x_2,\ldots,x_n]}{(x_2^N, f_3, \ldots, f_n)}$$

with $N \geq D - \gamma_2$. By Proposition(3.4.6), $D = \sum_{i=2}^{n} \gamma_i$ and therefore the socle degree of B is by construction $E = D - \gamma_2 + N - 1$. Now

$$\frac{E}{2} = \frac{D - \gamma_2 + N - 1}{2} \leq \frac{2N-1}{2} < N$$

and thus, Corollary(3.4.11) implies that B has the WLP.
Call $T := \dfrac{K[x_2, \dots, x_n]}{(f_3, \dots, f_n)}$. Since for every $i \neq 2$, $\deg(x_2, f_i) \leq \gamma_2$, we have $A \cong \dfrac{T}{(x_2^{\gamma_2+1})}$ and $B \cong \dfrac{T}{(x_2^N)}$. Hence

$$A \cong \frac{B}{(0 :_B x_2^{N-\gamma_2-1})}.$$

To prove the thesis of this theorem we need to show that $\dfrac{B}{(0 :_B x_2^C)}$ has the WLP for $1 \leq C \leq N - \gamma_2 - 1$. Use induction on C: when $C = 1$ this is Theorem(3.2.2). For $C > 1$, we can write

$$\frac{B}{(0 :_B x_2^C)} = \frac{\dfrac{B}{(0 :_B x_2^{C-1})}}{(0 :_{B'} \overline{x_2})}$$

with $B' := \dfrac{B}{(0 :_B x_2^{C-1})}$.
Now the result follows using Theorem(3.2.2) and inductive hypothesis. □

3.5 Codimension 3 Algebras

In this section we study the Gorenstein graded algebras associated to the Apéry Set of numerical semigroups in low codimension.

Let S be a M-pure symmetric numerical semigroup and let A be the graded algebra associated to $\mathrm{Ap}(S)$. Observe that if S is generated by n elements, then the codimension of the ring A is $n-1$. In [44] is proved that, when S is generated by 3 elements, it is M-pure symmetric if and only if $\mathrm{Ap}(S)$ is equal to the hyper-rectangle B and hence A is Gorenstein if and only if it is a monomial Complete Intersection. In this case it is known that A has the SLP.

A more interesting case that we are going to discuss is when S is generated by 4 natural numbers g_1, g_2, g_3, g_4. Write

$$A \cong \frac{K[y, z, w]}{I}.$$

In this context A has codimension 3 and, if it is a Complete Intersection, it has the WLP. But we recall that in general it is not known if a Gorenstein Artinian algebra of codimension 3 has the WLP.

Therefore for the rest of the section we assume that A is not a Complete Intersection. This means by Theorem(3.4.8) that $\mathrm{Ap}(S)$ is properly contained in the hyper-rectangle Γ. We want to characterize the defining ideal of A. In order to do this, we need some more results.

Lemma 3.5.1. *Let $S = \langle g_1, g_2, g_3, g_4 \rangle$ be a M-pure symmetric numerical semigroup and assume* $\mathrm{Ap}(S) \subsetneq \Gamma$. *Then $\gamma_3 < \beta_3$.*

Proof. Since $\mathrm{Ap}(S) \subsetneq \Gamma \subseteq B$, by Proposition(3.4.5), the maximal element of $\mathrm{Ap}(S)$, $f + g_1$ has more than one maximal representation. Subtracting common terms by two of this representations we obtain a double representation of an element of $\mathrm{Ap}(S)$

that has to be necessarily of the form $\lambda_3 g_3 = \mu_2 g_2 + \mu_4 g_4$ (since $g_2 < g_3 < g_4$) and the two different representations have the same order. This implies $\gamma_3 < \beta_3$ by definition. $\square$

Remark 3.5.2. The fact that A is not a Complete Intersection implies that there exists $\gamma_i < \beta_i$ for some $i = 2, \ldots, n$, is not true in codimension higher than 3. This is due to the fact that it may appear double maximal representations of elements of $\mathrm{Ap}(S)$ like $\sum_{i\in\mathcal{I}} \lambda_i g_i = \sum_{j\in\mathcal{J}} \mu_j g_j$ whit $\mathcal{I}, \mathcal{J} \subseteq \{2, \ldots, n\}$, $\mathcal{I} \cap \mathcal{J} = \emptyset$ and $|\mathcal{I}|, |\mathcal{J}| \geq 2$. For example consider the numerical semigroup $S = \langle g_1, g_2, g_3, g_4, g_5\rangle = \langle 6, 7, 8, 9, 10\rangle$. The Apéry Set of S is $\mathrm{Ap}(S) = \{0, 7, 8, 9, 10, 17\}$ and S is M-pure symmetric. Observe that $2g_i \notin \mathrm{Ap}(S)$ for every $i = 2, 3, 4, 5$ and hence $\gamma_i = \beta_i = 1$. Moreover $15 = 7 + 8 = 6 + 9 \in \Gamma \setminus \mathrm{Ap}(S)$. So in this case we have $\mathrm{Ap}(S) \subsetneq \Gamma = B$ and A is not a Complete Intersection.

Corollary 3.5.3. *There exists one element s of the Apéry Set of the numerical semigroup S which has the double representation*

$$s = (\gamma_3 + 1)g_3 = \mu_2 g_2 + \mu_4 g_4$$

with $1 \leq \mu_2 \leq \gamma_2$, $1 \leq \mu_4 \leq \gamma_4$ and $\mu_2 + \mu_4 = \gamma_3 + 1$. Hence the ideal

$$\widetilde{I} = (y^{\gamma_2+1}, z^{\gamma_3+1} - y^{\mu_2} w^{\mu_4}, w^{\gamma_4+1})$$

is properly contained in I.

Proof. It follows immediately from the previous Lemma(3.5.1) and from Proposition(3.4.7) and Remark(3.4.2). Notice that by Definition(3.4.1), this element $(\gamma_3 + 1)g_3$ is the minimal in $\mathrm{Ap}(S)$ with a double representation. The containment $\widetilde{I} \subsetneq I$ is proper since A is not a Complete Intersection. $\square$

Definition 3.5.4. Define the ring $G := \dfrac{K[y, z, w]}{\widetilde{I}}$.
We observe that G is a standard graded Artinian Complete Intersection algebra and

$$G = \langle x^\lambda \mid \sum_{i\geq 2}^{4} \lambda_i g_i \in \Gamma \text{ and } \lambda \text{ is maximal } \rangle_K.$$

Therefore A is isomorphic to a K-vector subspace of G.
As rings A is a quotient of G and we write

$$A \cong \frac{G}{J}$$

where $J := \widetilde{I}/I$.

We need to find which are the generators of the ideal J and for this reason, we want to characterize the elements of the set $\Gamma \setminus \mathrm{Ap}(S)$. Write $G = \bigoplus_{i\geq 0}^{D} G_i$ and $A = \bigoplus_{i\geq 0}^{D-C} A_i$ for a positive integer C. This two rings are equal if and only if A is a Complete Intersection and if and only if $C = 0$. The first "if and only if" follows from Theorem(3.4.8). We are going to explain the second one in the next proposition.

Call $\omega_D = \gamma_2 g_2 + \gamma_3 g_3 + \gamma_4 g_4$ the maximal element of Γ and $\omega_E = f + g_1$ the maximal element of $\mathrm{Ap}(S)$.

Proposition 3.5.5. *The following statements hold:*

1. $\omega_E = \omega_D - Cg_3$.

2. $C \leq \gamma_3$.

3. $\Gamma \setminus \mathrm{Ap}(S) = \{\omega \in \Gamma : \omega + Cg_3 \notin \Gamma\}$.

Proof. 1. We set, for each $\omega \in \Gamma$, the element $\omega' := \omega_E - \omega$. Clearly when $\omega \in \mathrm{Ap}(S)$, $\omega' \in \mathrm{Ap}(S)$ and $\mathrm{ord}(\omega)+\mathrm{ord}(\omega') = \mathrm{ord}(\omega_E)$ because S is M-pure symmetric. Instead, when $\omega \notin \mathrm{Ap}(S)$, also $\omega' \notin \mathrm{Ap}(S)$. Also the set Γ is clearly symmetric by construction: $\omega_D - \upsilon \in \Gamma$ for each $\upsilon \in \Gamma$ and also in this case an analogous equality for the orders of the elements holds, meaning that $\mathrm{ord}(\upsilon) + \mathrm{ord}(\omega_D - \upsilon) = \mathrm{ord}(\omega_D)$.

By way of contradiction assume $\omega_E = \omega_D - g_2 - \upsilon$ with $\upsilon \in \Gamma$. Thus $\omega_E + g_2 = \omega_D - \upsilon \in \Gamma$. But, by definition $\gamma_2 g_2 \in \mathrm{Ap}(S)$ and, since by Remark(3.4.2), $\gamma_2 = \beta_2$, it follows that $\omega_E = \gamma_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4$. This fact implies $\omega_E + g_2 \notin \Gamma$.
In the same way, using that $\gamma_4 g_4 \in \mathrm{Ap}(S)$ and $\gamma_4 = \beta_4$, it is possible to show that ω_E cannot be of the form $\omega_D - g_4 - \upsilon$ with $\upsilon \in \Gamma$ and hence $\omega_E = \omega_D - Cg_3$ with $C = \mathrm{ord}(\omega_D) - \mathrm{ord}(\omega_E)$.
2. Follows immediately from item 1 by definitions of ω_D.
3. Set $W := \{\omega \in \Gamma : \omega + Cg_3 \notin \Gamma\}$ and take $\omega \in W$. By item 1, we can write for each $\omega \in \Gamma$, $\omega' = \omega_D - Cg_3 - \omega$. Assuming $\omega \in \mathrm{Ap}(S)$, we also have $\omega' \in Ap(S) \subseteq \Gamma$. Therefore by definition, $\omega + Cg_3 = \omega_D - \omega' \in \Gamma$ and this is a contradiction since $\omega \in W$. This proves $W \subseteq \Gamma \setminus \mathrm{Ap}(S)$, now we prove the reverse inclusion.

Take now $\omega \in \Gamma \setminus \mathrm{Ap}(S)$. As said before, we have in this case $\omega' = \omega_E - \omega = \omega_D - Cg_3 - \omega \notin \mathrm{Ap}(S)$. If we assume $\omega \notin W$, we have $\omega + Cg_3 \in \Gamma$ and hence $\omega' \in \Gamma$. Thus $\omega' \in \Gamma \setminus \mathrm{Ap}(S)$ and, by definition of $\mathrm{Ap}(S)$, for every $\overline{\omega} \in \Gamma$, $\overline{\omega} + \omega' \notin \mathrm{Ap}(S)$. But certainly it must exists a minimal generator of the semigroup $g_j \in \mathrm{Ap}(S)$ (with $j \neq 1$) such that $\omega - g_j \in \Gamma$ and, setting $\overline{\omega} := \omega - g_j$, we have $\overline{\omega} + \omega' = \omega_D - Cg_3 - g_j = \omega_E - g_j \in \mathrm{Ap}(S)$. Therefore we must have $\omega \in W$. □

Theorem 3.5.6. *Assume with the same notations as before,* $G = \bigoplus_{i\geq 0}^{D} G_i$ *and* $A = \bigoplus_{i\geq 0}^{D-C} A_i$. *Set* $h_2 = \gamma_2 - \mu_2 + 1$, $h_3 = \gamma_3 - C + 1$ *and* $h_4 = \gamma_4 - \mu_4 + 1$ *where* μ_2 *and* μ_4 *are given in Corollary(3.5.3). Thus the defining ideal of* A *is*

$$I = \widetilde{I} + (z^{h_3} y^{h_2}, z^{h_3} w^{h_4}).$$

Moreover

$$A = \frac{G}{(0 :_G z^C)}.$$

Proof. We have seen that A is isomorphic to G modulo the ideal $J := \widetilde{I}/I$. By construction of these two algebras, the ideal J is generated by the elements $\{x^\lambda \mid \sum_{i\geq 2}^{4} \lambda_i g_i \in \Gamma \setminus \mathrm{Ap}(S)$ and λ is maximal $\}$.
Hence we need to show that $y^{h_2} z^{h_3}$ and $z^{h_3} w^{h_4}$ are the unique monomial representations of the minimal elements of $\Gamma \setminus \mathrm{Ap}(S)$ with respect to the standard partial order of $\Gamma \subseteq \mathbb{N}^3$.
Take $\omega = \lambda_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4 \in \Gamma \setminus \mathrm{Ap}(S)$, thus by Proposition(3.5.5) $\omega + Cg_3 \notin \Gamma$ and hence $\lambda_3 + C \geq \gamma_3 + 1$. Moreover, to be out of Γ we need either $\lambda_2 \geq \gamma_2 - \mu_2 + 1$ or $\lambda_4 \geq \gamma_4 - \mu_4 + 1$. Indeed $Cg_3 + h_2 g_2 + h_3 g_3 = Cg3 + (\gamma_2 - \mu_2 + 1) g_2 + (\gamma_3 - C + 1) g_3 = (\gamma_3 + 1) g_3 + (\gamma_2 - \mu_2 + 1) g_2 = \mu_2 g_2 + \mu_4 g_4 + (\gamma_2 - \mu_2 + 1) g_2 = \mu_4 g_4 + (\gamma_2 + 1) g_2 \notin \Gamma$.
Similarly we obtain $Cg_3 + h_4 g_4 + h_3 g_3 = \mu_2 g_2 + (\gamma_4 + 1) g_4 \notin \Gamma$.
Therefore the minimal elements of $\Gamma \setminus \mathrm{Ap}(S)$ are $h_2 g_2 + (\gamma_3 - C + 1) g_3$ and $h_4 g_4 +$

$(\gamma_3 - C + 1)g_3$ and this complete the proof of the first part of the Theorem.
For the "moreover" statement notice that, taken $f \in G$ homogeneous, then $f \in (0 :_G z^C)$ if and only if $z^C f \in \widetilde{I}$ and this happens if and only if the monomials of f are correnspondent to element of $\Gamma \setminus \mathrm{Ap}(S)$ that means $f \in (z^{h_3}y^{h_2}, z^{h_3}w^{h_4})$. □

Using the characterization of the defining ideal of A given by the last result we are able to prove that A has the WLP.

Theorem 3.5.7. *Let $S = \langle g_1, g_2, g_3, g_4 \rangle$ be an M-pure symmetric numerical semigroup generated by 4 elements. Then the ring A associated to $\mathrm{Ap}(S)$ has the WLP.*

Proof. By Theorem(3.4.8) if $\mathrm{Ap}(S) = \Gamma$, the ring A is a codimension 3 Complete Intersection and therefore it has the WLP. Using the same notation of all this Section, we call $z := x_3$ and we assume $\mathrm{Ap}(S) \subsetneq \Gamma$.
In this case, by Theorem(3.5.6) there exists a Complete Intersection Artinian standard graded algebra G such that $A \cong \dfrac{G}{(0 :_G z^C)}$ for $1 \leq C \leq \gamma_3$.
Now we can conclude applying Theorem(3.2.2) inductively on C as done in the proof of Theorem(3.4.12). □

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