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Arf Good Semigroups

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Contents

Introduction

In this Ph.D. thesis we give results regarding an important class of good subsemigroups of $\mathbb{N}^r$. The concept of good semigroup was introduced in [3]. Its definition depends on the properties of the value semigroups of one dimensional analytically unramified rings (for example the local rings of an algebraic curve), but in the same paper it is shown that the class of good semigroups is bigger than the class of value semigroups. Therefore the good semigroups can be seen as a natural generalization of the numerical semigroups and can be studied without referring to the ring theory context, with a more combinatorial approach. Here we focus on the local good semigroups, i.e good semigroups $S \subseteq \mathbb{N}^r$ such that the only element of S with zero component is the zero vector.

We focus on the class of local Arf good semigroups. This is motivated by the importance of the Arf numerical semigroups in the study of the equivalence between two algebroid branches. Given an algebroid branch R, its multiplicity sequence is defined to be the sequence of the multiplicities of the successive blowups R_i of R. Two algebroid branches are equivalent if and only if they have the same multiplicity sequence (cf. [5, Definition 1.5.11]). In [1] Cahit Arf introduced the concept of Arf ring and it is shown that for each algebroid branch R there exists a smallest Arf overring ${}^*R$, called the Arf closure of R, that has also the same multiplicity sequence of R, and it is described a procedure to compute it. In the same paper it is proved that two algebroid branches are equivalent if and only if their Arf closures have the same value semigroup, that is a numerical Arf semigroup, i.e. a numerical semigroup S such that $S(s) - s$ is a semigroup, for each $s \in S$, where $S(s) = \{n \in S; n \geq s\}$.

All these facts can be generalized to algebroid curves (with more than one branch), and this naturally leads to give a more general definition of Arf ring and of Arf good semigroup of $\mathbb{N}^r$.

In the numerical case an Arf semigroup $S = \{s_0 = 0 < s_1 < s_2, \ldots\}$ is completely described by its multiplicity sequence, that is the sequence of the differences $s_{i+1} - s_i$. Extending the concept of multiplicity sequence, in [3] it is also shown that to each local Arf good semigroup can be associated a multiplicity tree that characterizes the semigroup completely. A tree T of vectors of $\mathbb{N}^r$ has to satisfy some properties to be a multiplicity tree of a local Arf good semigroup. For instance it must have multiplicity sequences along its branches (since the projections are Arf numerical semigroups) and each node must be able to be expressed as a sum of nodes in a subtree of T rooted in it. Thus, taking in account this 1-1 correspondence, our aim is to study Arf good semigroups by characterizing their multiplicity trees, finding an unambiguous way to describe them. Using this approach, we can also deal with the problem of finding the

Arf closure of a good semigroup S, that is the smallest Arf semigroup containing S.

Given an algebroid curve R, it is still possible to consider the Arf closure of R as the smallest Arf ring ${}^*R$ containing R. In [4] it is proved that two algebroid curves are equivalent if and only if it is possible to permute their branches in a way that the value semigroups of their Arf closures (that are Arf good semigroups) have the same multiplicity tree. This stresses the importance to have a fast way to compute the Arf closure of an algebroid curve, and we see how the characterization of the properties of the multiplicity trees can be useful to this aim.

The structure of the thesis is the following.

In Chapter 1 we define and give the main properties of all the basic objects of this thesis. In order to motivate the definition of the concept of Arf good semigroup, we recall the main aspects of the Arf theory, the reasons behind its introduction in the study of algebroid branches and its generalization to algebroid curves.

In Chapter 2, we start considering the Arf good subsemigroups of $\mathbb{N}^r$ by focusing on the properties that arise from their combinatorial interpretation, in order to find an unambiguous way to describe them and to deal with the problem of finding the Arf closure of a good semigroup. Given a collection of r multiplicity sequences E, we define the set $\sigma(E)$ of all the Arf semigroups S such that the i-th projection S_i is an Arf numerical semigroup associated to the i-th multiplicity sequences of E. We define also the set $\tau(E)$ of the corresponding multiplicity trees and we describe a tree in $\tau(E)$ by an upper triangular matrix $(p_{i,j})$, where $p_{i,j}$ is the highest level where the i-th and j-th branches are glued, and we give a way to deduce from E the maximal value that can be assigned to $p_{i,j}$. This fact lets us to understand when the set $\sigma(E)$ is finite. We introduce the class of untwisted trees that are easier to study because they are completely described by the second diagonal of their matrix, and we notice that a tree can be always transformed into an untwisted one by permuting its branches.

In Section 2.2 we address the problem of understanding when a set of vectors $G \subseteq \mathbb{N}^r$ determines uniquely an Arf semigroup of $\mathbb{N}^r$. Thus, we define Arf(G) as the minimum of the set $S(G) = \{S : S \subseteq \mathbb{N}^n \text{ is an Arf semigroup and } G \subseteq S\}$, and we find the properties that G has to satisfy in order to have a good definition for Arf(G) (cf. Theorem 2.2.1). Finally, given a G satisfying these properties, we give a procedure for computing Arf(G).

In Section 2.3 we adapt the techniques learned in the previous section to the problem of determining the Arf closure of a good semigroup. In [8], the authors solved this problem for $r = 2$, leaving it open for larger dimensions. In this section, we use the fact that a good semigroup S can be completely described by its finite subset Small$(S) = \{s \in S : s \leq \delta\}$, where δ is the smallest element such that $\delta + \mathbb{N}^r \subseteq S$, whose existence is guaranteed by the properties of the good semigroups.

Finally, in Section 2.4, we address the inverse problem: given an Arf semigroup $S \subseteq \mathbb{N}^r$, find a set of vectors $G \subseteq \mathbb{N}^r$, called set of generators of S, such that Arf$(G) = S$, in order to find a possible generalization of the concept of characters introduced for the numerical case. In Theorem 2.4.1, we find the properties that such a G has to satisfy and we focus on the problem of finding a minimal one. From this point of view we are able to give a lower and an upper bound for the minimal cardinality for a set of generators of a given Arf semigroup (Corollary 2.4.8). With an example we also show that, given an Arf semigroup S, it is possible to find minimal sets of generators with distinct cardinalities. The contents of this chapter are based on [18].

In Chapter 3, we give useful procedures to compute all the Arf good semigroups satisfying some specific conditions.

In Section 3.1 we consider the problem of finding the set Cond($\mathbf{c}$) consisting of all the Arf good subsemigroups of $\mathbb{N}^r$ with a fixed conductor $\mathbf{c}$, where the conductor $\mathbf{c}$ of a good semigroup $S \subseteq \mathbb{N}^r$ is the minimal vector such that $\mathbf{c} + \mathbb{N}^r \subseteq S$. When $r = 1$, the problem is equivalent to finding the set of the multiplicity sequences of all the Arf numerical semigroups with a fixed conductor. This question was already addressed in [12], where the authors found a recursive algorithm for the computation of such a set. In Subsection 3.1.1, it is presented a non-recursive procedure to determine such a set, that is faster than the previous one, when used for large value of the conductor.

In Subsection 3.1.3 we address the general case and, using Lemma 3.1.4 and the base cases for $r = 1$ and $r = 2$, we are able to present a procedure that builds inductively the sets Cond($\mathbf{c}$) in all dimensions and for any value of the vector $\mathbf{c}$. We give a strategy for computing the set $\overline{\text{Cond}(\mathbf{c})}$ of all the possible multiplicity trees (twisted and untwisted) associated to an Arf semigroup with conductor $\mathbf{c}$. At the end of the section we give an example with $r = 3$ on the computation of this set and we present some tables containing the cardinalities of the constructed sets for particular values of the conductor $\mathbf{c}$.

In Section 3.2, we give a procedure that computes the set Gen(r, n) of the untwisted multiplicity trees of all the Arf good semigroups of $\mathbb{N}^r$ with genus n. The procedure works inductively and it is based on Theorem 3.2.1, that gives a way to compute the genus of an Arf good semigroup of $\mathbb{N}^r$ with an untwisted multiplicity tree from its representation T_E, and on the numerical case $r = 1$, that is solved in Subsection 3.2.1 by accordingly adapting the algorithm given for the conductor in the previous section (this problem was also considered in [12]). Finally, we give a strategy for computing the set $\overline{\text{Gen}(r, n)}$ of all the possible multiplicity trees (twisted and untwisted) associated to an Arf semigroup with genus n in $\mathbb{N}^r$. At the end of the section we give an example of the application of the developed procedure and we present some tables containing cardinalites of the sets Gen(r, n) for some values of r and n. The contents of this chapter are based on the papers [19] and [20]

In Chapter 4, we deal with the problem of finding an efficient algorithm for the computation of the Arf closure of an algebroid curve with more than one branch. In particular we generalize the procedure presented in [2], where Arslan and Sahin addressed the algebroid branch case.
In Section 4.1, we introduce an algorithm for the computation of the multiplicity tree of an algebroid curve with two branches R starting from its parametrization. This algorithm will return the parametrizations of all rings R_i in the Lipman sequence. Then we give a way to recover a presentation for the Arf Closure ${}^*R$ from the information contained in the multiplicity tree (cf. Discussion 4.2.2).
In Section 4.3 we see how to generalize the algorithm presented in the previous section to the case of curves with an arbitrary number of branches.
In Section 2.3, we give a way to improve the efficiency of our algorithm. In particular, we see that it is possible to compute the Arf closure of R by applying the algorithm to an alge-

broid curve with a simpler parametrization obtained by truncating all the monomials with order bigger than the conductor of the Arf semigroup $\nu(R^*)$ (cf.Theorem 4.4.1). Thus, in order to determine this bound, we need a way to estimate the conductor of $\nu(R^*)$ directly from the parametrization of R. We firstly analyse the case of curves with two branches having distinct multiplicity sequences along their branches (we can recover the multiplicity sequences by using the algorithm of Arslan and Sahin on each branch). In this case, it is possible to find a limitation for the conductor by using only the numerical properties given by the multiplicity sequences (cf.Proposition 2.1.2). Then, we study the case of two-branches algebroid curves with the same multiplicity sequence on their branches. In this case, we need to work on the parametrization of R to find a suitable bound (cf.Lemma 4.4.4 and Proposition 4.4.5). We conclude by seeing how it is possible to use the bound in the two-branches case to compute a bound in the general case (cf.Remark 22). In the end, we present an example that illustrates how the computation of the Arf closure is simplified by the truncation given by the given bound (cf.Example 4.4.6). The contents of this chapter are based on the results contained in [15].
All the procedures presented here have been implemented in GAP ([11]). The corresponding codes can be found in `https://github.com/pedritomelenas/Arf-semigroups`.

Chapter 1

Preliminaries

In this chapter we define all the main objects of this thesis, in order to explain the reasons behind the introduction of the concept of Arf good semigroup. In Section 1.1 we recall the definition and some basic properties of the numerical semigroups. In Section 1.2 we introduce the concept of algebroid branch explaining how it is related to the numerical semigroups and we give the definition of equivalence between these objects. Then, we introduce the Arf's theory, giving the concepts of Arf closure of an algebroid branch and of Arf numerical semigroup, showing their usefulness in the problem of establishing equivalence. Finally, in Section 1.4, we explain how it is possible to extend the aforementioned constructions for the algebroid branches to the more general context of algebroid curves. In particular, we present the main properties of the good subsemigroups of $\mathbb{N}^r$ that naturally arise as a generalization of numerical semigroup, and we study the Arf property in this case.

1.1 Generalities on numerical semigroups

Definition 1.1.1. A *numerical semigroup* S is a submonoid of $(\mathbb{N}, +)$ having finite complement in $\mathbb{N}$, that is, $|\mathbb{N} \setminus S| < \infty$.

Definition 1.1.2. Given a numerical semigroup S, the maximum $F(S)$ of the set $\mathbb{Z} \setminus S$ is known as the *Frobenius number* of S.

The *conductor* $c(S)$ of S is the smallest number such that $n \in S$ for all $n \geq c(S)$, and it is clear that we have $c(S) = F(S) + 1$.

Definition 1.1.3. Given a numerical semigroup S, the cardinality of the finite set $\mathbb{N} \setminus S$ is called *genus* of S, and denoted by $g(S)$. The elements of $\mathbb{N} \setminus S$ are called *gaps* of the semigroup.

Proposition 1.1.4. *The submonoid*

$$S = \langle g_1, \dots, g_k \rangle = \left\{ \sum_{i=1}^{k} n_i g_i | n_i \in \mathbb{N} \right\}$$

is a numerical semigroup if and only if $\gcd(g_1, \dots, g_k) = 1$.

Definition 1.1.5. $E \subseteq S$ is an *ideal* of S if for all $e \in E$ and for all $s \in S$ we have $e + s \in E$.

A *system of generators* of a numerical semigroup S is a set of elements A such that $\langle A \rangle = S$.

Proposition 1.1.6. *For every numerical semigroup S there exists a unique minimal system of generators (with respect to inclusion).*

Denoted by $M = S \setminus \{0\}$ the maximal ideal of S, and by $nM = \{m_1 + \cdots + m_n | m_i \in M\}$, we have that $M \setminus 2M$ is the required minimal system of generators.

Example 1.1.7. Let us consider the numerical semigroup

$$S = \{0, 4, 6, 8, 10, 12, 13, 14, 16, \rightarrow\},$$

where with $n \rightarrow$ we mean that all the integers larger than n are in S.

We have $M = \{4, 6, 8, 10, 12, 13, 14, 16, \rightarrow\}$ and $2M = \{8, 10, 12, 14, 16, \rightarrow\}$.

Thus $S = \langle 4, 6, 13 \rangle$.

Definition 1.1.8. We call *embedding dimension* of a numerical semigroup, and we denote it by $e.d.(S)$ the cardinality $|M \setminus 2M|$ of its minimal system of generators. The smallest number among the generators, is called *multiplicity* of the semigroup and denoted by $e(S)$.

Notice that the inequality $e.d.(S) \leq e(S)$ holds since if $x, y \in M \setminus 2M$ and $x \neq y$, then x and y have to be different modulo $e(S)$ for the minimality of the system of generators.

1.2 Algebroid branches

The concept of numerical semigroup plays a significant role in algebraic geometry and ring theory. Under certain circumstances, it is possible to associate to a ring R a numerical semigroup that can encode some of its properties. The following class of rings is an example of this situation.

Example 1.2.1. Let R be a one-dimensional local domain and suppose that R is also analytically irreducible, i.e., the completion $\hat{R}$ is a domain, or, equivalently, the integral closure $\overline{R}$ of R is a DVR, finite over R. Furthermore, denoted by $\mathfrak{M}$ and $\mathfrak{N}$ the maximal ideals, respectively of R and $\overline{R}$, we suppose that $R/\mathfrak{M} \cong \overline{R}/\mathfrak{N}$. Since $\overline{R}$ is a DVR, its maximal ideal $\mathfrak{N}$ has the form $\mathfrak{N} = (t)$, then if $r \in \overline{R}$ we can write $r = ut^n$ where u is an invertible element of $\overline{R}$. Hence there exists a valuation $\nu : \overline{R} \rightarrow \mathbb{N} \cup \{\infty\}$ such that $\nu(0) = \infty$ and $\nu(r = ut^n) = n$. Then the set

$$\nu(R) = \{\nu(R) | r \in R \setminus \{0\}\}$$

is a numerical semigroup (the fact that the complement $\mathbb{N} \setminus \nu(R)$ is finite follows from the fact that $\overline{R}$ is finitely generated as R-module).

An important example of rings satisfying the previous properties are the algebroid branches. We firstly give the definition of algebroid curve.

Definition 1.2.2. An *algebroid curve* R, is a one-dimensional local ring, complete for the $\mathfrak{M}$-adic topology (being $\mathfrak{M}$ is maximal ideal). We denote by $\mathbb{K} \cong R/\mathfrak{M}$ its coefficient field.

Definition 1.2.3. By an *algebroid branch* we mean an algebroid curve that is also a domain.

The following result, due to Cohen (cf. [6, Cohen 's Structure Theorem]) gives us important information regarding the structure of an algebroid curve.

Theorem 1.2.4. *Let R be an algebroid curve, then there exists an ideal I of $\mathbb{K}[[x_1, \ldots, x_k]]$ such that $R \cong \mathbb{K}[[x_1, \ldots, x_k]]/I$. Furthermore, there exists prime ideals $P_1, \ldots, P_r \subseteq \mathbb{K}[[x_1, \ldots, x_k]]$ such that $I = \bigcap_{i=1}^{r} P_i$. Notice also that $k = e.d.(R) = \dim(\mathfrak{M}/\mathfrak{M}^2)$.*

We firstly focus on the algebroid branches and their connection with numerical semigroups.

From the Cohen's Structure Theorem follows that if R is an algebroid branch, then $\overline{R} \cong \mathbb{K}[[t]]$. A consequence of this fact is that we can always associate to a branch a parametrization in power series.

Definition 1.2.5. Let R be an algebroid branch and $x_1, \ldots, x_N$ a system of generators for the maximal ideal $\mathfrak{M}$ of R. Let us consider the map

$$\mathfrak{C} : \mathbb{K}[[X_1, \ldots, X_N]] \to R,$$

such that $\mathfrak{C}(X_i) = x_i$ for all i. The map $\mathfrak{C}$ exists by Theorem 1.2.4. Then a *parametrization* of R is a $\mathbb{K}-$algebra homomorphism

$$\Psi : \mathbb{K}[[X_1, \ldots, X_N]] \to \mathbb{K}[[t]],$$

such that $\ker(\mathfrak{C}) \subseteq \ker(\Psi)$. Thus, we have

$$R \cong \mathbb{K}[[\phi_1(t), \ldots, \phi_N(t)]],$$

where $\phi_i(t) = \Psi(X_i) \in \mathbb{K}[[t]]$. Hence if $f \in \ker(\mathfrak{C})$ we have $f(\phi_1(t), \ldots, \phi_N(t)) = 0$.

Example 1.2.6. Let us consider the algebroid curve

$$R = \mathbb{K}[[x, y]]/(y^2 - x^3 - x^2).$$

The polynomial $y^2 - x^3 - x^2$ is irreducible in $\mathbb{K}[x, y]$ but not in its completion with respect the maximal ideal (x, y). In fact, we have

$$y^2 - x^3 - x^2 = (y - x\sqrt{1+x})(y + x\sqrt{1+x}),$$

where $\sqrt{1+x} \in \mathbb{K}[[x, y]]$. Thus, R is not an algebroid branch but an algebroid curve consisting of two algebroid branches. Specifically, they are $\mathbb{K}[[x, y]]/P_1$ and $\mathbb{K}[[x, y]]/P_2$, where $P_1 = \langle y - x - \frac{1}{2}x^2 + \frac{1}{8}x^3 + \ldots \rangle$ and $P_2 = \langle y + x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \ldots \rangle$.

The parametrizations corresponding to the two branches are:

- $\mathbb{K}[[x,y]]/P_1 = \mathbb{K}[[t, t + \frac{1}{2}t^2 - \frac{1}{8}t^3 + \dots]]$;
- $\mathbb{K}[[x,y]]/P_2 = \mathbb{K}[[t, -t - \frac{1}{2}t^2 + \frac{1}{8}t^3 + \dots]]$.

The fact that the integral closure $\bar{R}$ is isomorphic to the DVR $\mathbb{K}[[t]]$, let us also to consider, as in Example 1.2.1, a valuation ν and the numerical semigroup $\nu(R) = S$.

Definition 1.2.7. The multiplicity of an algebroid branch R, is given by the smallest positive value $e(R)$ in $S = \nu(R)$.

Example 1.2.8. Let us consider the algebroid branch

$$R = \mathbb{K}[[x,y,z]]/(x^3 - yz, y^3 - z^2) \cong \mathbb{K}[[t^5, t^6, t^9]].$$

We have $\overline{R} = \mathbb{K}[[t]]$ and

$$S = \nu(R) = \langle 5, 6, 9 \rangle = \{0, 5, 6, 9, 10, 11, 12, 14, \rightarrow\}.$$

We have $e(R) = 5$.

Notice that, if we represent an algebroid branch by a parametrization $\mathbb{K}[[\phi_1(t), \dots, \phi_r(t)]]$, then we can deduce that $e(R) = \min\{\text{ord}(\phi_i(t) - \phi_i(0)) : i = 1, \dots, r\}$.

Example 1.2.9. Let us consider the algebroid branch

$$R = \mathbb{K}[[t^4, t^6 + t^7]].$$

Then $\nu(R) = \langle 4, 6, 13 \rangle = \{0, 4, 6, 8, 10, 12, 13, 14, 16, \rightarrow\}$. Notice that the embedding dimension of R is two, while $e.d.(\nu(R)) = 3$.

1.2.1 Equivalence between algebroid branches

Consider an algebroid branch $R = \mathbb{K}[[\phi_1(t), \dots, \phi_r(t)]]$. We suppose that $\phi_i(0) = 0$ for all $i = 1, \dots, r$, thus the branch pass through the origin of $\mathbb{A}^r$. Without loss of generality, we can assume that $\phi_1(t)$ is such that $\text{ord}(\phi_1(t)) = e(R)$. Furthermore it is not restrictive to assume that $\text{ord}(\phi_i) > e(R)$ for all $i \neq 1$. We want to find a way to deduce from the parametrization of R the parametrization of the blowup of the branch at the origin (in the affine chart where it intersect the exceptional divisor). We recall that the blowup of the affine space $\mathbb{A}^r$ at the origin can be described in the following way:

$$\text{Bl}(\mathbb{A}^r) = \{((x_1, ..., x_r), [a_1 : \dots : a_r]) \in \mathbb{A}^r \times \mathbb{P}^{r-1} \mid x_i a_j = x_j a_i,\ 1 \leq i, j \leq r\ \}.$$

Denote by U_1 the affine chart corresponding to the points of $\text{Bl}(\mathbb{A}^r)$ such that $a_1 \neq 0$.

In U_1 we can therefore consider the local coordinates X_i:

$$X_1 = x_1,\ X_2 = a_2/a_1, ..., X_r = a_r/a_1.$$

U_1 is the affine chart in which the transformed branch intersects the exceptional divisor (it follows from the fact that $\phi_1(t)$ has the least order among the $\phi_i(t)$).

The parametrization of the transformed branch is therefore given by

$$\text{Bl}(R) = \mathbb{K}\left[\left[\phi_1(t), \frac{\phi_2(t)}{\phi_1(t)} \dots, \frac{\phi_r(t)}{\phi_1(t)}\right]\right].$$

Thus the blowup of an algebroid branch is still an algebroid branch and from the assumptions on the $\phi_i(t)$, it still passes through the origin. Notice that

$$e(\text{Bl(R)}) = \min\{\text{ord}(\phi_1(t)), \text{ord}(\phi_2(t)/\phi_1(t)), \dots, \text{ord}(\phi_r(t)/\phi_1(t))\} \leq \text{ord}(\phi_1(t)) = e(R),$$

so the multiplicity of the blowup of R is less or equal than the multiplicity of R. So, it is possible to repeat the previous process. If we set $B_2 = \text{Bl}(R)$ and $B_i = \text{Bl}(B_{i-1})$, we can consider the following sequence of algebroid branches:

$$R \subseteq B_2 \subseteq \dots B_n \subseteq \dots.$$

The fact that the integral closure $\overline{R} = \mathbb{K}[[t]]$ is finite over R guarantees that there exists a $N \in \mathbb{N}$, such that $B_n = \mathbb{K}[[t]]$ for all $n \geq N$. From the geometric point of view, it means that the singularity of the algebroid branch at the origin can be solved after considering a sufficient number of blowups.

Definition 1.2.10. If R is an algebroid branch, we can consider the previous chain of blowups

$$R \subseteq B_2 \subseteq \dots B_n \subseteq \dots \subseteq B_N = \mathbb{K}[[t]] = B_{N+1} = \dots$$

The non-increasing sequence

$$e(R) \geq e(B_2) \geq e(B_N) \dots 1, 1, \dots,$$

is known as the *multiplicity sequence of the algebroid branch* R.

The concept of multiplicity sequence plays a significant role in the study of the algebroid branches as we can see in the following definition.

Definition 1.2.11. Two algebroid branches are said to be *equivalent* if they have the same multiplicity sequence.

This equivalence extends the Zariski equivalence between plane branches (cf. [17]) to branches of any embedding dimension and has been studied by several authors (cf. e.g. [5, Definition 1.5.11]).

The problem of determining the equivalence between two algebroid branches through the calculation of their successive blowups, despite being more geometrical in nature, was completely solved by Cahit Arf by focusing on its algebraic aspect. In his paper [1], he answered to some open questions arised by Du Val in [10], regarding the computation of the multiplicity sequence of a branch starting from its parametrization. In the following section we summarize the theory developed by Arf.

1.3 Arf's theory

Let us consider an algebroid branch R. We know from Theorem 1.2.4 that it can be seen as a subring of the formal power series $\mathbb{K}[[t]]$ and we can consider the numerical semigroup $\nu(R)$. We denote by $c = C(\nu(R))$ the conductor of $\nu(R)$ and we describe the elements of R in the following way

$$\nu(R) = \{0 = i_0 < i_1 < \ldots < i_m, \rightarrow\},$$

where we are assuming $i_m = c$. For each element i_r of $\nu(R)$ we choose an arbitrary element S_{i_r} of R such that $\text{ord}(S_{i_r}) = i_r$. Notice that every element of R can be written as $\sum_{j=0}^{\infty} a_j S_{i_j}$ with $a_j \in \mathbb{K}$. Therefore it is not difficult to prove that

$$R = \mathbb{K} + \mathbb{K}S_{i_1} + \mathbb{K}S_{i_2} + \cdots + \mathbb{K}[[t]]S_{i_m}.$$

For each $n \in \mathbb{N}$ we consider the following ideal of R

$$I_n = \{r \in R : \text{ord}(r) \geq n\}.$$

For each $n \in \nu(R)$ we consider the set

$$I_n/S_n = \left\{ \frac{r}{S_n} : r \in I_n \right\}.$$

Notice that, in general, I_n/S_n is not a ring, so we can denote by $[I_n/S_n]$ the smallest subring of R inside $\mathbb{K}[[t]]$ that contains I_n/S_n. We have the following theorem proved in [1, Theorem 3, p.259].

Theorem 1.3.1. *The ring $[I_n/S_n]$ does not depend on the choice of the elements $S_n \in R$.*

From the previous theorem, it follows that, in the following, we can simply denote $[I_n/S_n]$ by $[I_n]$. The following example, due to Arf (cf. [1, p.260]), shows that, in general, we have $\nu([I_n]) \neq \langle \nu(I_n/S_n) \rangle$.

Example 1.3.2. Notice that the containment $\nu(I_n/S_n) \subseteq \nu([I_n])$ is trivial, thus we also have $\langle \nu(I_n/S_n) \rangle \subseteq \nu([I_n])$. Let us consider the algebroid branch $R = \mathbb{K}[[t^4, t^{10} + t^{15}]]$. We have that

$$\nu(R) = \{0, 4, 8, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26, 28, 29, 30, 32, 33, \rightarrow\}.$$

Suppose $n = 4$ and consider $S_4 = t^4 \in R$. We have

$$\nu(I_4/t^4) = \{0, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22, 24, 25, 26, 28, 29, \rightarrow\},$$

then

$$\langle \nu(I_4/t^4) \rangle = \{0, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, \rightarrow\}.$$

But in $[I_4]$ we can find the element

$$h(t) = \left(\frac{t^{10} + t^{15}}{t^4} \right)^2 - (t^4)^3,$$

such that $\text{ord}(h(t)) = 17 \notin \langle \nu(I_4/t^4) \rangle$.

We are ready to define the main objects of this section.

Definition 1.3.3. An algebroid branch R is called an *Arf ring* if $[I_n] = I_n/S_n$ for each $n \in \nu(R)$.

Example 1.3.4. From the definition, $\mathbb{K}[[t]]$ is an Arf ring.

Notice that, if R is an Arf ring, then for each n, the integers

$$i_n - i_n = 0, i_{n+1} - i_n, i_{n+2} - i_n \dots,$$

must form a numerical semigroup. This leads to the following important definition.

Definition 1.3.5. Let S be a numerical semigroup. If $s \in S$, denote by $S(s) = \{n \in S; n \geq s\}$. Then S is an *Arf numerical semigroup* if

$$S(s) - s = \{n - s | n \in S(s)\}$$

is a numerical semigroup for each $s \in S$.

Example 1.3.6. From the definition, $\mathbb{N}$ is an Arf numerical semigroup.

Given an Arf numerical semigroup $S = \{i_0 = 0 < i_1 < i_2 < \cdots\}$, *the multiplicity sequence of* S is the sequence $m = \{m_j = i_j - i_{j-1} | j \geq 1\}$.

It is evident from the definition that the multiplicity sequence of an Arf numerical semigroup satisfies the following properties.

Proposition 1.3.7. *Let S be an Arf numerical semigroup, and let m be its multiplicity sequence. Then we have:*

- *$m = \{m_i | i \geq 1\}$ is a non-increasing sequence of positive integers;*
- *there exists $k \in \mathbb{N}^*$ such that $m_n = 1$ for all $n \geq k$;*
- *for all $n \in \mathbb{N}^*$ there exists $s(n) \geq n+1$ such that $m_n = \sum_{k=n+1}^{s(n)} m_k$.*

We will call *multiplicity sequence* any sequence that satisfies the conditions of the previous proposition. We fix some notation regarding the representation of a multiplicity sequence m. Since $m = \{m_n : n \geq 1\}$ is a sequence of integers that stabilizes to 1, we can describe it by a finite list

$$m = [m_1, \dots, m_{l(m)}],$$

with the convention that $m_j = 1$ for all $j > l(m)$ and $m_{l(m)} \neq 1$. The integer $l(m)$, that appears in the previous description, is the *length of the multiplicity sequence* m. Notice that the multiplicity sequence $m = \{m_i = 1 : i \in \mathbb{N}\}$, will be represented by the empty list [ ], and we set by definition $l([\,]) = 0$.

In [16, Corollary 39] it is proved the following result.

Proposition 1.3.8. *A non-empty subset of $\mathbb{N}$ is an Arf numerical semigroup if and only if there exists a multiplicity sequence $m = [m_1, \dots, m_{l(m)}]$ such that*
$S = \left\{0, m_1, m_1 + m_2, \dots, m_1 + \cdots + m_{l(m)}, \rightarrow\right\}$.

So the multiplicity sequence of an Arf semigroup characterizes the semigroup completely, and to give an Arf numerical semigroup is equivalent to give its multiplicity sequence. Throughout this thesis, given a multiplicity sequence m, we denote by $\mathrm{AS}(m)$ the Arf numerical semigroup corresponding to m.

Notice that, if R is an Arf ring, then the numerical semigroup $\nu(R)$ is an Arf semigroup, while the converse is not true in general.

We give now some further results on the properties of Arf rings that can be found in [1].

Theorem 1.3.9. *If R is an Arf ring, then $[I_{i_n}]$ is also an Arf ring for all $n \in \mathbb{N}$.*

Theorem 1.3.10.

- *The intersection of a finite number of Arf rings is an Arf ring.*
- *The intersection of a finite number of Arf numerical semigroups is an Arf numerical semigroup.*

Theorem 1.3.10 and Examples 1.3.4 and 1.3.6 ensure that the following definition is not void.

Definition 1.3.11. Let S be a numerical semigroup, we call *Arf closure* of S, and denote it by ${}^*S$, the smallest Arf numerical semigroup containing S.

Similarly, given an algebroid branch R, we call *Arf closure* of R, and denote it by ${}^*R$, the smallest Arf ring containing R.

Now we give an algorithm for computing the Arf closure of a numerical semigroup. It is based on the following procedure, known as the modified Jacobian algorithm of Du Val (cf. [10]).

Definition 1.3.12 (Modified Jacobian algorithm)**.** The input of the algorithm is a finite set of non-negative integers $J_1 = \{j_{1,1} < \ldots < j_{1,n_1}\}$, with $\gcd(j_{1,1}, \ldots, j_{1,n_1}) = d$.

Suppose that $j_{1,2} = q_1 j_{1,1} + r_1$, with $r_1 < j_{1,1}$.

We consider the set

$$J_2 = \{j_{1,1}, j_{1,2} - q_1 j_{1,1}, \ldots, j_{1,n_1} - q_1 j_{1,1}\} \setminus \{0\} = \{j_{2,1} < \ldots < j_{2,n_2}\}.$$

Suppose that $j_{2,2} = q_2 j_{2,1} + r_2$, with $r_2 < j_{2,1}$, and we repeat the construction for J_3, subtracting $q_2 j_{2,1}$.

The algorithm stops when we reach a N such that $d \in J_N$, and this will eventually happen because we started with a set J_1 with $\gcd(J_1) = d$ and we are essentially performing an euclidean algorithm. The output is the sequence

$$\underbrace{j_{1,1}, \ldots, j_{1,1}}_{q_1 \text{ times}}, \underbrace{j_{2,1}, \ldots, j_{2,1}}_{q_2 \text{ times}}, \ldots, \underbrace{j_{N-1,1}, \ldots, j_{N-1,1}}_{q_{N-1} \text{ times}}, d, d, \ldots,$$

that is a multiplicity sequence when $d = 1$.

Example 1.3.13. Let us consider $J_1 = \{8, 12, 18, 23, 25\}$. We have $\gcd(8, 12, 18, 23, 25) = 1$ so we will find a multiplicity sequence. We have:

1. $12 = 1 \cdot 8 + 4$, then $j_{1,1} = 8$ and $q_1 = 1$. Then
$$J_2 = \{8, 12 - 8, 18 - 8, 23 - 8, 25 - 8\} \setminus \{0\} = \{4, 8, 10, 15, 17\}.$$

2. $8 = 2 \cdot 4$, then $j_{2,1} = 4$ and $q_2 = 2$. Then
$$J_3 = \{4, 8 - 8, 10 - 8, 15 - 8, 17 - 8\} \setminus \{0\} = \{2, 4, 7, 9\}.$$

3. $4 = 2 \cdot 2$, then $j_{3,1} = 2$ and $q_3 = 2$. Then
$$J_4 = \{2, 4 - 4, 7 - 4, 9 - 4\} \setminus \{0\} = \{2, 3, 5\}.$$

4. $3 = 1 \cdot 2 + 1$, then $j_{4,1} = 2$ and $q_4 = 1$. Then
$$J_5 = \{2, 3 - 2, 5 - 2\} \setminus \{0\} = \{1, 2, 3\}.$$

   We found $1 \in J_5$, so the procedure stops.

   The output is the multiplicity sequence $m = [8, 4, 4, 2, 2, 2]$.

To compute the Arf closure of a numerical semigroup S, it suffices to apply the modified Jacobian algorithm of Du Val to a minimal system of generators of S.

Example 1.3.14. Suppose that
$$S = \langle 8, 12, 18, 23, 25 \rangle = \{0, 8, 12, 16, 18, 20, 23, 24, 25, 26, 28, 30, \rightarrow\}.$$

If we apply the modified Jacobian algorithm to the minimal system of generators of S, we obtain the multiplicity sequence $m = [8, 4, 4, 2, 2, 2]$, then the Arf closure of S is the Arf semigroup
$$\text{AS}(m) = \{0, 8, 12, 16, 18, 20, 22 \rightarrow\}.$$

Now, we explain how we can construct the Arf closure of an algebroid branch. From the construction, we will deduce the connection between the multiplicity sequence of an algebroid branch and the multiplicity sequence of an Arf semigroup, finding the algebraic answer to the problem of Du Val.

We follow the construction explained in [1, p.267].

Let R be an algebroid branch, such that $\nu(R) = \{0 = i_0 < i_1 < \ldots, < i_m, \rightarrow\}$. We have already noticed that it can be presented as
$$R = \mathbb{K} + \mathbb{K}S_{i_1} + \mathbb{K}S_{i_2} + \cdots + \mathbb{K}[[t]]S_{i_{i_m}}.$$

Let R_1 denote the ring $[I_{i_1}]$, that is the smallest ring containing I_{i_1}/S_{i_1}. Using the previous presentation we can deduce that

$$R_2 = [I_{i_1}] = \sum \mathbb{K} \left(\frac{S_{i_2}}{S_{i_1}}\right)^{\alpha_2} \cdot \left(\frac{S_{i_3}}{S_{i_1}}\right)^{\alpha_3} \dots \left(\frac{S_{i_{m-1}}}{S_{i_1}}\right)^{\alpha_{m-1}} + \mathbb{K}[[t]]\frac{S_{i_m}}{S_{i_1}},$$

where the summation is taken over all the α_i with $i = 2, \dots, m-1$ such that

$$\alpha_2(i_2 - i_1) + \alpha_3(i_3 - i_1) + \ldots + \alpha_{m-1}(i_{m-1} - i_1) < i_m - i_1,$$

in order to prevent redundancy with elements arising by the term $\mathbb{K}[[t]]\frac{S_{i_m}}{S_{i_1}}$.

The Arf closure ${}^*R$ of R clearly contains $\mathbb{K} + R_2 S_{i_1}$. On the other hand, $\mathbb{K} + R_2 S_{i_1}$ contains R, thus, from the definition of Arf closure we can deduce that

$${}^*R = \mathbb{K} + {}^*R_2 S_{i_1}.$$

Now we can repeat the previous procedure, deriving from a ring R_i the ring R_{i+1}. If N is sufficiently large, we will find $R_N = \mathbb{K}[[t]]$. Now, if we denote by T_i an arbitrary element of minimial valuation in R_i, (we can set $T_1 = S_{i_1}$), we can write:

$$\begin{array}{rcl} {}^*R & = & \mathbb{K} + {}^*R_2 T_1 = \\ & = & \mathbb{K} + (\mathbb{K} + {}^*R_3 T_2)T_1 = \mathbb{K} + \mathbb{K}T_1 + {}^*R_3 T_1 T_2 = \\ & & \dots\dots\dots \\ & = & \mathbb{K} + \mathbb{K}T_1 + \mathbb{K}T_1T_2 + \dots + \mathbb{K}T_1T_2\dots T_{N-1} + {}^*R_N T_1 T_2 \dots T_{N-1}T_N = \\ & = & \mathbb{K} + \mathbb{K}T_1 + \mathbb{K}T_1T_2 + \dots + \mathbb{K}T_1T_2\dots T_{N-1} + \mathbb{K}[[t]]T_1T_2\dots T_{N-1}T_N. \end{array}$$

If we denote by $m_i = \text{ord}(T_i)$ we find that

$$\nu({}^*R) = \{0, m_1, m_1 + m_2, \dots, m_1 + \dots + m_N, \rightarrow\},$$

thus $[m_1, m_2, \dots, m_N]$ is the multiplicity sequence of the Arf semigroup $\nu({}^*R)$. Notice that R and ${}^*R$ share the same multiplicity $e(R) = e({}^*R) = \text{ord}(S_{i_1})$. The following lemma lets us to simplify the construction presented above.

Lemma 1.3.15. *Consider the algebroid branch $R = \mathbb{K}[[\phi_1(t), \dots, \phi_r(t)]]$, where, without loss of generality, we can assume that $\phi_1(t)$ has order equal to the multiplicity of $\nu(R)$.*

Then

$$[I_{i_1}] = \left[\left[\phi_1(t), \frac{\phi_2(t)}{\phi_1(t)} \dots, \frac{\phi_r(t)}{\phi_1(t)}\right]\right] = \textbf{\textit{Bl}}(R).$$

From the previous lemma, it follows that, in the construction of the Arf Closure we have $R_i = B_i$ then

$$m_i = \text{ord}(T_i) = \min\{\text{ord}(r) | r \in R_i\} = \min\{\text{ord}(r) | r \in B_i\} = e(B_i).$$

Thus the multiplicity sequence of the algebroid branch R equals the multiplicity sequence of the Arf numerical semigroup $\nu(^*R)$. To compute the multiplicity sequence of an algebroid branch, it suffices to compute its Arf closure and consider the associated Arf numerical semigroup. We can also deduce the following important corollary that gives the answer that we were looking for.

Corollary 1.3.16. *Two algebroid branches R and T are equivalent if and only if $\nu(^*R) = \nu(^*T)$.*

Example 1.3.17. Let $R = \mathbb{K}[[t^4, t^6 + t^9, t^{14}]]$ be an algebroid branch. Let us compute its Arf closure. We need to consider the chain of blowups and their respective elements T_i of minimal value.

1. We choose $T_1 = t^4$ as an element of minimal value in R. Then

$$B_2 = \mathbb{K}\left[\left[t^4, \frac{t^6 + t^9}{t^4}, \frac{t^{14}}{t^4}\right]\right] = \mathbb{K}[[t^2 + t^5, t^4, t^{10}]].$$

2. We choose $T_2 = t^2 + t^5$ as an element of minimal value in B_2. Then

$$B_3 = \mathbb{K}\left[\left[t^2 + t^5, \frac{t^2}{1 + t^3}, \frac{t^8}{1 + t^3}\right]\right].$$

3. We choose $T_3 = \frac{t^2}{1+t^3}$ as an element of minimal value in B_3. Then

$$B_4 = \mathbb{K}\left[\left[\frac{t^2}{1 + t^3}, (1 + t^3)^2, t^6\right]\right] = \mathbb{K}\left[\left[\frac{t^2}{1 + t^3}, 2t^3 + t^6, t^6\right]\right].$$

4. We choose $T_4 = \frac{t^2}{1+t^3}$ as an element of minimal value in B_4. Then

$$B_5 = \mathbb{K}\left[\left[\frac{t^2}{1 + t^3}, (2t + t^4)(1 + t^3), t^4(1 + t^3)\right]\right] = \mathbb{K}\left[[t]\right].$$

Thus the Arf closure $^*R$ can be presented in the following way:

$$^*R = \mathbb{K} + \mathbb{K}t^4 + \mathbb{K}(t^6 + t^9) + \mathbb{K}t^8 + \mathbb{K}[[t]]t^{10},$$

and we have $\nu(^*R) = \{0, 4, 6, 8, 10, \rightarrow\}$, with the multiplicity sequence $[4, 2, 2, 2]$. Notice also that $\nu(R) = \{0, 4, 6, 8, 10, 12, 14, \rightarrow\}$, that is already an Arf numerical semigroup. This shows that, in general can happen $\nu(^*R) \neq {}^*\nu(R)$.

Remark 1. Determining the Arf closure following the procedure explained above can be computationally demanding. In fact, at each step the parametrizations of the blowups can gain higher order terms, arising from the divisions, that can considerably slow down the process. In [2, Theorem 2.4] Arslan and Sahin proved that, if R is an algebroid branch, then all the monomials appearing in the parametrization of R with degree strictly greater than $c^* + 1$, where c^* is the conductor of $\nu(^*R)$, do not actually affect the computation of the Arf Closure of R. They also found a method to estimate an upper bound for c^*, by only looking at the initial parametrization of R. This means that we can find the Arf Closure of R by applying our procedure to a ring with a simpler parametrization, obtained by R deleting all the terms with degree greater than the determined bound. In Chapter 4 we will present a generalization of this result.

In [1, Theorem 1, p.264] Arf proved the following theorem.

Theorem 1.3.18. *Given an Arf numerical semigroup G, the intersection of all the numerical semigroups S, such that $^*S = G$ is a semigroup G_χ, called characteristic sub-semigroup of G, and we have $^*G_\chi = G$.*

Theorem 1.3.18 let us to give the following definition.

Definition 1.3.19. Given an Arf numerical semigroup G, we call *characters* of G a minimal system of generators for the characteristic sub-semigroup G_χ.

Notice that, if we know the characters $\{\chi_1, \dots, \chi_h\}$ of an Arf semigroup G, we can easily compute G finding its multiplicity sequence by applying the modified Jacobian algorithm to the set $\{\chi_1, \dots, \chi_h\}$ (we have $\gcd(\chi_i) = 1$ because the characters are a system of generators of a numerical semigroup). So, through the characters, we have a way to represent an Arf semigroup with less data than through its multiplicity sequence.

We explain, now a way to find the characters (cf. [4, Lemma 3.1]).

Lemma 1.3.20. *Let G be an Arf numerical semigroup, and $m = [m_1, \dots, m_k]$ its multiplicity sequence. We denote by $r(m_j)$, and we call it restriction number of m_j, the number of sums $m_q = \sum_{h=1}^{k} m_{q+h}$ where m_j appears as a summand. Then the characters of G are the integers*

$$\chi_j = m_1 + \dots + m_j,$$

where j is such that $r(m_j) < r(m_{j+1})$ (when it happens we have $r(m_j) = r(m_{j+1}) - 1$).

Example 1.3.21. Let us consider the Arf semigroup

$$G = \{0, 8, 12, 16, 18, 20, 22 \rightarrow\},$$

and compute its characters. We have, that $m = [8, 4, 4, 2, 2, 2]$ is the multiplicity sequence of G. In order to compute the characters, it is useful to consider also the first two one entries in m, writing

$$m = [8, 4, 4, 2, 2, 2, 1, 1],$$

in fact it is easy to realize that we cannot find $r(m_j) < r(m_{j+1})$ if $j \geq l(m) + 2$. We have

1. $m_1 = 8$ is clearly not a summand, so $r(m_1) = 0$;
2. $m_2 = 4$ appears as a summand in the sum $m_1 = m_2 + m_3$, therefore $r(m_2) = 1$;
3. $m_3 = 4$ appears as a summand in the sums $m_1 = m_2 + m_3$, and $m_2 = m_3$, therefore $r(m_3) = 2$;
4. $m_4 = 5$ appears as a summand in the sum $m_3 = m_4 + m_5$, therefore $r(m_4) = 1$;
5. $m_5 = 2$ appears as a summand in the sums $m_3 = m_4 + m_5$, and $m_4 = m_5$, therefore $r(m_5) = 2$;
6. $m_6 = 2$ appears as a summand in the sum $m_5 = m_6$, therefore $r(m_6) = 1$;
7. $m_7 = 1$ appears as a summand in the sum $m_6 = m_7 + m_8$, therefore $r(m_7) = 1$;
8. $m_8 = 1$ appears as a summand in the sums $m_6 = m_7 + m_8$, and $m_7 = m_8$, therefore $r(m_8) = 2$.

The indices where we get an increase in the restriction numbers are $J = \{1, 2, 4, 7\}$ so we get the characters:

1. $\chi_1 = m_1 = 8$;
2. $\chi_2 = m_1 + m_2 = 12$;
3. $\chi_3 = m_1 + m_2 + m_3 + m_4 = 18$;
4. $\chi_4 = m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 = 23$.

If we apply the modified Jacobian algorithm to these characters we will obtain the multiplicity sequence m. Notice that we did this on Example 1.3.13, where the element $25 \in J_1$ had no impact on the procedure, because it is not a character for the Arf closure of the numerical semigroup generated by J_1.

Definition 1.3.22. Given an algebroid branch R, *the Arf characters* of R are the characters of the Arf numerical semigroup $\nu(^*R)$.

The following immediate corollary stresses the importance of knowing the Arf characters of an algebroid branch.

Corollary 1.3.23. *Two algebroid branches are equivalent if and only if they have the same set of Arf characters.*

1.4 Good semigroups

In this section we define the good semigroups of $\mathbb{N}^r$ and we show why they can be regarded as a natural extension of the numerical semigroups.

The concept of good semigroup firstly arises in [3] where the authors studied the properties of the value semigroup of a one-dimensional analytically unramified local ring.

Definition 1.4.1. A one-dimensional reduced Noetherian local ring $(R, \mathfrak{M})$ is called *analytically unramified* if it satisfies any of the following equivalent conditions (cf. [14, Chapter 10])

- the integral closure $\overline{R}$ is finite over R;
- the completion $\hat{R}$ is reduced.

Let R be a ring satisfying Definition 1.4.1. Denote by $Q(R)$ its total ring of fractions, and by $P_1, \ldots, P_n \in \mathrm{Ass}(R)$ its minimal primes. We have that

$$\begin{array}{rcl} R & \subset & R/P_1 \times \ldots \times R/P_n \\ \overline{R} & \cong & \overline{R/P_1} \times \ldots \overline{R/P_n} \\ Q(R) & \cong & Q(R/P_1) \times \ldots Q(R/P_n). \end{array}$$

We can associate to R a subsemigroup of $\mathbb{N}^r$ (where r is the number of maximal ideal of $\overline{R}$), in the following way. If $q \in Q(R)$ we can see it as an element of $Q(R/P_1) \times \ldots Q(R/P_n)$, therefore we can consider

$$\nu(q) = (\nu_{1,1}(q_1), \ldots, \nu_{1,h_1}(q_1), \nu_{2,1}(q_2), \ldots, \nu_{n,h_n}(q_n)),$$

where $\nu_{i,j}$ is the valuation associated to the DVR $V_{i,j} = \left(\overline{R/P_i}\right)_{\mathfrak{M}_{i,j}}$, obtained localizing at the maximal ideal $\mathfrak{M}_{i,j}$. Notice that $\sum_{i=1}^{n} h_i = r$. Then

$$S = \nu(R) = \{\nu(q) : q \in R \setminus Z(R)\},$$

where $Z(R)$ is the set of zero divisors of R, is the required subsemigroup of $\mathbb{N}^r$.

Example 1.4.2. An important example of rings satisfying the previous conditions is the class of the local rings of an algebraic curve. We will mainly focus on the previously defined algebroid curves that can be obtained as the completion of local rings of algebraic curves at a singular point. Given an algebroid curve, $R = \mathbb{K}[[x_1, x_2, \ldots, x_k]]/(P_1 \cap \ldots \cap P_r)$, where the P_i's are prime ideals, we have the inclusion

$$\tau : R \hookrightarrow \mathbb{K}[[x_1, \ldots, x_k]]/P_1 \times \ldots \times \mathbb{K}[[x_1, \ldots, x_k]]/P_r.$$

The $R^i = \mathbb{K}[[x_1, \ldots, x_k]]/P_i$ for $i = 1, \ldots r$ are the algebroid branches of the algebroid curve.

Considering the integral closures in $Q(R)$, we have $\overline{R} \cong \overline{R^1} \times \ldots \times \overline{R^r}$ where each $\overline{R^i}$ is isomorphic to the ring $\mathbb{K}[[t_i]]$, thus we can consider the valuation $\nu_i : \mathbb{K}[[t_i]] \to \mathbb{N} \cup \{\infty\}$, such that $\nu_i(0) = \infty$ and $\nu_i(\phi(t_i)) = \mathrm{ord}(\phi(t_i))$. Finally, because $R \subseteq \mathbb{K}[[t_1]] \times \mathbb{K}[[t_2]] \times \ldots \times \mathbb{K}[[t_r]]$, we can define the valuation of an non-zero divisor $y \in R$, such that $\tau(y) = (\phi_1(t_1), \ldots, \phi_r(t_r))$, as the vector $\nu(y) = (\nu_1(\phi_1(t_1)), \ldots, \nu_r(\phi_r(t_r)))$.

Therefore the set of values of non-zero divisors in R constitutes a subsemigroup of $\mathbb{N}^r$.

The subsemigroups $\nu(R) = S$, arising from the construction explained above always satisfy the following conditions, cf. [3]:

Proposition 1.4.3. *Let $S = \nu(R) \subseteq \mathbb{N}^r$. Then*

1. *for all $a, b \in S$, $\min(a, b) = (\min(a[1], b[1]), \ldots, \min(a[r], b[r])) \in S$;*
2. *if $a, b \in S$ and $a[i] = b[i]$ for some $i \in \{1, \ldots, r\}$, then there exists $c \in S$ such that $c[i] > a[i] = b[i]$, $c[j] \geq \min(a[j], b[j])$ for $j \in \{1, \ldots, r\} \setminus \{i\}$ and $c[j] = \min(a[j], b[j])$ if $a[j] \neq b[j]$;*
3. *there exists $\delta \in S$ such that $\delta + \mathbb{N}^r \subseteq S$.*

In the following we will consider the usual partial ordering in $\mathbb{N}^r$: $a \leq b$ if $a[i] \leq b[i]$ for each $i = 1, \ldots, r$.

Example 1.4.4. Let R be the algebroid curve

$$R = \mathbb{K}[[x, y, z]]/(x^3 - z^2, y) \cap (x^3 - y^2, z).$$

There is an isomorphism between R and the subring $k[[(t^2, u^2), (0, u^3), (t^3, 0)]]$ of $k[[t]] \times k[[u]]$. We have that

$$\nu(R) = \{(0,0), (2,2), (3,3), (4,4)\} \cup \{(3, n), (n, 3) : n \geq 4\} \cup \left\{(5 + m, 5 + n) : (m, n) \in \mathbb{N}^2\right\},$$

and it is easy to check that all the conditions of Proposition 1.4.3 are satisfied.

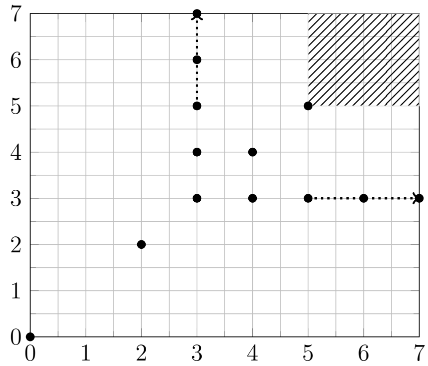


Figure 1.1: $\nu(R)$

Definition 1.4.5. Any subsemigroup of $\mathbb{N}^r$ that satisfies the conditions of Proposition 1.4.3 is called *good semigroup*.

A good semigroup is *local* if $\mathbf{0}$ is the only element of the semigroup which has some coordinate equal to 0. In fact, it is easy to notice that the ring R is local if and only if the semigroup $v(R)$ is local. However, it can be shown that every good semigroup is the direct product of local semigroups (cf. [3, Theorem 2.5]).

Although, as we have just seen, the definition of good semigroup naturally arises from the ring theory context, it was proved in [3, Example 2.16] that not all good semigroups are value semigroups of rings. Hence, these objects represent a natural generalization of the numerical semigroups and it makes sense to study them without taking in account the ring theory context, focusing only on their combinatorial properties.

Definition 1.4.6. Let S be a good subsemigroup of $\mathbb{N}^r$. The *conductor* of S is the least vector $C(S) \in S$, according to the component-wise partial ordering of $\mathbb{N}^r$, such that $C(S) + \mathbb{N}^r \subseteq S$.

The existence of such a vector is guaranteed by the properties 1. and 3. in Proposition 1.4.3.

Given a good semigroup S we can consider the following set

$$\text{Small}(S) = \{s \in S : s \leq C(S)\},$$

that is known as the set of *small elements* of S. Using the small elements we can represent a good semigroup by a finite number of information, as it is shown in the following proposition proved in [8, Proposition 2].

Proposition 1.4.7. *Let $a \in \mathbb{N}^r$. Then $a \in S$ if and only if $\min(a, C(S)) \in \text{Small}(S)$. Notice that if we know $\text{Small}(S)$, we also know $C(S)$ that is the maximum of $\text{Small}(S)$.*

Definition 1.4.8. Let S be a good subsemigroup of $\mathbb{N}^r$. A set $\emptyset \neq E \subseteq \mathbb{Z}^r$ is called a *relative ideal* of S if

1. $E + S \subseteq E$;

2. there exists $\alpha \in S$ such that $\alpha + E \subseteq S$.

A relative ideal E of S does not need to satisfy the conditions 1. and 2. of Proposition 1.4.3. However it always satisfy the third condition. If a relative ideal also satisfies the first two conditions, it will be called *good relative ideal*. We have the following proposition proved in [7, Proposition 2.3].

Proposition 1.4.9. *Let E be a good relative ideal of S. Consider $\alpha, \beta \in E$ with $\alpha < \beta$. A chain*

$$\alpha = \alpha^{(0)} < \alpha^{(1)} < \ldots < \alpha^{(n)} = \beta,$$

with $\alpha^{(i)} \in E$ for all i, is said to be saturated if it cannot be extended to a longer one between α and β in E.

Then all the saturated chains between α and β in E have the same length.

Notice that the length is computed considering the "edges" in the chain; for instance the chain

$$\alpha = \alpha^{(0)} < \alpha^{(1)} < \ldots < \alpha^{(n)} = \beta,$$

has length n.

Let E be a good relative ideal of S and suppose that $\alpha, \beta \in E$, with $\alpha < \beta$. We denote by $d_E(\alpha, \beta)$ the common length of a saturated chain in E from α to β. If $\alpha = \beta$ we set $d_E(\alpha, \beta) = 0$. The definition is well defined due to the previous proposition.

Definition 1.4.10. Let $F \subseteq E$ be two good relative ideals of S. Consider m_F and m_E the minimal elements in F and E respectively. Then for any sufficiently large $\alpha \in F$ we set $d(E \setminus F) = d_E(m_E, \alpha) - d_F(m_F, \alpha)$. In [7] it is shown that this definition does not depend on the choice of α.

The function $d(\_ \setminus \_)$ has some good properties as it was proved in [7, Proposition 2.7 and Corollary 2.5].

Proposition 1.4.11.

1. *If $G \subseteq F \subseteq E$ are good relative ideals of S, then $d(E \setminus G) = d(E \setminus F) + d(F \setminus G)$.*
2. *If $F \subseteq E$ are good relative ideals of S, then $d(E \setminus F) = 0$ if and only if $E = F$.*
3. *If R is a ring, $J \subseteq I$ fractional ideals of R, then $l_R(I \setminus J) = d(\nu(I) \setminus \nu(J))$, where l_R is the length function of R-modules.*

The function $d(\_ \setminus \_)$ lets us also to extend the concept of genus to the good semigroups of $\mathbb{N}^r$. If S is a numerical semigroup with conductor c, then $C = \{c, \rightarrow\}$ is an ideal of S. Thus the genus of S can be also obtained in the following way:

$$g(S) = |\mathbb{N} \setminus C| - |S \setminus C| = c - |S \setminus C|.$$

So we have a natural way to extend this concept to the good semigroups of $\mathbb{N}^r$. If S is a good semigroup of $\mathbb{N}^r$ with conductor $C(S)$, then $C = C(S) + \mathbb{N}^r$ is a good ideal of S and we can define the genus of S as:

$$g(S) = d(\mathbb{N}^r \setminus C) - d(S \setminus C).$$

Since $d(\mathbb{N}^r \setminus C)$ is the length of a saturated chain in $\mathbb{N}^r$ from the vector $\mathbf{0} \in \mathbb{N}^r$ to the conductor $C(S) = (c[1], \ldots, c[r])$, it is easy to show that

$$d(\mathbb{N}^r \setminus C) = \sum_{k=1}^{r} c[k].$$

On the other hand, $d(S \setminus C)$ is the length of a sautared chain in S from $\mathbf{0} \in S$ to $C(S) \in S$. In other words the genus is computed by considering the number of unoccupied places in an arbitrary saturated path linking the zero vector with the conductor, where an unoccupied place denotes any lattice point belonging to the complement of S in $\mathbb{N}^r$.

1.4.1 The multiplicity tree of an algebroid curve R

We want to extend the concept of multiplicity sequence given for the algebroid branches to the algebroid curve case. We will associate to a an algebroid curve R with r branches a tree of vectors of $\mathbb{N}^r$ which we will call the *multiplicity tree* of R.

Let R be an algebroid curve. In the following , we will only consider algebroid curves given through their parametrization.
Thus we assume that there exists

$$x_1 = (\phi_{11}(t_1), \ldots, \phi_{1r}(t_r)), \ldots, x_k = (\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))$$

such that

$$R \cong \mathbb{K}[[(\phi_{11}(t_1), \ldots, \phi_{1r}(t_r)), \ldots, (\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))]].$$

Since R is a local ring we can define its blow-up as $Bl(R) = \cup_{i=0}^{\infty}(\mathfrak{m}^n : \mathfrak{m}^n)$, where $\mathfrak{m}$ is its maximal ideal (the chain of ideals $\mathfrak{m}^n : \mathfrak{m}^n$ has to stabilize because we are in the Noetherian case).
If R is an algebroid curve with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_k)$, then $Bl(R) = R[x, \frac{x_1}{x}, \ldots, \frac{x_k}{x}]$ (see [13, Prop 1.1]), where x is an element of R with minimal valuation (this follows from the fact that $R \subseteq \bar{R}$ is a finite integral extension on R).

Thus, it is easy to see that

$$Bl(R) = \mathbb{K}\left[\left[x, \frac{(\phi_{11}(t_1), \ldots, \phi_{1r}(t_r))}{x}, \ldots, \frac{(\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))}{x}\right]\right].$$

In the following we denote this ring with the symbol $[x^{-1}R]$ (it is the smallest ring containing $x^{-1}R$). In general the blowup of a local ring is a semilocal ring, that is a ring with a finite number of maximal ideals. If S is a semilocal ring then $\mathrm{Bl}(S) = \cup_{n=0}^{\infty}(\mathrm{rad}(S)^n : \mathrm{rad}(S)^n)$, where $\mathrm{rad}(S)$ is the Jacobson ideal of S. Thus to R we can associate (cf. [13]) the following sequence

$$R = R_1 \subseteq R_2 \subseteq R_3 \subseteq \dots,$$

called the Lipman sequence, where $R_i = Bl(R_{i-1})$. Since $\bar{R}$ is a finite R-module, there exists an integer $N \in \mathbb{N}$ such that $R_N = \overline{R} = \mathbb{K}[[t_1]] \times \ldots \times \mathbb{K}[[t_r]]$.
The rings R_i are semilocal rings. We can always see a semilocal ring $S \subseteq \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$, parametrized by

$$S = \mathbb{K}[[(\phi_{11}(t_1), \dots, \phi_{1r}(t_r)), \dots, (\phi_{k1}(t_1), \dots, \phi_{kr}(t_r))]],$$

as a product of local rings (that are the localizations at its maximal ideals). In other words there exists a partition $\mathfrak{P}(S) = \{P_1, \dots, P_t\}$ of $\{1, \dots, r\}$, with

$$P_i = \left\{q_{i,1}, \dots, q_{i,k(i)}\right\},$$

such that

$$\mathfrak{M}_j = \prod_{i=1}^{r} t_i^{\delta_{i,j}} \mathbb{K}[[t_i]] \text{ for } j = 1, \dots, t \text{ and } \delta_{i,j} = \begin{cases} 1 \text{ if } i \in P_j \\ 0 \text{ if } i \notin P_j \end{cases},$$

and the $\mathfrak{M}_j$'s are the maximal ideals of S.

Then

$$S = \prod_{i=1}^{t} S_{\mathfrak{M}_i} \cong \prod_{i=1}^{t} \pi_{P_i}(S),$$

where

$$\pi_{P_i} : S \to \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]],$$

with

$$\pi_{P_i}(s)[j] = \begin{cases} s[j] \text{ if } j \in P_i \\ 0 \text{ if } j \notin P_i \end{cases},$$

and such that the rings $\pi_{P_i}(S)$ are isomorphic to the local subrings of $\mathbb{K}[[t_{q_{i,1}}]] \times \cdots \times \mathbb{K}[[t_{q_{i,k(i)}}]]$. given by the parametrization

$$\mathbb{K}\left[\left[\left(\phi_{1q_{i,1}}(t_{q_{i,1}}), \dots, \phi_{1q_{i,k(i)}}(t_{q_{i,k(i)}})\right), \dots, \left(\phi_{kq_{i,1}}(t_{q_{i,1}}), \dots, \phi_{kq_{i,k(i)}}(t_{q_{i,k(i)}})\right)\right]\right].$$

In (cf. [3]) the authors associated to R a *blowing up tree* in the following way.

The nodes of the tree are all the local rings appearing in the Lipman sequence of R. We say that a node is at the level j of the tree if it is one of the local rings that appears as a factor in the expression of R_j.

Furthermore, considered the maximal ideal $\mathfrak{N}_i = \mathbb{K}[[t_1]] \times \cdots \times t_i\mathbb{K}[[t_i]] \times \cdots \times \mathbb{K}[[t_r]]$ of $\overline{R}$, the branch sequence of R along $\mathfrak{N}_i$ is the sequence of the rings $(R_t)_{\mathfrak{N}_i \cap R_t}$. Following our notation, the node at the j-th level and on the i-th branch is the local ring $\pi_{P_{j,k}}(R_j)$, such that $i \in P_{j,k}$.

A node at level j is linked to a node at level $j+1$ if and only if the corresponding local rings are in the same branch sequence. The multiplicity sequence of R along $\mathfrak{N}_i$ is given by the multiplicities of the rings appearing on the i-th branch.

Now we give a way to associate a multiplicity vector to each ring on the blowing up tree of R.

If $P = \{q_1, \dots, q_k\} \subseteq \{1, \dots, r\}$ and S is a local ring in $\mathbb{K}[[t_{q_1}]] \times \cdots \times \mathbb{K}[[t_{q_k}]]$, we define $\text{mult}(S) = \min\{\nu(s) : s \in S\}$, where u is the valuation defined in $\mathbb{K}[[t_{q_1}]] \times \cdots \times \mathbb{K}[[t_{q_k}]]$. It is easy to see that if

$$S = \mathbb{K}[[(\phi_{11}(t_{q_1}), \dots, \phi_{1k}(t_{q_k})), \dots, (\phi_{n1}(t_{q_1}), \dots, \phi_{nr}(t_{q_1}))]],$$

then

$$\text{mult}(S)[i] = \min\{\text{ord}(\phi_{1i}(t_{q_i}) - \phi_{1i}(0)), \dots, \text{ord}(\phi_{ki}(t_{q_i}) - \phi_{ki}(0))\} \text{ for all } i = 1, \dots, k.$$

Because the field $\mathbb{K}$ is infinite we can always find a linear combination x_S of the generators of S, such that $\nu(x_S) = \text{mult}(S)$. Note that the multiplicity of S as a local ring is given by the sum of components of $\text{mult}(S)$.

Suppose now that S is a ring on the blowing up tree. Then there exists an integer j and a partition $\mathfrak{P}(R_j) = \{P_1, \dots, P_t\}$ of $\{1, \dots, r\}$ such that $S = \pi_{P_i}(R_j)$, with $P_i = \{q_{i,1}, \dots, q_{i,k(i)}\}$. We denote by R_j^i the local subring of $\mathbb{K}[[t_{q_{i,1}}]] \times \cdots \times \mathbb{K}[[t_{q_{i,k(i)}}]]$ isomorphic to S.

Then we define the *fine multiplicity* of S as the r-vector $\text{mult}(S)$ such that

- $\text{mult}(\pi_{P_i}(S))[j] = 0$ if $j \notin P_i$;
- $\text{mult}(\pi_{P_i}(S))[q_{i,j}] = \text{mult}(R_j^i)[j]$, for $j = 1, \dots, k(i)$.

Furthermore we define:

$$\text{multset}(R_j) = \{\text{mult}(\pi_{P_i}(R_j)) : P_i \in \mathfrak{P}(R_j)\}.$$

To each element of $\text{multset}(R_j)$ we can associate an element $x_{\pi_{P_i}(R_j)}$ of R_j with $\nu(x_{\pi_{P_i}(R_j)}) = \text{mult}(\pi_{P_i}(R_j))$. Notice that we can always choose $x_{\pi_{P_i}(R_j)}$ such that $x_{\pi_{P_i}(R_j)}[j] = 1$ if $j \notin P_i$.

If we replace the local rings in the tree with their fine multiplicities, we get the *multiplicity tree* of R. In the following we will denote by $T(R)$ the multiplicity tree of the algebroid curve R. We can also define the *minimal tree* by assigning to each node of the multiplicity tree the corresponding element $x_{\pi_{P_i}(R_j)}$ of minimal value.

Example 1.4.12. Consider the algebroid curve $R = \mathbb{K}[[(t^2, u^2), (0, u^3), (t^3, 0)]] \subset \mathbb{K}[[t]] \times \mathbb{K}[[u]]$ of Example 1.4.4. Let us compute the multiplicity tree of R.

1. We can choose $x_{R_1} = (t^2, u^2)$ as an element of minimal value in $R = R_1$. Thus multset$(R_1) = \{(2,2)\}$. We have

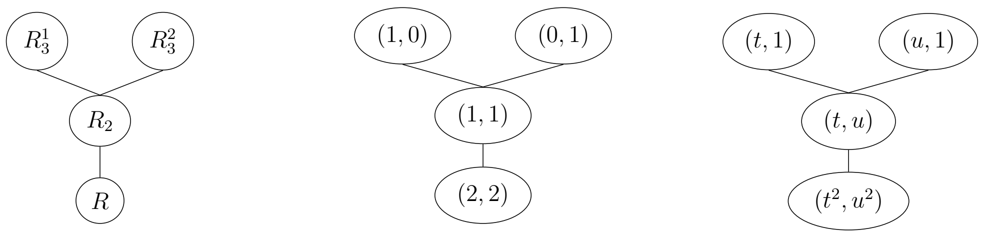
$$R_2 = \mathbb{K}\left[\left[x_{R_1} = (t^2, u^2), \frac{(0,u^3)}{x_{R_1}}, \frac{(t^3,0)}{x_{R_1}}\right]\right] = \mathbb{K}[[(t^2,u^2),(0,u),(t,0)]],$$

   that is still local.

2. We can choose $x_{R_2} = (t,u)$ as an element of minimal value in R_2. Thus multset$(R_2) = \{(1,1)\}$. We have

$$R_3 = \mathbb{K}\left[\left[(t,u), \frac{(0,u)}{x_{R_2}}, \frac{(t,0)}{x_{R_2}}\right]\right] = \mathbb{K}[[(t,u),(0,1),(1,0)]] = k[[t]] \times k[[u]].$$

   Thus multset$(R_3) = \{(1,0),(0,1)\}$ and the multiplicity tree and the minimal tree have the following form:



blowing up tree of R Multiplicity tree of R Minimal tree of R

In [4] the authors gave the following definition of equivalence between algebroid curves that naturally extends the definition given for the algebroid branches.

Definition 1.4.13. Let R_1 and R_2 be two algebroid curve. Then R_1 and R_2 are said to be *equivalent* if they have the same number of branches and the branches can be reordered in a way such that the multiplicity trees $T(R_1)$ and $T(R_2)$ coincide.

1.4.2 Arf's theory for the algebroid curves

We saw in the previous section how the concept of Arf ring was introduced in the special context of algebroid branches in order to find a suitable criterion to establish equivalence. In [13] Lipman gave a general definition of Arf ring.

Definition 1.4.14. Let R be a one-dimensional semilocal Noetherian ring, such that the Jacobson ideal contains a regular element.

Then R is an *Arf ring* if and only if every regular integrally closed ideal of R is stable, where an ideal I of R is said to be *stable* if $z(I : I) = I$ for some $z \in I$.

In our context, it is possible to prove (cf. [3, Lemmas 3.18 and 3.22]) that an algebroid curve R is Arf if, for every $\alpha \in \nu(R)$ and $x \in R$ with $\nu(x) = \alpha$, we have that

$$x^{-1}R(\alpha) \text{ is a ring,}$$

where $R(\alpha) = \{r \in R : \nu(r) \geq \alpha\}$. Also in this more general case it is possible to define the *Arf closure* of a ring R as the smallest Arf overring $\overline{R} \supseteq {}^*R$ of R (cf. [13, Proposition-Definition 3.1]). In Chapter 4 we will present a procedure for computing the Arf closure of an algebroid curve.

In [3, Proposition 5.3] it is proved the following important result that lets us to generalize Corollary 1.3.16.

Proposition 1.4.15. *Let R be an algebroid curve. Then the multiplicity trees of R and of its Arf closure ${}^*R$ are the same.*

We can also give the following definition for an Arf good semigroup.

Definition 1.4.16. A good semigroup S of $\mathbb{N}^r$ is an *Arf semigroup* if $S(\alpha) - \alpha$ is a semigroup for each $\alpha \in S$ where $S(\alpha) = \{\beta \in S; \beta \geq \alpha\}$.

If R is an Arf ring then $\nu(R)$ is an Arf good semigroup, while the converse is not true in general. Furthermore any local Arf good semigroup is the semigroup value of local ring (cf. [3, Proposition 3.19 and Corollary 5.8]).

This means that to each local Arf good semigroup $S \subseteq \mathbb{N}^r$ it is possible to associate a multiplicity tree and, if $S = v(R')$, $T(R) = T(S)$ (we denote with $T(S)$ the multiplicity tree of the semigroup S). Thus the multiplicity tree of a local Arf semigroup characterizes the semigroup completely and we have the following corollary.

Corollary 1.4.17. *Two algebroid curves R and U are equivalent if and only if it is possible to reorder the branches in such a way that $\nu({}^*R) = \nu({}^*U)$.*

The following proposition, cf. [3, Theorem 5.11], gives us the properties that a tree of vectors of $\mathbb{N}^r$ has to satisfy in order to be a multiplicity tree of an Arf subsemigroup of $\mathbb{N}^r$.

Proposition 1.4.18. *A tree $T = \{\mathbf{n}_i^j\}$ of vectors of $\mathbb{N}^r$, where $\mathbf{n}_i^j$ is a node at the j-th level and on the i-th branch, is the multiplicity tree of a local Arf semigroup s if and only if it satisfies the following conditions:*

a) Two nodes $\mathbf{n}_{i_1}^{j_1}$ and $\mathbf{n}_{i_2}^{j_2}$ are linked if and only if $i_1 = i_2$ and $|j_2 - j_1| = 1$.

b) There exists $n \in \mathbb{N}$ such that, for $m \geq n$, $\mathbf{n}_j^m = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the non-zero coordinate in the j-th position) for any $j = 1, \ldots, r$.

c) The h-th coordinate of $\mathbf{n}_j^i$ is 0 if and only if $\mathbf{n}_j^i$ is not in the h-th branch of the tree (the h-th branch of the tree is the unique maximal path containing the h-th unit vector) and $\mathbf{n}_{j_1}^i \equiv \mathbf{n}_{j_2}^i$ (i.e. the two vectors give the same node in the tree) if and only if the j_1-th and j_2-th branches are glued in a node at level i.

d) $\mathbf{n}_j^i = \sum_{\mathbf{n} \in T' \setminus \mathbf{n}_j^i} \mathbf{n}$, for some finite subtree T' of T, rooted in $\mathbf{n}_j^i$.

Notice that we must have multiplicity sequences along each branch. If $T = \{\mathbf{n}_i^j\}$ is the multiplicity tree T of a local Arf semigroup S then the root of the tree is $\mathbf{n}_1^1 = \mathbf{n}_i^1$ for all i (at level one all the branches must be glued in order to have a local semigroup). Furthermore we have

$$S = \{\mathbf{0}\} \bigcup_{T'} \left\{ \sum_{\mathbf{n}_i^j \in T'} \mathbf{n}_i^j \right\},$$

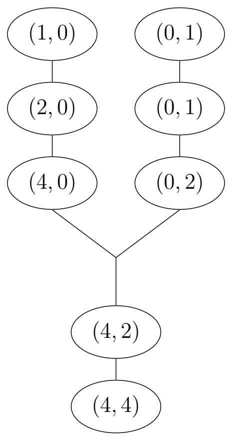
where T' ranges over all finite subtrees of T rooted in $\mathbf{n}_1^1$.

Example 1.4.19. Let us consider the following subset of $\mathbb{N}^2$,

$$S = \{(0,0),(4,4),(8,6),(12,6)\} \cup \{(8,8+n),(12,8+n),(14+n,6); n \in \mathbb{N}\} \cup$$

$$\cup \{(14+m,8+n); m,n \in \mathbb{N}\}.$$

It is possible to verify that S ia an Arf good semigroup with the following multiplicity tree:



Chapter 2

Arf good semigroups

In this chapter we focus on the class of local Arf good subsemigroups of $\mathbb{N}^r$. In the following we will always assume without mentioning that a semigroup is local. We want to use the 1-1 correspondence between Arf good semigroups and multiplicity trees in order to find an unambiguous way to describe them. As we will see in the following sections, finding a good way to represent an Arf semigroup will help us to answer some interesting questions about the determination of the Arf closure of a good semigroup and regarding a possible extension of the concept of Arf characters to these more general objects.

2.1 Arf semigroups with a given collection of multiplicity branches

In this section we determine a way to find all the local Arf good subsemigroups of $\mathbb{N}^r$ having the same collection of multiplicity branches.

Suppose that $E = \{m_i : i = 1, \dots, r\}$ is an ordered collection of r multiplicity sequences. Denote by $\tau(E)$ the set of all multiplicity trees having the r branches in E (specifically, we want that the multiplicity sequence along the i-th branch of the tree corresponds to the multiplicity sequence m_i of E) and by $\sigma(E)$ the set of the corresponding Arf semigroups. Our aim is to find an unambiguous way to describe distinct trees of $\tau(E)$.

If $N = \max\{l(m_i) + 2 : i = 1, \dots, r\}$ we can also write for all $i = 1, \dots, r$

$$m_i = [m_{i,1}, \dots, m_{i,N}].$$

In this way, we manage to describe the multiplicity sequences of E by using finite vectors of the same length. It will be clear later why it is useful to consider also the first two integers j such that $m_{i,j} = 1$. Since m_i represents a multiplicity sequence of an Arf numerical semigroup, it must satisfy the following property:

$$\forall j \geq 1 \text{ there exists } s_{i,j} \in \mathbb{N}, \text{ such that } s_{i,j} \geq j+1 \text{ and } m_{i,j} = \sum_{k=j+1}^{s_{i,j}} m_{i,k}.$$

We define, for all $i = 1, \ldots, r$, the following vectors

$$s_i = [s_{i,1}, \ldots s_{i,N}].$$

Notice that, because we have $m_{i,j} = 1$ for all $j \geq N-1$, it follows $s_{i,j} = j+1$ for all $j \geq N-1$.

Example 2.1.1. Let $m_1 = [14, 7, 5]$ be a multiplicity sequence. In this case $N = 5$ so we write:

$$m_1 = [14, 7, 5, 1, 1],$$

thus s_1 is:

$$s_1 = [5, 5, 8, 5, 6].$$

Notice that, with this notation, from the vectors s_i we can easily reconstruct the sequences m_i. It suffices to set $m_{i,N} = 1$ and then to compute the values of $m_{i,j}$ using the information contained in the integers $s_{i,j}$.

In the next proposition, the vectors s_i, arising from the collection E, are used to determine the level, in a tree of $\tau(E)$, where two branches are forced to split up in order to maintain fulfilled the conditions for a well defined multiplicity tree.
For each pair of integers i, j such that $i < j$ and $i, j = 1, \ldots, r$ we consider the set $D(i, j) = \{k : s_{i,k} \neq s_{j,k}\}$. If $D(i, j) \neq \emptyset$, we denote by $k_E(i, j)$ the integer

$$k_E(i, j) = \min \{\min(s_{i,k}, s_{j,k}), k \in D(i, j)\},$$

while if $D(i, j) = \emptyset$, i.e. $m_i = m_j$, we set $k_E(i, j) = +\infty$.

Proposition 2.1.2. *Consider a collection of multiplicity sequences E and let $T \in \tau(E)$. Then $k_E(i, j) + 1$ is the lowest level where the i-th and the j-th branches are prevented from being glued in T (if $k_E(i, j)$ is infinite there are no limitations on the level where the branches have to split up).*

Proof

Suppose $k_E(i, j) \neq +\infty$ and, by contradiction, that the i-th and the j-th branches are glued at level $k_E(i, j)+1$. From the definition of $k_E(i, j)$, there exists $\overline{k} \in D(i, j)$ such that $k_E(i, j) = \min(s_{i,\overline{k}}, s_{j,\overline{k}})$. Without loss of generality suppose that $\min(s_{i,\overline{k}}, s_{j,\overline{k}}) = s_{i,\overline{k}} \neq s_{j,\overline{k}}$ (since $\overline{k} \in D(i, j)$).

So in the tree we have the following nodes,

$$(\ldots, m_{i,\overline{k}}, \ldots, m_{j,\overline{k}}, \ldots), \ldots, (\ldots, m_{i,k_E(i,j)}, \ldots, m_{j,k_E(i,j)}, \ldots),$$

$$, (\ldots, m_{i,k_E(i,j)+1}, \ldots, m_{j,k_E(i,j)+1}, \ldots).$$

We have that $k_E(i, j) = s_{i,\overline{k}}$ so

$$m_{i,\overline{k}} = \sum_{t=\overline{k}+1}^{k_E(i,j)} m_{i,t},$$

while $k_E(i,j)+1 = s_{i,\overline{k}}+1 \leq s_{j,\overline{k}}$ so

$$m_{j,\overline{k}} \geq \sum_{t=\overline{k}+1}^{k_E(i,j)+1} m_{j,t}.$$

These facts easily imply that the first node cannot be expressed as a sum of the nodes of a subtree rooted in it, so we have a contradiction because the property d) of Proposition 1.4.18 is not fulfilled for the node $\mathbf{n}_i^{\overline{k}}$ of T. Two branches are forced to split up only when we have this kind of problem, so the minimality of $k_E(i,j)$ guarantees that they can be glued at level $k_E(i,j)$ (and obviously at lower levels).

The case $k_E(i,j) = +\infty$ is trivial, because we have the same sequence along two distinct branches and therefore no discrepancies that force the two branches to split up at a certain level. $\square$

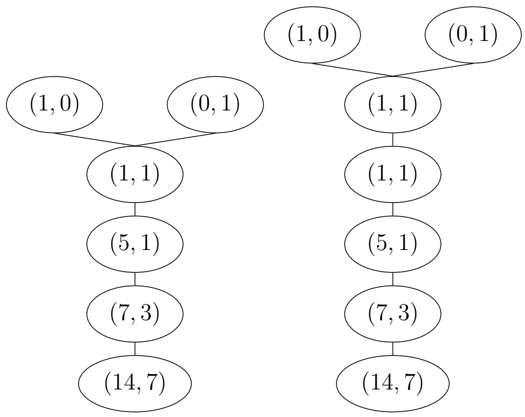
Example 2.1.3. Suppose that we have

$$E = \{m_1 = [14,7,5], m_2 = [7,3]\}.$$

So we can compute the vectors s_1 and s_2:

$$s_1 = [5,5,8,5,6] \text{ and } s_2 = [6,5,4,5,6].$$

We have $D(1,2) = \{1,3\}$, then $k_E(1,2) = \min\{\min(5,6), \min(4,8)\} = \min\{5,4\} = 4$. Then the branches have to be separated at the fifth level.



Notice that the first tree in the previous picture fulfils the properties of the multiplicity trees of an Arf semigroup. The second one cannot be the multiplicity tree of an Arf semigroup because the third node $(5,1)$ cannot be expressed as a sum of nodes in a subtree rooted in it.

Now we prove a general lemma that will be useful in the following.

Lemma 2.1.4. *Consider v_1, v_2 and v_3 in $\mathbb{N}^r$. If $i,j \in \{1,2,3\}$ with $i \neq j$ we define:*

- *MIN*$(v_i, v_j) = +\infty$ *if* $v_i = v_j$*;*
- *MIN*$(v_i, v_j) = \min\{\min(v_i[k], v_j[k]), k \in \{1, \ldots, r\} : v_i[k] \neq v_j[k]\}$.

Then there exists a permutation $\delta \in S^3$ *such that*

$$\mathit{MIN}(v_{\delta(1)}, v_{\delta(2)}) = \mathit{MIN}(v_{\delta(2)}, v_{\delta(3)}) \leq \mathit{MIN}(v_{\delta(1)}, v_{\delta(3)}).$$

Proof Suppose by contradiction that the thesis is not true. Then, renaming the indices if necessary, we have

$$\mathrm{MIN}(v_1, v_2) < \mathrm{MIN}(v_1, v_3) \leq \mathrm{MIN}(v_2, v_3).$$

From the definition of $\mathrm{MIN}(v_1, v_2) = l_{1,2}$ it follows that there exists a $k \in \{1, \ldots, r\}$ such that $v_1[k] \neq v_2[k]$ and $\min(v_1[k], v_2[k]) = l_{1,2}$. We have two cases:

- If $v_1[k] = l_{1,2} \Rightarrow v_2[k] > l_{1,2}$. Then we must have $v_3[k] = l_{1,2}$, in fact otherwise we would have $\mathrm{MIN}(v_1, v_3) \leq l_{1,2} < \mathrm{MIN}(v_1, v_3)$. But then

  $$l_{1,2} < \mathrm{MIN}(v_2, v_3) \leq \min(v_2[k], v_3[k]) = l_{1,2},$$

  and we have a contradiction.

- If $v_2[k] = l_{1,2} \Rightarrow v_1[k] > l_{1,2}$. Then we must have $v_3[k] = l_{1,2}$, in fact otherwise we would have $\mathrm{MIN}(v_2, v_3) \leq l_{1,2} < \mathrm{MIN}(v_2, v_3)$. But then

  $$l_{1,2} < \mathrm{MIN}(v_1, v_3) \leq \min(v_1[k], v_3[k]) = l_{1,2},$$

  and we have a contradiction. □

Remark 2. If we have three multiplicity sequences m_1, m_2 and m_3 then, if $E = \{m_1, m_2, m_3\}$, there exists a permutation $\delta \in S^3$ such that

$$k_E(\delta(1), \delta(2)) = k_E(\delta(2), \delta(3)) \leq k_E(\delta(1), \delta(3)).$$

In fact the integers $k_E(i, j)$ are of the same type of the integers $\mathrm{MIN}(v_i, v_j)$ of the previous lemma with $v_i = s_i$.

Now we give a way to describe a tree of $\tau(E)$. If $T \in \tau(E)$, it can be represented by an upper triangular matrix $r \times r$

$$M(T)_E = \begin{pmatrix} 0 & p_{1,2} & p_{1,3} & \ldots & p_{1,r} \\ 0 & 0 & p_{2,3} & \ldots & p_{2,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & p_{r-1,r} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

where $p_{i,j}$ is the highest level such that the i-th and the j-th branches are glued in T. We will call $M(T)_E$ the *ramification matrix* of T.

Remark 3. If $M(T)_E$ is the matrix of a $T \in \tau(E)$, it is clear that every time we consider three indices $i < j < k$ we must have:

$$p_{i,j} \geq \min(p_{i,k}, p_{j,k}), p_{j,k} \geq \min(p_{i,j}, p_{i,k}) \text{ and } p_{i,k} \geq \min(p_{i,j}, p_{j,k}),$$

when we are using the obvious fact that the relation of being glued has the transitive property. From the previous inequalities it follows that $\{p_{i,j}, p_{j,k}, p_{i,k}\} = \{x, x, y\}$, with $x \leq y$ (independently of the order).

From Proposition 2.1.2 we have that $p_{i,j} \in \{1, \ldots, k_E(i,j)\}$ for all $i, j = 1, \ldots, r$ with $i < j$. In the following, with an abuse of notation, we will identify a tree with its representation.

We call a tree T of $\tau(E)$ *untwisted* if two non-consecutive branches are glued at level l if and only if all the consecutive branches between them are glued at a level greater than or equal to l. We will call *twisted* a tree that it is not untwisted.

From the definition it follows that the matrix of an untwisted tree $T \in \tau(E)$ is such that:

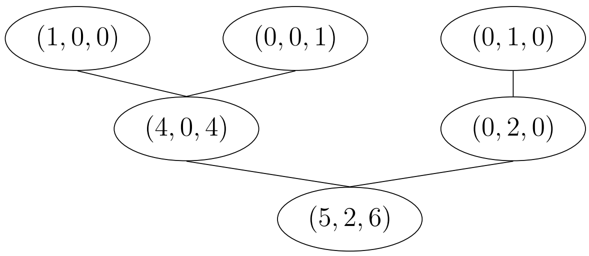
$$p_{i,j} = \min\{p_{i,i+1}, p_{i+1,i+2}, \ldots, p_{j-1,j}\} \text{ for all } i < j.$$

So an untwisted tree can be completely described by the second diagonal of its matrix. Thus, in the following, we will indicate an untwisted tree by a vector $T_E = (d_1, \ldots, d_{r-1})$ where $d_i = p_{i,i+1}$, called the *ramification vector* of T.

Remark 4. It is easy to see that a twisted tree can be converted to an untwisted one by accordingly permuting its branches (the corresponding Arf semigroups are therefore equivalent). So in the following we can focus, when it is possible, only on the properties of the untwisted trees, that are easier to study, obtaining the twisted one by permutation.

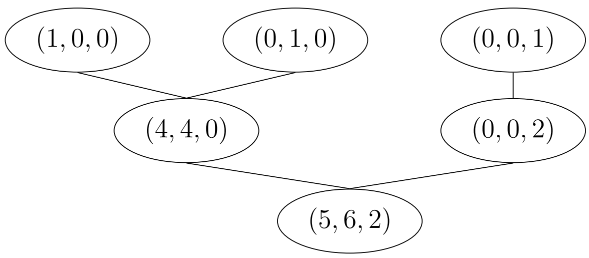
Example 2.1.5. Let us consider the following tree of $\tau(E)$ with

$$E = \{m_1 = [5,4], m_2 = [2,2], m_3 = [6,4]\}$$



This tree is twisted because the first and the third branches are glued at level two, while the first and the second are not.

If we consider the permutation $(2,3)$ on the branches we obtain the tree



that is untwisted, even if it belongs to a different set $\tau(E')$ where

$$E' = \{m_1 = [5,4], m_2 = [6,4], m_3 = [2,2]\},$$

and it can be represented by the ramification vector $T_{E'} = (2,1)$.

Denote by $S(T)$ the semigroup determined by the tree T. In [4, Lemma 5.1] it is shown that if T^1 and T^2 are untwisted trees of $\tau(E)$, then $S(T^1) \subseteq S(T^2)$ if and only if T^2_E is coordinatewise less than or equal to T^1_E. The previous result can be easily extended to the twisted trees. Then, in the general case we have that $S(T^1) \subseteq S(T^2)$, where $S(T^1)$ and $S(T^2)$ belong to $\sigma(E)$, if and only if each entry of $M(T^2)_E$ is less than or equal to the corresponding entry of $M(T^1)_E$. If $k_E(i,j) \neq +\infty$ for all $i < j$, we can consider T^{MIN} such that

$$M(T^{\text{MIN}})_E = \begin{pmatrix} 0 & k_E(1,2) & k_E(1,3) & \dots & k_E(1,r) \\ 0 & 0 & k_E(2,3) & \dots & k_E(2,r) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & k_E(r-1,r) \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

that is well defined for Remark 2. Then $S(T^{\text{MIN}})$ is the smallest Arf semigroup belonging to $\sigma(E)$.

Remark 5. If in the collection E there are two branches with the same multiplicity sequence then $|\sigma(E)| = +\infty$.

Example 2.1.6. We can count the number of untwisted trees of $\tau(E)$ by using their representation. If we call $\tau^*(E)$ the set of all the untwisted trees of $\tau(E)$, these trees are completely determined by the elements in the second diagonal of their matrix, that are bounded by $k_E(j, j+1)$. Hence the number of untwisted trees is:

$$|\tau^*(E)| = \prod_{j=1}^{r-1} k_E(j, j+1).$$

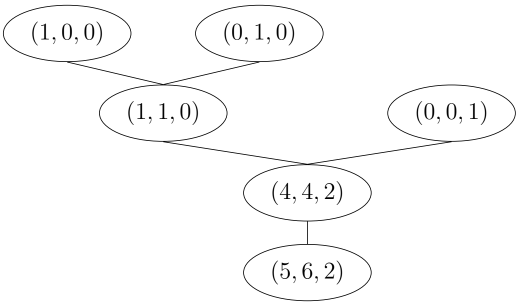
Suppose that $E = \{m_1, m_2, m_3\}$, where

$$m_1 = [5,4],\ m_2 = [6,4],\ m_3 = [2,2].$$

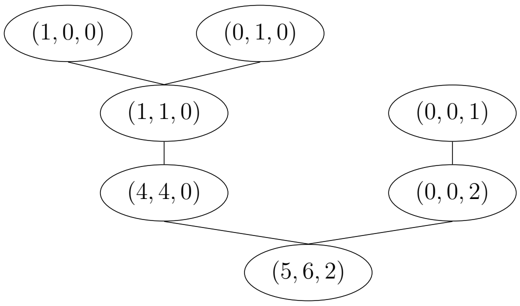
We have:

$$s_1 = [3,6,4,5],\ s_2 = [4,6,4,5],\ s_3 = [2,4,4,5].$$

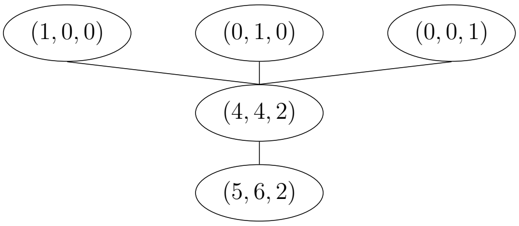
Then $D(1,2) = \{1\}$, $D(2,3) = \{1,2\}$ and $k_E(1,2) = \min(3,4) = 3$ and $k_E(2,3) = \min\{\min(2,4), \min(4,6)\} = 2$. There are $k_E(1,2) \cdot k_E(2,3) = 6$ trees in $\tau^*(E)$. They are:



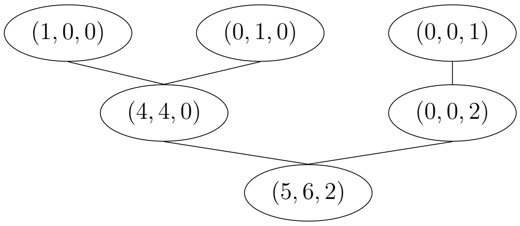
$T_{\text{MIN}} = T_E = (3, 2)$



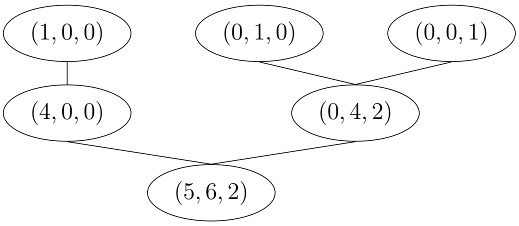
$T_E = (3, 1)$



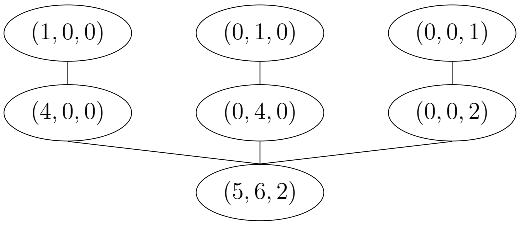
$T_E = (2, 2)$



$T_E = (2, 1)$



$T_E = (1, 2)$



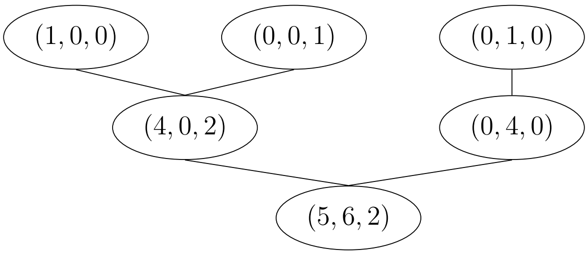
$T_E = (1, 1)$

Remark 6. Because we are able to determine completely $\tau^*(E)$ for each E collection of multiplicity sequences we have a way to determine $\tau(E)$. If $\delta \in S_r$ is a permutation of the symmetric group S_r we can consider $\delta^{-1}(\tau^*(\delta(E))) \subseteq \tau(E)$. It is trivial to see that

$$\bigcup_{\delta \in S_r} \delta^{-1}(\tau^*(\delta(E))) = \tau(E).$$

If we apply this strategy to find $\tau(E)$ with the E of the previous example we find that in $\tau(E)$ there is only one twisted tree T with

$$M(T)_E = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$



2.2 When a set of vectors determines an Arf semigroup

In this section we want to understand when a set $G \subseteq \mathbb{N}^r$ determines uniquely an Arf semigroup of $\mathbb{N}^r$. First of all we need to fix some notations.

Given $G \subseteq \mathbb{N}^r$ we denote by $S(G)$ the following set

$$S(G) = \{S : S \subseteq \mathbb{N}^r \text{ is an Arf semigroup and } G \subseteq S\}.$$

If the set $S(G)$ has a minimum (with the partial order given by the inclusion), we will denote such a minimum by Arf(G). Hence, we have to understand when Arf(G) is well defined and, in this case, how to determine it.

If $i \in \{1, \ldots, r\}$, and $S \in S(G)$ we denote by S_i the projection on the i-th coordinate. We know that S_i is an Arf numerical semigroup and it contains the set $G[i] = \{g[i] : g \in G\}$ where with $g[i]$ we indicate the i-th coordinate of the vector g. We recall also that, if we have a set of integers I such that $\gcd(I) = 1$, we can compute the smallest Arf semigroup containing I, that is the Arf closure of the numerical semigroup generated by the elements of I, by using the modified Jacobian algorithm of Du Val (cf. Section 1.3).

We have the following theorem:

Theorem 2.2.1. *Suppose that we have $G \subseteq \mathbb{N}^r$. Then Arf(G) is well defined if and only if the following conditions hold:*

- $\gcd\{g[i], g \in G\} = 1$ *for* $i = 1, \ldots, r$;
- *For all $i, j \in \{1, \ldots, r\}$ such that $i < j$ there exists $g \in G$ such that $g[i] \neq g[j]$.*

Proof ($\Rightarrow$) Suppose that Arf(G) is well defined and suppose by contradiction that the two conditions of the theorem are not simultaneously fulfilled.

We have two cases.

- **Case 1**: The first condition is not fulfilled.

  Then there exists an i such that $\gcd(G[i]) = d \neq 1$. When we apply the Jacobian algorithm to the elements of $G[i]$ we will produce a sequence of the following type:

$$[m_{i,1}, \ldots, m_{i,k}, \ldots]$$

where there exists a k such that $m_{i,j} = d$ for all $j \geq k$. Denote by $\overline{k}$ the minimum k such that the Arf semigroup associated to the sequence

$$[m_{i,1}, \ldots, m_{i,\overline{k}} = d],$$

contains $G[i]$ (such minimum exists for the properties of the algorithm of Du Val). Then for all $z \geq \overline{k}$ we can consider the multiplicity sequence

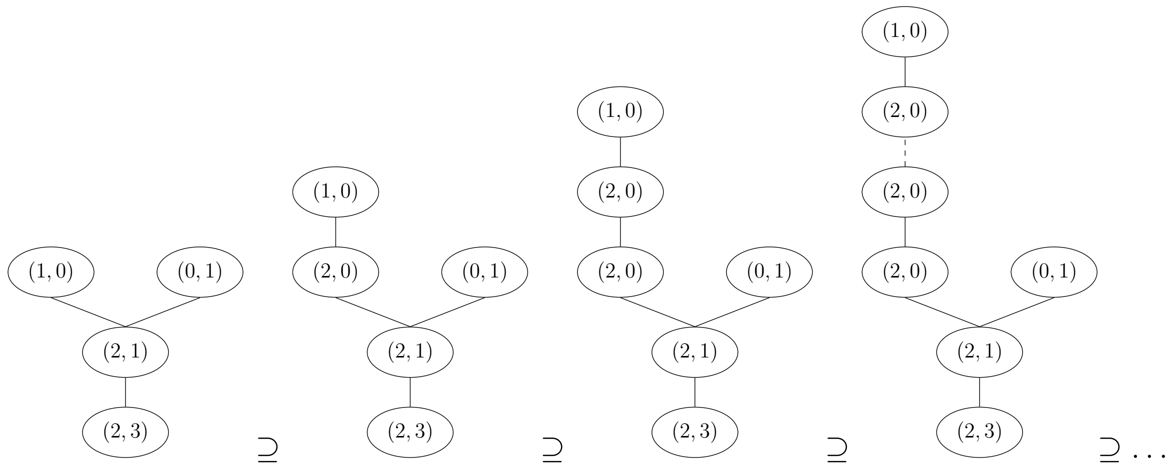
$$m_i(z) = [m_{i,1}, \ldots, m_{i,\overline{k}} = d, \ldots, m_{i,z} = d]$$

and if AS$(m_i(z))$ is the Arf numerical semigroup associated to $m_i(z)$ then $G[i] \subseteq$ AS$(m_i(z))$. Now it is trivial to show that AS$(m_i(z_1)) \subseteq$ AS$(m_i(z_2))$ if $z_1 \geq z_2$. Then we have an infinite decreasing chain of Arf semigroup containing the set $G[i]$. This means that the projection on the i-th branch can be smaller and smaller, therefore we cannot find a minimum in the set $S(G)$.

Thus we have found a contradiction in this case.

An example illustrating **Case 1** is the following.

If we consider $G = \{[2,3],[4,4]\}$, we have no information on the multiplicity sequence along the first branch and so we can obtain the following infinite decreasing chain of Arf semigroups containing G:



- **Case 2**: The first condition is fulfilled.

  So in this case the second condition is not fulfilled. The fact that $\gcd\{g[i], g \in G\} = 1$ for $i = 1, \ldots, r$ implies that we can compute the smallest Arf numerical semigroup $S(i)$ containing $G[i]$ for all $i = 1, \ldots, r$.

  Therefore if we denote by m_i the multiplicity sequence of $S(i)$ we clearly have that

Arf$(G) \in \sigma(E)$, where $E = \{m_i, i = 1, \ldots, r\}$. Suppose that it is defined by the matrix

$$M(T)_E = \begin{pmatrix} 0 & p_{1,2} & p_{1,3} & \ldots & p_{1,r} \\ 0 & 0 & p_{2,3} & \ldots & p_{2,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & p_{r-1,r} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Now if we consider an element $h \in G[i]$ we have that $h \in S(i)$ and therefore there exists an index $\mathrm{pos}_E(i, h)$ such that

$$h = \sum_{k=1}^{\mathrm{pos}_E(i,h)} m_{i,k}.$$

If $g \in G$ we can define $\mathrm{pos}_E(g) = [\mathrm{pos}_E(1, g[1]), \ldots, \mathrm{pos}_E(r, g[r])]$.

Notice that, if we consider $i, j \in \{1, \ldots, r\}$, with $i < j$ and $g \in G$ such that $\mathrm{pos}_E(i, g[i]) \neq \mathrm{pos}_E(j, g[j])$, we can easily deduce that in a multiplicity tree of an Arf semigroup of $\sigma(E)$ containing G the i-th and j-th branches cannot be glued at a level greater than $\min(\mathrm{pos}_E(i, g[i]), \mathrm{pos}_E(j, g[j]))$.

Then $p_{i,j}$ is at most $\min(\mathrm{pos}_E(i, g[i]), \mathrm{pos}_E(j, g[j]))$, and we also have to recall that $p_{i,j}$ is at most $k_E(i, j)$.

So denote by

$$U_E(G) = \left\{(i, j) \in \{1, \ldots, r\}^2 : i < j; \mathrm{pos}_E(i, g[i]) = \mathrm{pos}_E(j, g[j]) \text{ for all } g \in G\right\}.$$

For each $(i, j) \notin U_E(G)$ we define

$$\mathrm{MIN}_E(i, j, G) = \min\left(k_E(i, j), \min\{\min(\mathrm{pos}_E(i, g[i]), \mathrm{pos}_E(j, g[j])) : g \in G,\right.$$
$$\left.\mathrm{pos}_E(i, g[i]) \neq \mathrm{pos}_E(j, g[j])\}\right).$$

Notice that we need $(i, j) \notin U_E(G)$ to have the previous integers well defined.

So from the previous remark it follows that an Arf semigroup $S(T^1)$ of $\sigma(E)$ containing G with

$$M(T^1)_E = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \ldots & a_{1,r} \\ 0 & 0 & a_{2,3} & \ldots & a_{2,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_{r-1,r} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

is such that $a_{i,j}$ is at most $k_E(i, j)$ for $(i, j) \in U_E(G)$ and $a_{i,j}$ is at most $\mathrm{MIN}_E(i, j, G)$ for $(i, j) \notin U_E(G)$. In order to obtain the Arf closure we want to choose the biggest possible values of the a_i, thus from the previous bounds we obtain:

$$p_{i,j} = k_E(i, j) \text{ for } (i, j) \in U_E(G) \text{ and } p_{i,j} = \mathrm{MIN}_E(i, j, G) \text{ for } (i, j) \notin U_E(G).$$

We need to prove that this integers are compatible with the transitive property of the ramification matrix of an Arf semigroup tree. Therefore we consider a triad of integers $i < j < k$ and we want to show that $p_{i,j}, p_{j,k}$ and $p_{i,k}$ are in a $\{x, x, y\}$ configuration. We have the following cases:

1. $(i, j), (j, k), (k, i) \in U_E(G)$. Then $p_{i,j} = k_E(i, j)$,$p_{i,k} = k_E(i, k)$ and $p_{j,k} = k_E(j, k)$ and for the Remark 2 they satisfy our condition;
2. $(i, j), (j, k), (k, i) \notin U_E(G)$. We consider the vectors

   $$v_l = [\mathrm{pos}_E(l, g_1[l]), \ldots, \mathrm{pos}_E(l, g_m[l])],$$

   where $l \in \{i, j, k\}$ and $G = \{g_1, \ldots, g_m\}$. Then, using the notations of Lemma 2.1.4, we have that

   $$p_{i,j} = \min(k_E(i, j), \mathrm{MIN}(v_i, v_j)), p_{i,k} = \min(k_E(i, k), \mathrm{MIN}(v_i, v_k)) \text{ and}$$

   $$p_{j,k} = \min(k_E(j, k), \mathrm{MIN}(v_j, v_k)).$$

   Then suppose by contradiction that they are not compatible. Without loss of generality, we can assume that

   $$p_{i,j} < p_{i,k} \leq p_{j,k}.$$

   We have two cases

   - $p_{i,j} = k_E(i, j)$. Then we would have

     $$k_E(i, j) = p_{i,j} < p_{j,k} \leq k_E(j, k) \text{ and } k_E(i, j) = p_{i,j} < p_{i,k} \leq k_E(i, k),$$

     and this is absurd for the Remark 2;
   - $p_{i,j} = \mathrm{MIN}(v_i, v_j)$. Then we would have

     $$\mathrm{MIN}(v_i, v_j) = p_{i,j} < p_{j,k} \leq \mathrm{MIN}(v_j, v_k) \text{ and } \mathrm{MIN}(v_i, v_j) = p_{i,j} < p_{i,k} \leq \mathrm{MIN}(v_i, v_k),$$

     and this is absurd against Lemma 2.1.4 applied to the vectors v_i, v_j and v_k.
3. $(i, j) \in U_E(G)$ and $(j, k), (k, i) \notin U_E(G)$ (and the similar configurations). In this case we have that $v_i = v_j$. Then

   $$p_{i,j} = k_E(i, j), p_{i,k} = \min(k_E(i, k), x), \text{ and } p_{j,k} = \min(k_E(j, k), x),$$

   where $x = \mathrm{MIN}(v_i, v_k) = \mathrm{MIN}(v_j, v_k)$. We have two cases:

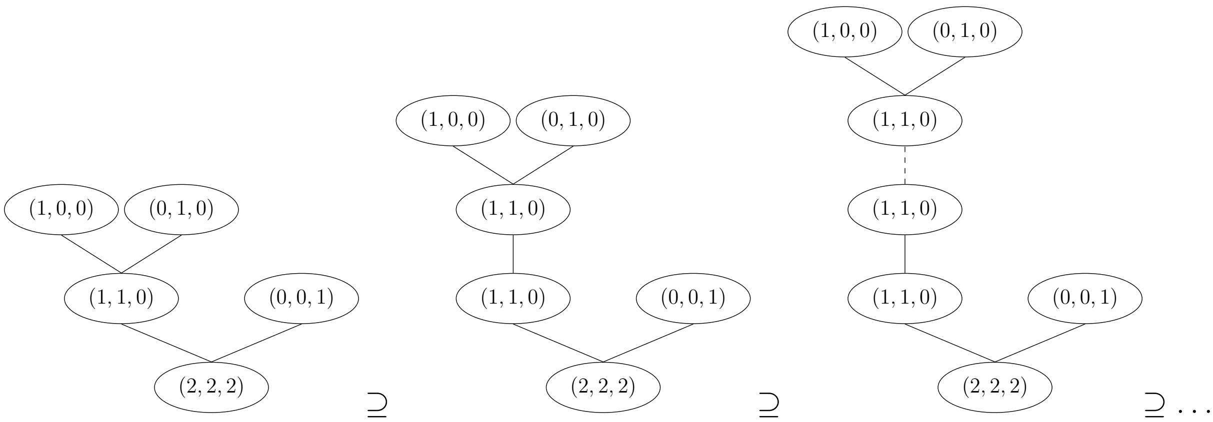
   - $k_E(i, j) = k_E(j, k) \leq k_E(i, k)$ (or equivalently $k_E(i, j) = k_E(i, k) \leq k_E(j, k)$). If $x < k_E(j, k) \leq k_E(i, k)$ then we have $p_{j,k} = p_{i,k} = x < k_E(i, j)$ and it is fine. If $x \geq k_E(j, k)$ then $p_{j,k} = k_E(j, k) = p_{i,j} \leq p_{i,k}$ that is compatible too.
   - $k_E(i, k) = k(j, k) < k_E(i, j)$. In this case we have $p_{i,k} = p_{j,k} < k_E(i, j) = p_{i,j}$ and it is fine.

So we actually have a well defined tree.

Anyway, because the second condition is not fulfilled, then there exists a pair $(i, j) \in \{1, \dots, r\}^2$ such that for all $g \in G$ we have $g[i] = g[j]$. So $(i, j) \in U_E(G)$, and, since in this case the two sequences are the same, we obtain $p_{i,j} = k_E(i, j) = +\infty$.

Thus we have found a contradiction because $\mathrm{Arf}(G)$ is not well defined.

An example illustrating **Case 2** is the following. If we consider $G = \{[3, 3, 2], [2, 2, 3]\}$, we will have the same multiplicity sequences in the first two branches, with no clues about the splitting point so we can obtain the following infinite decreasing chain in $S(G)$:



($\Leftarrow$) The previous proof gives us a way to compute $\mathrm{Arf}(G)$. We have to compute, using the modified Jacobian algorithm of Du Val, the Arf closure of each $G[i]$, finding the collection E (the first condition guarantees that it is possible to do that). Then we can find the matrix describing the semigroup using the set $U_E(G)$ and the integers $\mathrm{MIN}_E(i, j, G)$ with the procedure present in the first part (we cannot have $p_{i,j} = +\infty$ for the second condition). $\square$

Example 2.2.2. Suppose that we have $G = \{\mathbf{G}(1) = [5, 6, 5], \mathbf{G}(2) = [9, 11, 4], \mathbf{G}(3) = [9, 10, 2]\}$, that satisfies the conditions of the theorem. Then we have to apply the modified Jacobian algorithm to the sets

$$G[1] = \{5, 9\}\,, G[2] = \{6, 10, 11\} \text{ and } G[3] = \{2, 4, 5\}\,.$$

We find the following collection of multiplicity sequences:

$$E = \{m_1 = [5, 4], m_2 = [6, 4], m_3 = [2, 2]\}\,.$$

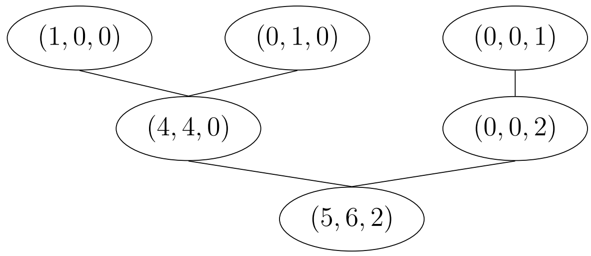
We have $k_E(1, 2) = 3$, $k_E(2, 3) = 2$ and $k_E(1, 3) = 2$.
Thus $\mathrm{pos}_E(\mathbf{G}(1)) = [1, 1, 3]$, $\mathrm{pos}_E(\mathbf{G}(2)) = [2, 3, 2]$ and $\mathrm{pos}_E(\mathbf{G}(3)) = [2, 2, 1]$. In this case $U_E(G) = \emptyset$.

Hence $\mathrm{MIN}_E(1, 2, G) = \min(2, k_E(1, 2)) = 2$, $\mathrm{MIN}_E(2, 3, G) = \min(1, k_E(2, 3)) = 1$ and $\mathrm{MIN}_E(1, 3, G) = \min(1, k_E(1, 3)) = 1$.

So the Arf closure is described by the matrix

$$M(T)_E = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that in this case we find that the Arf closure is an untwisted tree of $\tau(E)$ represented by the vector $T_E = (2, 1)$.



Example 2.2.3. Suppose that we have $G = \{\mathrm{G}(1) = [8, 6, 10], \mathrm{G}(2) = [5, 10, 5], \mathrm{G}(3) = [10, 13, 8]\}$, that satisfies the conditions of the theorem. By applying the modified Jacobian algorithm to the sets

$$G[1] = \{5, 8, 10\}, G[2] = \{6, 10, 13\} \text{ and } G[3] = \{5, 8, 10\},$$

we find the following collection of multiplicity sequences:

$$E = \{m_1 = [5, 3, 2], m_2 = [6, 4, 2] \text{ and } m_3 = [5, 3, 2]\}.$$

We have $k_E(1, 2) = 4$, $k_E(2, 3) = 4$ and $k_E(1, 3) = +\infty$.

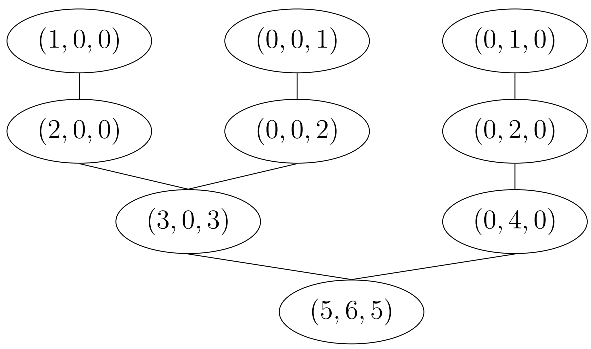
Thus $\mathrm{pos}_E(\mathrm{G}(1)) = [2, 1, 3]$, $\mathrm{pos}_E(\mathrm{G}(2)) = [1, 2, 1]$ and $\mathrm{pos}_E(\mathrm{G}(3)) = [3, 4, 2]$.

In this case $U_E(G) = \emptyset$, so we get $\mathrm{MIN}_E(1, 2, G) = \min(1, k_E(1, 2)) = 1$, $\mathrm{MIN}_E(2, 3, G) = \min(1, k_E(2, 3)) = 1$ and $\mathrm{MIN}_E(1, 3, G) = \min(2, k_E(1, 3)) = 2$.

The Arf closure is therefore described by the matrix

$$M(T)_E = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that in this case we find that the Arf closure is a twisted tree.



2.3 Arf closure of a good semigroup of $\mathbb{N}^r$

Denote by S a good semigroup of $\mathbb{N}^r$. In this section we describe a way to find the smallest Arf semigroup of $\mathbb{N}^r$ containing S, that is the Arf closure of S (the existence of the Arf closure is proved in [8]). We denote this semigroup by Arf(S). If S is a good semigroup of $\mathbb{N}^r$, we denote by S_i the projection on the i-th coordinate. The properties of the good semigroups guarantee that S_i is a numerical semigroup. Thus, it is clear that an Arf semigroup T containing S is such that Arf$(S_i) \subseteq T_i$ for all $i = 1, \dots, r$, where Arf(S_i) is the Arf closure of the numerical semigroup S_i (we can compute it using the algorithm of Du Val on a minimal set of generators of S_i).

Therefore, in order to have the smallest Arf semigroup containing S, we must have Arf$(S) \in \sigma(E)$ where $E = \{m_1, \dots, m_r\}$ and m_i is the multiplicity sequence associated to the Arf numerical semigroup Arf(S_i) (this follows from the fact, proved in [8, Proposition 31] , that the intersection of two Arf good semigroups containing S is stll an Arf good semigroup).
Now we need to find the matrix

$$M(T)_E = \begin{pmatrix} 0 & p_{1,2} & p_{1,3} & \dots & p_{1,r} \\ 0 & 0 & p_{2,3} & \dots & p_{2,r} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p_{r-1,r} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

that describes the tree of Arf(S).

Denote by $\delta = (c[1], \dots, c[r])$ the conductor of S, and consider the set

$$\text{Small}^*(S) = \text{Small}(S) \setminus \{\mathbf{0}\}$$

of the small elements of S with the exclusion of the zero vector.

Remark 7. We can recover the collection E from Small$^*(S)$. In fact, the multiplicity sequence m_i can be determined applying the Du Val algorithm to the set $\{s[i], s \in \text{Small}^*(S)\} \cup \{c[i]+1\} \subseteq S_i$. In order to find a multiplicity sequence we may have to add $c[i]+1$ because we can have $\gcd(\{s[i], s \in \text{Small}(S)\}) \neq 1$. Because $c[i]$ and $c[i]+1$ belong to S_i, we know that Arf(S_i) has conductor smaller than $c[i]$ and this implies that we only have to consider the elements that are smaller than $c[i]+1$.

Now, we notice that $p_{i,j} \leq \min(\text{pos}_E(i, c[i]), \text{pos}_E(j, c[j]))$ for all $i, j \in \{1, \dots, r\}$, with $i < j$, where we are using the notations of the previous section. In fact, if $\text{pos}_E(i, c[i]) \neq \text{pos}_E(j, c[j])$, we have already noticed that in an Arf semigroup containing δ the i-th and the j-th branches cannot be glued at a level greater than $\min(\text{pos}_E(i, c[i]), \text{pos}_E(j, c[j]))$, then $p_{i,j} \leq \min(\text{pos}_E(i, c[i]), \text{pos}_E(j, c[j]))$. If $\text{pos}_E(i, c[i]) = \text{pos}_E(j, c[j])$ then we have $\delta_1 = (c[1], \dots, c[i]+1, c[i+1], \dots, c[r]) \in S$, and $\text{pos}_E(i, c[i]+1) = \text{pos}_E(i, c[i])+1 > \text{pos}_E(j, c[j])$.

Therefore in an Arf semigroup containing δ_1 the i-th and the j-th branches cannot be glued at a level greater than

$$\min(\text{pos}_E(i, c[i]) + 1, \text{pos}_E(j, c[j])) = \text{pos}_E(j, c[j]) =$$

$$= \min(\text{pos}_E(i, c[i]), \text{pos}_E(j, c[j]),$$

hence we have again $p_{i,j} \leq \min(\text{pos}_E(i, c[i]), \text{pos}_E(j, c[j]))$.
Furthermore, we always have to take in account that $p_{i,j} \leq k_E(i, j)$ for all $i, j \in \{1, \dots, r\}$.
Now let us consider the following subset of $\{1, \dots, r\}^2$,

$$U_E(\text{Small}^*(S)) = \{(i, j) : \text{pos}_E(i, s[i]) = \text{pos}_E(j, s[j]) \text{ for all } s \in \text{Small}^*(S)\}.$$

If $(i, j) \in \{1, \dots, r\}^2 \setminus U_E(\text{Small}^*(S))$, and $i < j$ we can consider the following integers

$$\text{MIN}_E(i, j, \text{Small}^*(S)) = \min\left(k_E(i, j), \min\{\min(\text{pos}_E(i, s[i]), \text{pos}_E(j, s[j])) : s \in \text{Small}^*(S),\right.$$

$$\left.\text{pos}_E(i, s[i]) \neq \text{pos}_E(j, s[j])\}\right).$$

Notice that we need only to consider the vectors of Small$^*(S)$ because if $s \in S$ then $s_1 = \min(s, \delta) \in \text{Small}^*(S)$ and we clearly have

$$\min(\text{pos}_E(i, s[i]), \text{pos}_E(j, s[j])) \geq \min(\text{pos}_E(i, s_1[i]), \text{pos}_E(j, s_1[j])),$$

therefore $s_1 \in \text{Small}^*(S)$ gives us more accurate information on the ramification level than s (it can happen that $\text{pos}_E(i, s_1[i]) = \text{pos}_E(j, s_1[j])$ and $\text{pos}_E(i, s[i]) \neq \text{pos}_E(j, s[j])$, but only when $\min(\text{pos}_E(i, s[i]), \text{pos}_E(j, s[j])) \geq \min(\text{pos}_E(i, c[i]), \text{pos}_E(j, c[j]))$).

Thus, if T^1 is an Arf semigroup of $\sigma(E)$ containing S, represented by

$$M(T^1)_E = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ 0 & 0 & a_{2,3} & \dots & a_{2,r} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{r-1,r} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

we have:

- $a_{i,j} \leq \text{MIN}_E(i, j, \text{Small}^*(S))$ for $(i, j) \in \{1, \dots, n\}^2 \setminus U_E(\text{Small}^*(S))$;
- $a_{i,j} \leq \min(k_E(i, j), \text{pos}_E(i, c[i]))$, for $i \in U_E(\text{Small}^*(S))$ (we have $\text{pos}_E(i, c[i]) = \text{pos}_E(j, c[j])$).

Then we can finally deduce that the $p_{i,j}$ that define the matrix of Arf(S) are such that

- $p_{i,j} = \text{MIN}_E(i, j, \text{Small}^*(S))$, for $(i, j) \in \{1, \dots, r\}^2 \setminus U_E(\text{Small}^*(S))$;
- $p_{i,j} = \min(k_E(i, j), \text{pos}_E(i, c[i]))$, for $i \in U_E(\text{Small}^*(S))$ (we have $\text{pos}_E(i, c[i]) = \text{pos}_E(j, c[j])$),

and it is easy to see that the $p_{i,j}$ fulfil the condition of Remark 3.

Remark 8. In other words we proved that $\mathrm{Arf}(S)$ can be found by computing $\mathrm{Arf}(G)$ where:

$$G = \mathrm{Small}^*(S) \bigcup \{(c[1]+1,\dots,c[i],c[i+1],\dots,c[r]),\dots,(c[1],\dots,c[i]+1,c[i+1],\dots,c[r]),\dots,$$
$$(c[1],\dots,c[i],c[i+1],\dots,c[r]+1)\}\,.$$

Example 2.3.1. Let us consider the good semigroup S with the following set of small elements,

$$\mathrm{Small}^*(S) = \{[5,6,5],[5,10,5],[5,12,5],[8,6,8],[8,10,8],[8,12,8],[8,6,10],[8,10,10],$$
$$[8,12,10],[10,6,8],[10,10,8],[10,12,8],[10,6,10],[10,10,10],[10,12,10]\}\,.$$

The conductor is $\delta = [10,12,10]$. First of all we need to recover from $\mathrm{Small}^*(S)$ the collection of multiplicity sequences E. We have to apply the Du Val algorithm to the following sets:

$$\{5,8,10,11\}\,,\{6,10,12,13\}\ \text{and}\ \{5,8,10,11\}\,,$$

therefore we find that $E = \{[5,3,2],[6,4,2],[5,3,2]\}$.

We have

$$\mathrm{pos}(\mathrm{Small}^*(S)) = \{\mathrm{pos}_E(s) : s \in \mathrm{Small}(S)\} = \{[1,1,1],[1,2,1],[1,3,1],[2,1,2],[2,2,2],$$
$$[2,3,2],[2,1,3],[2,2,3],[2,3,3],[3,1,2],[3,2,2],[3,3,2],[3,1,3],[3,2,3],[3,3,3]\}\,.$$

It is easy to check that $U_E(\mathrm{Small}^*(S)) = \emptyset$. Thus we have

- $p_{1,2} = \mathrm{MIN}_E(1,2,\mathrm{Small}^*(S)) = \min(k_E(1,2)=4,1) = 1$, because we have the element $[1,2,1] \in \mathrm{pos}(\mathrm{Small}^*(S))$ corresponding to $s = [5,10,5] \in \mathrm{Small}^*(S)$ such that $1 = \mathrm{pos}_E(1,s[1]) \neq \mathrm{pos}_E(2,s[2]) = 2$ and $\min(\mathrm{pos}_E(1,s[1]),\mathrm{pos}_E(2,s[2])) = 1$.
- $p_{2,3} = \mathrm{MIN}_E(2,3,\mathrm{Small}^*(S)) = \min(k_E(2,3)=4,1) = 1$, because we have the element $[1,2,1] \in \mathrm{pos}(\mathrm{Small}^*(S))$ corresponding to $s = [5,10,5] \in \mathrm{Small}^*(S)$ such that $2 = \mathrm{pos}_E(2,s[2]) \neq \mathrm{pos}_E(3,s[3]) = 1$ and $\min(\mathrm{pos}_E(2,s[2]),\mathrm{pos}_E(3,s[3])) = 1$.
- $p_{1,3} = \mathrm{MIN}_E(1,3,\mathrm{Small}^*(S)) = \min(k_E(1,3)=+\infty,2) = 2$, because we have the element $[2,2,3] \in \mathrm{pos}(\mathrm{Small}^*(S))$ corresponding to $s = [8,10,10] \in \mathrm{Small}^*(S)$ such that $2 = \mathrm{pos}_E(1,s[1]) \neq \mathrm{pos}_E(3,s[3]) = 3$ and $\min(\mathrm{pos}_E(1,s[1]),\mathrm{pos}_E(3,s[3])) = 2$, and we cannot find a smaller discrepancy.

  So the Arf closure of S is described by the matrix

$$M(T)_E = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

  with

$$E = \{m_1 = [5,3,2], m_2 = [6,4,2]\ \text{and}\ m_3 = [5,3,2]\}\,.$$

The following procedure, implemented in GAP, has as argument the set of small elements of a good semigroup and give as a result the Arf Closure of the given good semigroup. The Arf closure is described by a list $[E, M(T)_E]$.

```
gap> S:=[[5,6,5],[5,10,5],[5,12,5],[8,6,8],[8,10,8],[8,12,8],
[8,6,10],[8,10,10],[8,12,10],[10,6,8],[10,10,8],[10,12,8],
[10,6,10],[10,10,10],[10,12,10]];

[ [ 5, 6, 5 ], [ 5, 10, 5 ], [ 5, 12, 5 ], [ 8, 6, 8 ],
 [ 8, 10, 8 ], [ 8, 12, 8 ], [ 8, 6, 10 ], [ 8, 10, 10 ],
  [ 8, 12, 10 ], [ 10, 6, 8 ], [ 10, 10, 8 ], [ 10, 12, 8 ],
   [ 10, 6, 10 ], [ 10, 10, 10 ], [ 10, 12, 10 ] ]

gap> ArfClosureOfGoodsemigroup(S);
 [ [ [ 5, 3, 2 ], [ 6, 4, 2 ], [ 5, 3, 2 ] ],
  [ [ 0, 1, 2 ], [ 0, 0, 1 ], [ 0, 0, 0 ] ] ]
```

2.4 Bounds on the minimal number of vectors determining a given Arf semigroup

Suppose that E is a collection of r multiplicity sequences. Let $T \in \tau(E)$ and consider the corresponding semigroup $S(T)$ in $\sigma(E)$, we want to study the properties that a set of vectors $G(T) \subseteq \mathbb{N}^r$ has to satisfy to have $S(T) = \text{Arf}(G(T))$, with the notations given in the previous section. We call such a $G(T)$ a set of generators for $S(T)$. In particular we want to find bounds on the cardinality of a minimal set of generators for a $S(T) \in \sigma(E)$.

Since we want to find a $G(T)$ such that $\text{Arf}(G(T))$ is well defined, it has to satisfy the following properties:

- For all $i = 1, \dots, r$
$$\gcd(v[i]; v \in G(T)) = 1,$$
where $v[i]$ is the i-th coordinate of the vector $v \in G(T)$.

- For all $i, j \in \{1, \dots, r\}$, with $i < j$ there exists $v \in G(T)$ such that $v[i] \neq v[j]$.

Now we want that $\text{Arf}(G(T))$ is an element of $\sigma(E)$. This implies that, when we apply the algorithm of Du Val to $G(T)[i]$, we have to find the i-th multiplicity sequence of E. This means that, if we call S_i the Arf numerical semigroup corresponding to the projection on the i-th coordinate, we must have $G(T)[i] \subseteq S_i$ and furthermore $G(T)[i]$ has to contain the characters of S_i. In fact, in [1] it is proved that if we have $G = \{g_1, \dots, g_m\} \subseteq \mathbb{N}$ with $\gcd(G) = 1$ then G must contain the set of characters of the Arf closure of the numerical semigroup $N = \langle G \rangle$.

We suppose that $E = \{m_1, \ldots, m_r\}$. Given

$$m_i = [m_{i,1}, \ldots, m_{i,N}],$$

we consider the restricion number $r(m_{i,j})$ of $m_{i,j}$. With this notation we have that the characters of the multiplicity sequence m_i are the elements of the set

$$\text{Char}_E(i) = \left\{ \sum_{k=1}^{j} m_{i,k} : r(m_{i,j}) < r(m_{i,j+1}) \right\}.$$

Notice that, from our assumptions on N, it follows that the last two entries in each m_i are 1, and it is easy to see how it guarantees that we cannot find characters in correspondence of indices greater than N. We define $\text{PChar}_E(i) = \{j : r(m_{i,j}) < r(m_{i,j+1})\}$.

Given the collection E, we denote by

$$V_E(j_1, j_2, \ldots, j_r) = \left[\sum_{k=1}^{j_1} m_{1,k}, \sum_{k=1}^{j_2} m_{2,k}, \ldots, \sum_{k=1}^{j_r} m_{r,k} \right].$$

Now, the elements of $G(T)$ must be of the type $V_E(j_1, j_2, \ldots, j_r)$ (in fact we noticed that when we project on the k-th coordinate we must find an element of the corresponding numerical semigroup that has the previous representation for some j_k).

From the previous remarks and notations we have the following property:

$$G(T) = \{\text{Gen}(1) = V_E(j_{1,1}, \ldots, j_{1,r}), \ldots, \text{Gen}(t) = V_E(j_{t,1}, \ldots, j_{t,r})\}$$

are generators of a semigroup of $\sigma(E)$ if and only if

$$\text{PChar}_E(i) \subseteq \{j_{1,i}, \ldots, j_{t,i}\} \text{ for all } i = 1, \ldots, r.$$

In particular we have found a lower bound for the cardinality of a minimal set of generators for a $S(T) \in \sigma(E)$. In fact $G(T)$ has at least $C_E = \max\{|\text{PChar}_E(i)|, i = 1, \ldots, r\}$ elements.

Now we want to determine the generators of a given semigroup $S(T) \in \sigma(E)$. We have the following theorem.

Theorem 2.4.1. *Let $S(T) \in \sigma(E)$ with*

$$M(T)_E = \begin{pmatrix} 0 & p_{1,2} & p_{1,3} & \ldots & p_{1,r} \\ 0 & 0 & p_{2,3} & \ldots & p_{2,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & p_{r-1,r} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Denote by $P = \{(q,u) : p_{q,u} = k_E(q,u)\}$. Then $G(T) = \{\mathit{Gen}(1), \ldots, \mathit{Gen}(t)\} \subseteq \mathbb{N}^r$ is such that $\mathit{Arf}(G(T)) = S(T)$ if and only if the following conditions hold

- *Gen*$(1) = V_E(j_{1,1}, \dots, j_{1,r}), \dots,$ *Gen*$(t) = V_E(j_{t,1}, \dots, j_{t,r})$ *for some values of the indices* $j_{1,1}, \dots, j_{t,r}$*;*

- *PChar*$_E(i) \subseteq \{j_{1,i}, \dots, j_{t,i}\}$ *for all* $i = 1, \dots, r$*.*

  Furthermore, if we consider the following integer

$$\mathit{MIN}_{G(T)}(q, u) = \min\left(k_E(q, u), \min\left\{\min(j_{p,q}, j_{p,u}) : j_{p,q} \neq j_{p,u}, p = 1, \dots, t\right\}\right),$$

  for the $(q, u) \in \{1, \dots, r\}^2$ *with* $q < u$ *and where it is well defined, we have:*

- *for* $(q, u) \in P$ *we have either* $j_{p,q} = j_{p,u}$ *for all* $p = 1, \dots, t$*, or* $\mathit{MIN}_{G(T)}(q, u)$ *is well defined and it equals* $k_E(q, u)$*;*

- $\mathit{MIN}_{G(T)}(q, u)$ *is well defined and it equals* $p_{q,u}$*, for all* $(q, u) \notin P$*.*

Proof ($\Leftarrow$) Suppose that we have $G(T) = \{\mathrm{Gen}(1), \dots, \mathrm{Gen}(t)\} \subseteq \mathbb{N}^r$ satisfying the conditions of the theorem. The first two conditions ensure that if we apply the algorithm defined in the previous section on $G(T)$ it will produce an element of $\sigma(E)$.

Now it is easy, using the notations of Theorem 2.2.1, to show that $j_{p,q} = \mathrm{pos}_E(q, \mathrm{Gen}(p)[q])$ and from this it follows that, when $\mathrm{MIN}_{G(T)}(q, u)$ is well defined, it is equal to $\mathrm{MIN}_E(q, u, G(T))$. Furthermore we have $U_E(G(T)) \subseteq P$. In fact we have

$$U_E(G(T)) = \left\{(q, u) \in \{1, \dots, r\}^2 : \mathrm{pos}_E(q, \mathrm{Gen}(p)[q]) = \mathrm{pos}_E(u, \mathrm{Gen}(p)[u])\right.$$

$$\left.\text{for all } p = 1, \dots, t\right\} = \left\{(q, u) \in \{1, \dots, r\}^2 : j_{p,q} = j_{p,u} \text{ for all } p = 1, \dots, t\right\},$$

therefore if $(q, u) \in U_E(G(T))$ then $(q, u) \in P$, since $G(T)$ satisfies the fourth condition in the statement of the theorem (we cannot have $(q, u) \notin P$ because in this case $\mathrm{MIN}_{G(T)}(q, u) = \mathrm{MIN}_E(q, u, G(T))$ has to be well defined). So it will follow that, if $S(T^1)$ is $\mathrm{Arf}(G(T))$ then

$$M(T^1)_E = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ 0 & 0 & a_{2,3} & \dots & a_{2,r} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{r-1,r} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where

- $a_{i,j} = \mathrm{MIN}_E(i, j, G(T))$ if $(i, j) \notin U_E(G(T))$;

- $a_{i,j} = k_E(i, j)$ if $(i, j) \in U_E(G(T))$.

Therefore if $(i, j) \notin P$ then $(i, j) \notin U_E(G(T))$ and we have $a_{i,j} = \mathrm{MIN}_E(i, j, G(T)) = \mathrm{MIN}_{G(T)}(i, j) = p_{i,j}$. If $(i, j) \in P$ then

- if $(i, j) \in U_E(G(T))$ then $a_{i,j} = k_E(i, j)$;

- if $(i,j) \notin U_E(G(T))$ then $a_{i,j} = \text{MIN}_E(i,j,G(T)) = \text{MIN}_{G(T)}(i,j) = k_E(i,j)$, for the properties of the set $G(T)$ ($(i,j) \in P$).

So we showed that $\text{Arf}(G(T)) = S(T)$. Thus the proof of this implication is complete.

($\Rightarrow$) It follows immediately by contradiction, using the first part of the proof. $\square$

Example 2.4.2. Suppose that we have $E = \{m_1, m_2, m_3\}$, where

$$m_1 = [5,4], m_2 = [6,4], m_3 = [2,2].$$

We have, $k_E(1,2) = 3, k_E(2,3) = 2$ and $k_E(1,3) = 2$.

We can define:

$$R(i) = [r(m_{i,1}), r(m_{i,2}), \ldots, r(m_{i,N})].$$

Notice that $r(m_{i,1}) = 0, r(m_{i,2}) = 1$. The values of $\text{PChar}(i)$ are the indices where this sequence has an increase (it can be easily shown that when the sequence has an increase we have $r(m_{i,j}) = r(m_{i,j+1}) - 1$ cf. [4, Lemma 3.2]). Furthermore $R(1) = [0,1,2,2,2,2]$, $R(2) = [0,1,2,3,2,2]$ and $R(3) = [0,1,1,2]$. So $\text{PChar}_E(1) = \{1,2\}, \text{PChar}_E(2) = \{1,2,3\}$ and $\text{PChar}_E(3) = \{1,3\}$.

Suppose that we want to find generators for the untwisted tree T^1 such that $T^1_E = (2,1)$. We need at least three vectors because $C_E = 3$. Consider the vectors $\text{Gen}(1) = V_E(1,1,3), \text{Gen}(2) = V_E(2,3,2)$ and $\text{Gen}(3) = V_E(2,2,1)$. The second condition, that guarantees that we have a tree belonging to $\tau(E)$, is satisfied. Furthermore $\text{MIN}_{G(T)}(1,2) = \min(k_E(1,2),2) = 2$, $\text{MIN}_{G(T)}(2,3) = \min(k_E(2,3),1) = 1$, and $\text{MIN}_{G(T)}(1,3) = \min(k_E(1,3),1) = 1 = \min(p_{1,2}, p_{2,3})$ where $G(T) = \{\text{Gen}(1), \text{Gen}(2), \text{Gen}(3)\}$. Thus we have $\text{Arf}(G(T)) = S(T^1)$. They are the vectors $\text{Gen}(1) = [5,6,5], \text{Gen}(2) = [9,11,4], \text{Gen}(3) = [9,10,2]$ that appeared in the Example 2.2.2.

Now, we want to find an upper bound for the cardinality of a minimal set $G(T)$ such that $\text{Arf}(G(T)) \in \sigma(E)$.

Remark 9. Suppose that T^1 is a twisted tree of $\tau(E)$, where E is a collection of r multiplicity sequences. Then, there exists a permutation $\delta \in S^r$ such that $\delta(T^1)$ is an untwisted tree of $\tau(\delta(E))$. If G is a set of generators for $\delta(T^1)$, it is clear that we have

$$\delta^{-1}(G) = \left\{\delta^{-1}(g); g \in G\right\},$$

is a set of generators for the twisted tree T^1.

From the previous remark it follows that we can focus only on the untwisted trees of $\tau(E)$ to find an upper bound for the cardinality of $G(T)$.

Our problem is clearly linked to the following question.

Question 2.4.3. *Let us consider a vector $\boldsymbol{d} = [d_1, \ldots, d_r] \in \mathbb{N}^r$. For all the $G \subseteq \mathbb{N}^{r+1}$ we denote by MIN(G,i,j) the integers (if they are well defined)*

$$MIN(G,i,j) = \min\{\min(g[i], g[j]) : g \in G \textit{ with } g[i] \neq g[j]\},$$

for all the $i < j$ and $i, j \in \{1, \ldots, r+1\}$.

*We define a solution for the vector **d** as a set $G \subseteq \mathbb{N}^{r+1}$ such that:*

$$\mathit{MIN}(G, i, j) = \min\{d_i, \ldots, d_{j-1}\} \textit{ for all } i < j.$$

Consider $r \in \mathbb{N}$ with $r \geq 1$. Find the smallest $t \in \mathbb{N}$, such that for all $[d_1, \ldots, d_r] \in \mathbb{N}^r$ there exists a solution with t vectors. We denote such a number t by $NS(r)$.

Theorem 2.4.4. *Consider $r \in \mathbb{N}$ with $r \geq 1$. Then $NS(r) = \lceil \log_2(r+1) \rceil$, where $\lceil d \rceil = \min\{m \in \mathbb{N} : m \geq d\}$.*

Proof First of all we show that given an arbitrary vector **d** of $\mathbb{N}^r$ we are able to find a solution of **d** consisting of $N = \lceil \log_2(r+1) \rceil$ vectors.

We will do it by induction on r. The base of induction is trivial. In fact if $r = 1$ then for each vector $[d_1]$ we find the solution $G = \{[d_1, d_1 + 1]\}$ that has cardinality $\lceil \log_2(1+1) \rceil = 1$. Thus we suppose that the theorem is true for all the $m < r$ and we prove it for r. Let **d** an arbitrary vector of $\mathbb{N}^r$. We fix some notations. Given a vector **d**, we will denote by $\mathrm{sol}(\mathbf{d})$ a solution with $\lceil \log_2(r+1) \rceil$ vectors. We denote by $\mathrm{Inf}(\mathbf{d}) = \min\{d_i : i = 1, \ldots, r\}$ and by $\mathrm{Pinf}(\mathbf{d}) = \{i \in \{1, \ldots, r\} : d_i = \mathrm{Inf}(\mathbf{d})\}$. We have $1 \leq |\mathrm{Pinf}(\mathbf{d})| = k(\mathbf{d}) \leq r$.

Suppose that $\mathrm{Pinf}(\mathbf{d}) = \{i_1 < i_2 < \cdots < i_{k(\mathbf{d})}\}$. Then we can split the vector **d** in the following $k(\mathbf{d}) + 1$ subvectors:

$$\begin{cases} \mathbf{d}_1 = \mathbf{d}(1 \ldots i_1 - 1), \\ \mathbf{d}_j = \mathbf{d}(i_{j-1} + 1 \ldots i_j - 1) \text{ for } j = 2, \ldots, k(\mathbf{d}), \\ \mathbf{d}_{k(\mathbf{d})+1} = \mathbf{d}(i_{k(\mathbf{d})} + 1 \ldots r), \end{cases}$$

where with $\mathbf{d}(a \ldots b)$ we mean

- $\emptyset$ if $b < a$;
- The subvector of **d** with components between a and b if $a \leq b$.

Then the subvectors $\mathbf{d}_j$ are either empty or with all the components greater than $\mathrm{Inf}(\mathbf{d})$. We briefly illustrate with an example the construction of the subvectors $\mathbf{d}_j$.

Example 2.4.5. Suppose that $\mathbf{d} = [2, 3, 2, 2, 5, 4, 5]$. Then $\mathrm{Inf}(\mathbf{d}) = 2$, $\mathrm{Pinf}(\mathbf{d}) = \{1, 3, 4\}$ and then we have the four subvectors:

- $\mathbf{d}_1 = \mathbf{d}(1 \ldots 0) = \emptyset$,
- $\mathbf{d}_2 = \mathbf{d}(2 \ldots 2) = [3]$,
- $\mathbf{d}_3 = \mathbf{d}(4 \ldots 3) = \emptyset$,
- $\mathbf{d}_4 = \mathbf{d}(5 \ldots 7) = [5, 4, 5]$.

Then we can consider the list of $k(\mathbf{d})+1$ subvectors:

$$p(\mathbf{d}) = [\mathbf{d}_1, \dots, \mathbf{d}_{k(\mathbf{d})+1}],$$

and, because all the $\mathbf{d}_i$ have length strictly less than r we can find a solution for each of them with $N = \lceil \log_2 (r+1) \rceil$ or less vectors. For the $\mathbf{d}_i = \emptyset$ we will set $\mathrm{sol}(\emptyset) = \{[x]\}$, where x is an arbitrary integer that is strictly greater than all the entries of $\mathbf{d}$. It is quite easy to check that the following equality holds:

$$r = k(\mathbf{d}) + \sum_{i=1}^{k(\mathbf{d})+1} \mathrm{Length}(\mathbf{d}_i). \tag{2.1}$$

We associate to the list of vectors $p(\mathbf{d})$ another list of vector $c(\mathbf{d})$ such that

$$c(\mathbf{d}) = [\mathbf{c}_1, \dots, \mathbf{c}_{k(\mathbf{d})+1}],$$

where $\mathrm{Length}(\mathbf{c}_j) = \mathrm{Length}(\mathbf{d}_j) + 1$ and the entries of $\mathbf{c}_j$ are all equal to $\mathrm{Inf}(\mathbf{d})$ for all $j = 1, \dots, k(\mathbf{d})+1$.

Now we consider the set $I(N) = \{0,1\}^N$. For each $\mathbf{t} \in I(N)$ we denote by $O(\mathbf{t})$ the number of ones that appear in $\mathbf{t}$. Because we have $N = \lceil \log_2 (r+1) \rceil$, it follows

$$k(\mathbf{d}) + 1 \leq r + 1 \leq 2^N = |I(N)|,$$

therefore it is always possible to associate to each subvectors of the list $p(\mathbf{d})$ distinct elements of $I(N)$. We actually want to show that it is possible to associate to all the subvectors $\mathbf{d}_i$ distinct vectors of $\mathbf{t} \in I(N)$ such that $O(\mathbf{t}) \geq |\mathrm{sol}(\mathbf{d}_i)|$ (for $\mathbf{d}_i = \emptyset$ we can also associate the zero vector). We already know for the inductive step that all the $\mathbf{d}_i$ have solutions with at most N vectors. Suppose therefore that $m \leq N$.

It is easy to see that

$$|\{\mathbf{t} \in I(N) : O(t) \geq m\}| = \sum_{k=m}^{N} \binom{N}{k}.$$

Then we suppose by contradiction that in $p(\mathbf{d})$ we have $\sum_{k=m}^{N} \binom{N}{k} + 1$ subvectors with solution with cardinality m. From the inductive step it follows that all these subvectors have at least length 2^{m-1}, and from the formula 2.1 it follows:

$$r \geq \sum_{k=m}^{N} \binom{N}{k} + \left(\sum_{k=m}^{N} \binom{N}{k} + 1 \right) 2^{m-1} \Rightarrow r + 1 \geq \left(\sum_{k=m}^{N} \binom{N}{k} + 1 \right) (1 + 2^{m-1}).$$

But we also have that:

$$\sum_{k=m}^{N} \binom{N}{k} + 1 \geq 2^{N-m+1},$$

in fact $\sum_{k=m}^{N} \binom{N}{k}$ is the number of ways to select a subset of $\{1, \dots, N\}$ of at least m elements while there are $2^{N-m+1} - 1$ ways to select a subset which contains at least m elements and contains $\{1, 2, \dots, m-1\}$.

Therefore we can continue the inequality:

$$r + 1 \geq 2^{N-m+1}(1 + 2^{m-1}) = 2^N + 2^{N-m+1} > 2^N.$$

But $N = \lceil \log_2 (r+1) \rceil$ and therefore $r + 1 \leq 2^N$ and we find a contradiction. Then in $\{\mathbf{t} \in I(N) : O(t) \geq m\}$ we have enough vectors to cover all the subvectors with solution with cardinality m. We still also have to exclude the following possibility. Suppose that we have x subvectors with solutions of cardinality m_1 and y subvectors with solutions of cardinality $m_2 > m_1$. If $| \{\mathbf{t} \in I(N) : O(t) \geq m_1\} | - x < y$ then it would not be possible to associate to all the subvectors of the second type an element $\mathbf{t}$ of $I(N)$ with $O(\mathbf{t}) \geq m_2$. Indeed if this happen we would have:

$$r \geq x + y - 1 + x \cdot 2^{m_1 - 1} + y \cdot 2^{m_2 - 1} > x + y - 1 + (x+y)2^{m_1 - 1} \Rightarrow$$

$$\Rightarrow r + 1 \geq (x+y)(1 + 2^{m_1 - 1}) \geq \left(\sum_{k=m_1}^{N} \binom{N}{k} + 1 \right) (1 + 2^{m_1 - 1}),$$

and we already have seen that this is not possible.

Therefore we proved that we can consider a matrix A with N rows and $k(\mathbf{d}) + 1$ distinct columns with only zeroes and ones as entries and such that the i-th column of A is a vector $\mathbf{t}$ of $I(N)$ such that $O(\mathbf{t}) \geq |\mathrm{sol}(\mathbf{d}_i)|$ for each $1 \leq i \leq k(\mathbf{d}) + 1$.

Now we can complete the construction of a solution for $\mathbf{d}$. We consider a matrix B with N rows and $k(\mathbf{d}) + 1$ columns. We fill the matrix B following these rules:

- If $A[i,j] = 0$ then in $B[i,j]$ we put the vector $\mathbf{c}_j$;
- If $A[i,j] = 1$ then in $B[i,j]$ we put an element of $\mathrm{sol}(\mathbf{d}_j)$;
- All the elements of $\mathrm{sol}(\mathbf{d}_j)$ have to appear in the j-th column for all $j = 1, \dots, k(\mathbf{d}) + 1$.

Then if we glue all the vectors appearing in the i-th row of B for each $i = 1, \dots, N$ we obtain a solution G for the vector $\mathbf{d}$. In fact if we consider i_1, j_1 such that $i_1 < j_1$ we have two possibilities:

- i_1 and j_1 both correspond to elements in the j-th column of B. Then because in this column we have either vectors of a solution for $\mathbf{d}_j$ or constant vectors, it follows that they fulfil our conditions.

- i_1 and j_1 correspond to elements in distinct columns. This implies that we must have MIN(G, i_1, j_1) = Inf($\mathbf{d}$). In fact, for construction, between two distinct subvectors we have an element equal to Inf($\mathbf{d}$) in $\mathbf{d}$ forcing MIN$(G, i_1, j_1) =$ Inf($\mathbf{d}$). Now suppose that i_1 and j_1 correspond respectively to elements in the i-th and j-th columns of B. Because we suppose $i \neq j$ we have that the i-th column and the j-th column of the matrix A are distinct so there exists a k such that $A[k, i] = 0$ and $A[k, j] = 1$ (or vice versa). This implies that in B we have a row where in the i-th column there is the constant vector equal to Inf($\mathbf{d}$) while in the j-th column we have a vector corresponding to a solution of a subvectors of $\mathbf{d}$ (that has all the components greater than Inf($\mathbf{d}$) by construction). This easily implies that MIN$(G, i_1, j_1) =$ Inf($\mathbf{d}$).

Example 2.4.6. Suppose that $\mathbf{d} = [2, 3, 2, 2, 5, 4, 5]$. We have $r = 7$, then we want to show that there exists a solution with three vectors. We have already seen that in this case we have:

$$p(\mathbf{d}) = [\emptyset, [3], \emptyset, [5, 4, 5]].$$

We need to compute a solution for each entry of $p(\mathbf{d})$. We have:

- sol$(\emptyset) = \{[6]\}$ (6 is greater than all the entries of $\mathbf{d}$);
- sol$([3]) = \{[3, 4]\}$;
- Let us compute a solution for $\mathbf{f} = [5, 4, 5]$ with the same techniques. Because Length($\mathbf{f}$) = 3 we expect to find a solution with at most two vectors. We have:

  $$p(\mathbf{f}) = [[5], [5]],$$

  and we have sol$([5]) = \{[5, 6]\}$. Then in $I(2)$ we want to find two distinct vectors with at least an entry equal to one. We can choose $[1, 1]$ and $[0, 1]$. Therefore we have:

  $$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} [5,6] & [4,4] \\ [5,6] & [5,6] \end{pmatrix}.$$

  Then sol$([5, 4, 5]) = \{[5, 6, 4, 4], [5, 6, 5, 6]\}$.

Now we want to find in $I(3)$ four vectors $\mathbf{t}_i$ for $i = 1, \ldots, 4$. We have free choice for the $\mathbf{t}_1$ and $\mathbf{t}_3$, while we need $O(\mathbf{t}_2) \geq 1$ and $O(\mathbf{t}_4) \geq 2$. For instance we choose $\mathbf{t}_1 = [0, 0, 0], \mathbf{t}_2 = [1, 0, 0], \mathbf{t}_3 = [1, 1, 0], \mathbf{t}_4 = [1, 0, 1]$. Then we have:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} [2] & [3,4] & [6] & [5,6,4,4] \\ [2] & [2,2] & [6] & [2,2,2,2] \\ [2] & [2,2] & [2] & [5,6,5,6] \end{pmatrix}.$$

Then a solution for $\mathbf{d}$ is the set

$$G = \{[2, 3, 4, 6, 5, 6, 4, 4], [2, 2, 2, 6, 2, 2, 2, 2], [2, 2, 2, 2, 5, 6, 5, 6]\}.$$

So we proved that $\mathrm{NS}(r) \leq \lceil \log_2 (r+1) \rceil$. To prove that the equality holds we notice that for each r a constant vector needs exactly $\lceil \log_2 (r+1) \rceil$ vectors in its solutions. $\square$

Now we can return to the problem of determining an upper bound for the cardinality of $G(T)$. We need another lemma:

Lemma 2.4.7. *Let $E = \{m_1, m_2\}$ be a collection of two multiplicity sequences. Then, with the previous notations we have:*

$$k_E(1,2) \leq \min \{j : j \in (PChar_E(1) \cup PChar_E(2)) \setminus (PChar_E(1) \cap PChar_E(2))\}.$$

Proof Let us choose an arbitrary element $t \in (\mathrm{PChar}_E(1) \cup \mathrm{PChar}_E(2)) \setminus (\mathrm{PChar}_E(1) \cap \mathrm{PChar}_E(2))$. We want to show that $k_E(1,2) \leq t$. Suppose by contradiction that $t < k_E(1,2)$. Without loss of generality we suppose that $t \in \mathrm{PChar}_E(1)$. It follows that $t \notin \mathrm{PChar}_E(2)$ and we have:

$$r(m_{1,t}) < r(m_{1,t+1}) \text{ and } r(m_{2,t}) \geq r(m_{2,t+1}).$$

Notice that if an entry of m_1 has $m_{1,t+1}$ as a summand and it is not $m_{1,t}$, it is forced to have $m_{1,t}$ as a summand too. So from $r(m_{1,t}) < r(m_{1,t+1})$ we deduce that in m_1 there are no entries involving only $m_{1,t}$. Similarly from $r(m_{2,t}) \geq r(m_{2,t+1})$ we deduce that in m_2 we must have at least one entry $m_{2,s}$ that involves $m_{2,t}$ as a summand but not $m_{2,t+1}$.

Namely

$$m_{2,s} = \sum_{k=s+1}^{t} m_{2,k}. \tag{2.2}$$

Now, we have assumed that $t < k_E(1,2)$ hence $t+1 \leq k_E(1,2)$. This implies that the untwisted tree T such that $T_E = (t+1)$ is well defined. In T we have the following nodes:

$$(m_{1,s}, m_{2,s}), \ldots, (m_{1,t}, m_{2,t}), (m_{1,t+1}, m_{2,t+1}).$$

Then from (2.2) and from the fact that the two branches are still glued at level $t+1$ it must follow that

$$m_{1,s} = \sum_{k=s+1}^{t} m_{1,k}$$

and we have still noticed how it contradicts the assumption $r(m_{1,t}) < r(m_{1,t+1})$. $\square$

Now we can prove the following result:

Proposition 2.4.8. *Let E be a collection of r multiplicity sequences. Then, if $S(T) \in \sigma(E)$, there exists $G(T) \subseteq \mathbb{N}^r$ with $Arf(G(T)) = S(T)$ and $|G(T)| = C_E + \lceil \log_2 (r) \rceil$.*

Proof For the Remark 9 it suffices to prove the theorem only for the untwisted trees. Therefore we suppose that $T_E = (d_1, \ldots, d_{r-1})$. First of all we have to satisfy the condition on the characters to ensure that $\mathrm{Arf}(G(T)) \in \sigma(E)$. From the Lemma (2.4.7) it follows that we can use C_E vectors to satisfy all the conditions. To see it, let us fix some notations.

Denote by $\tau(i) = |\text{PChar}_E(i)|$ for all $i = 1, \ldots, r$. Therefore $C_E = \max\{\tau(i), i = 1, \ldots, r\}$. Suppose that

$$\text{PChar}_E(i) = \left\{a_{i,1} < \cdots < a_{i,\tau(i)}\right\},$$

and we define

$$L = \max\left(\bigcup_{i=1}^{r} \text{PChar}_E(i)\right) + 1.$$

For all $i = 1, \ldots, r$ we consider the vector $J(i) = [a_{i,1}, \ldots, a_{i,\tau(i)}, L, \ldots, L] \in \mathbb{N}^{C_E}$. Thus we can use the following set of vectors to satisfy the condition on the characters,

$$G = \text{Gen}(1) = V_E(j_{1,1}, \ldots, j_{1,r}), \ldots, \text{Gen}(C_E) = V_E(j_{C_E,1}, \ldots, j_{C_E,r}),$$

where $j_{p,q} = J(q)[p]$ for all $p = 1, \ldots, C_E$ and $q = 1, \ldots, r$. Now it is clear that we have $\text{PChar}_E(i) \subseteq \{j_{1,i}, \ldots, j_{C_E,i}\}$ for all $i = 1, \ldots, r$.

We also need to show that this choice does not affect the condition on $(d_1, \ldots, d_{r-1})$. We define $P = \left\{(q,u) \in \{1, \ldots, r\}^2 : j_{p,q} = j_{p,u} \text{ for all } p = 1, \ldots, C_E\right\}$. Thus for each $(q,u) \in P$ the previous vectors are compatible with the conditions on the element $p_{q,u}$ of $M(T)_E$.

For each $(q,u) \notin P$, we consider

$$p(q,u) = \min\{p : j_{p,q} \neq j_{p,u}\}.$$

Now, because the entries of the vectors $J(q)$ are in an increasing order, it is clear that we have

$$\text{MIN}_G(q,u) = \min\left(k_E(q,u), \min\{\min(j_{p,q}, j_{p,u}) : j_{p,q} \neq j_{p,u}\}\right) =$$

$$= \min\left(k_E(q,u), \min(j_{p(q,u),q}, j_{p(q,u),u})\right), \text{ for all } (q,u) \notin P.$$

Furthermore, for the particular choice of the vectors $\text{Gen}(i)$ and of L, it is clear that from $j_{p(q,u),q} \neq j_{p(q,u),u}$, it follows that

$$\min(j_{p(q,u),q}, j_{p(q,u),u}) \in (\text{PChar}_E(q) \cup \text{PChar}_E(u)) \setminus (\text{PChar}_E(q) \cap \text{PChar}_E(u)),$$

and from the Lemma 2.4.7, we finally have

$$\min(j_{p(q,u),q}, j_{p(q,u),u}) \geq k_E(q,u) \text{ for all } (q,u) \notin P,$$

so the vectors $\text{Gen}(i)$ are compatible with our tree.

Now from the Theorem 2.4.4 it follows that we can use $\lceil \log_2(r) \rceil$ vectors to have a solution for the vector $[d_1, \ldots, d_{r-1}]$. Adding the vectors corresponding to this solution to the previous C_E we obtain a set $G(T)$ such that $\text{Arf}(G(T)) = S(T)$. $\square$

Notice that the first C_E vectors may satisfy some conditions on the d_i, therefore it is possible to find $G(T)$ with smaller cardinality than the previous upper bound.

Remark 10. Let us consider the Arf semigroup of the Example 2.4.2.

It was $T = T_E = (2,1)$, where

$$E = \{m_1 = [5,4], m_2 = [6,4], m_3 = [2,2]\},$$

with

$$\mathrm{PChar}_E(1) = \{1,2\}, \mathrm{PChar}_E(2) = \{1,2,3\} \text{ and } \mathrm{PChar}_E(3) = \{1,3\}.$$

We found $G = \{V_E(1,1,3), V_E(2,3,2), V_E(2,2,1)\}$ as a set such that $\mathrm{Arf}(G) = S(T)$, and it is also minimal because we have $|G| = C_E$ and we clearly cannot take off any vector from it. Using the strategy of the previous corollary we would find the vectors:

$$\mathrm{Gen}(1) = V_E(1,1,1), \mathrm{Gen}(2) = V_E(2,2,3) \text{ and } \mathrm{Gen}(3) = V_E(4,3,4),$$

that satisfy the conditions on the characters ($L = 4$).

We have to add vectors that correspond to a solution for the vector $[2,1]$. For instance it suffices to consider $[3,2,1]$ and therefore we will add the vector $\mathrm{Gen}(4) = V_E(3,2,1)$. Notice how the set $G' = \{V_E(1,1,1), V_E(2,2,3), V_E(4,3,4), V_E(3,2,1)\}$, with $|G'| > |G|$, is still minimal because we cannot remove any vector from it without disrupting the condition on the tree. Therefore we can have minimal sets of generators with distinct cardinalities.

Example 2.4.9. Let us consider

$$E = \{m_1 = [4,4], m_2 = [6,4], m_3 = [2,2], m_4 = [3,2]\}.$$

We want to find a set of generators for the twisted tree T of $\tau(E)$ such that:

$$M(T)_E = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

First of all we notice that it is well defined because it satisfies the conditions given by the Remark 2 and we have

$$k(1,2) = 2, k(1,3) = 4, k(1,4) = 2, k(2,3) = 2, k(2,4) = 3 \text{ and } k(3,4) = 2.$$

We consider the permutation $\delta = (3,4)$ of S^4. Then $\delta(T)$ is an untwisted tree of $\tau(\delta(E))$ and it is described by the vector $T_{\delta(E)} = (2,3,1)$. We have:

- $\mathrm{PChar}_{\delta(E)}(1) = \{1,3\}$;
- $\mathrm{PChar}_{\delta(E)}(2) = \{1,2,3\}$;
- $\mathrm{PChar}_{\delta(E)}(3) = \{1,2\}$;

- $\mathrm{PChar}_{\delta(E)}(4) = \{1, 3\}$.

Then with the vectors $V_{\delta(E)}(1,1,1,1), V_{\delta(E)}(3,2,2,3), V_{\delta(E)}(4,3,4,4)$, we satisfy the condition on the characters. We need to add the vectors corresponding to a solution for $[2,3,1]$. It suffices to add $V_{\delta(E)}(2,4,3,1)$. Then

$$G(T) = \{[4,6,3,2], [9,10,5,5], [10,11,7,6], [8,12,6,2]\}\,,$$

is a set of generators for $\delta(T)$. Because $\delta^{-1} = (3,4)$, we have that

$$\delta^{-1}(G(T)) = \{[4,6,2,3], [9,10,5,5], [10,11,6,7], [8,12,2,6]\}$$

is a set of generators for the twisted tree T.

Chapter 3

Algorithms for Arf good semigroups

In this chapter, we present some procedures regarding the computation of sets consisting of Arf good semigroups satisfying some specific conditions.

3.1 Finding Arf semigroups with a fixed conductor

The aim of this section is to find an algorithm for computing the set of all the Arf good semigroups of $\mathbb{N}^r$ having as conductor a fixed vector $\mathbf{c} \in \mathbb{N}^r$. We will develop a procedure that works inductively on the dimension r, thus the following two subsections are dedicated to the solution of the required base cases for $r = 1$ and $r = 2$.

3.1.1 An algorithm for Cond(n) where $n \in \mathbb{N}$

In [12] it is presented an algorithm for the computation of the set of the Arf numerical semigroups with a given conductor. In this section we give a new procedure for the computation of such a set. It has already replaced the older one in the GAP package Numericalsgps of which the author is one of the contributors [9].

Now, given a multiplicity sequence $m = [m_1, \ldots, m_{l(m)}]$, it is clear that the conductor of the associated Arf numerical semigroup AS(m) is $\sum_{i=1}^{l(m)} m_i$. In particular notice that the conductor of AS$([\,]) = \mathbb{N}$ is 0.

We denote by Cond(n) the set of the multiplicity sequences of Arf numerical semigroups with conductor n, and we want to find a procedure to compute this set for all $n \in \mathbb{N}$.

If $n = 0$, then Cond$(n) = \{[\,]\}$, while if $n = 1$, Cond$(n) = \emptyset$. Thus, we suppose $n > 1$. Denote by $T^n(i) = \{m \in \text{Cond}(i) : m_1 + i \leq n\}$ for all $i = 2, \ldots, n-2$. Now suppose that $m = [m_1, \ldots, m_k] \in$ Cond(n). If $k = 1$ then $m = [n]$, otherwise we have the following situation:

- $2 \leq m_1 < n - 1$ and $[m_2, \ldots, m_k] \in$ Cond$(n - m_1)$;
- $m_1 \in$ AS$([m_2, \ldots, m_k])$;
- $m_2 - m_1 \leq 0 \Rightarrow m_2 + n - m_1 \leq n \Rightarrow [m_2, \ldots, m_k] \in T^n(n - m_1)$.

Hence, if we know $T^n(i)$ for $i = 2, \ldots, n-2$, we can compute $\mathrm{Cond}(n)$ in the following way:

$$\mathrm{Cond}(n) = \bigcup_{i=2}^{n-2} \{(n-i) :: m | m \in T^n(i), n-i \in \mathrm{AS}(m)\} \cup \{[n]\},$$

where we denote by $(n-i) :: m$ the list obtained by appending $n-i$ at the beginning of m. Now we need a way to compute $T^n(i)$. Suppose that $m = [m_1, \ldots, m_k] \in T^n(i)$. If $k = 1$, and $2 \cdot i \leq n$ then $m = [i]$, otherwise we have the following situation:

- $2 \leq m_1 < i - 1$ and $[m_2, \ldots, m_k] \in \mathrm{Cond}(i - m_1) = \mathrm{Cond}(q)$;
- $q + m_2 = i - m_1 + m_2 \leq i \leq n \Rightarrow [m_2, \ldots, m_k] \in T^n(q)$;
- $m_1 \in \mathrm{AS}([m_2, \ldots, m_k])$;
- $m_1 + i \leq n \Rightarrow 2m_1 \leq n + m_1 - i \Rightarrow 2m_1 \leq n - q \Rightarrow m_1 \leq \left\lfloor \frac{n-q}{2} \right\rfloor$.

So each $T^n(i)$ can be constructed using $T^n(q)$ with $2 \leq q < i$. Thus, we have the following algorithm for $\mathrm{Cond}(n)$ for $n > 1$.

input : An integer $n > 1$
output: The set $\mathrm{Cond}(n)$ of all the multiplicity sequences of Arf semigroups with conductor n

$\mathrm{Cond}(n) \longleftarrow \{[n]\}$
for $i \leftarrow 2$ **to** $n - 2$ **do**
    if $i \leq \left\lfloor \frac{n}{2} \right\rfloor$ **then**
        $T^n(i) \longleftarrow \{[i]\}$
    end
    else
        $T^n(i) \longleftarrow \emptyset$
    end
end
for $i \leftarrow 2$ **to** $n - 2$ **do**
    for $m \in T^n(i)$ **do**
        if $n - i \in AS(m)$ **then**
            $\mathrm{Cond}(n) \longleftarrow \mathrm{Cond}(n) \cup \{(n-i) :: m\}$
        end
        for $k \in AS(m) \cap \left\{2, \ldots, \left\lfloor \frac{n-i}{2} \right\rfloor\right\}$ **do**
            $T^n(i+k) \longleftarrow T^n(i+k) \cup \{(k) :: m\}$
        end
    end
end
$\mathrm{Cond}(n)$

Algorithm 1:

3.1.2 Arf good semigroups of $\mathbb{N}^2$ with given conductor

From this section we begin to deal with Arf good semigroups of $\mathbb{N}^r$. The aim of this and the following section is to find a procedure that lets us to determine all the local Arf semigroups $S \subseteq \mathbb{N}^r$ with a given conductor $\mathbf{c} \in \mathbb{N}^r$. For the Remark 4, we can focus only on the untwisted trees.

We denote by $\text{Cond}(\mathbf{c})$ the set of all the untwisted multiplicity trees of Arf semigroups in $\mathbb{N}^r$ with conductor $\mathbf{c} \in \mathbb{N}^r$ (in the case $r = 1$ we have the multiplicity sequences and from the previous section we have a procedure to determine such a set).

We notice the following general fact.

Proposition 3.1.1. *Let S be an Arf semigroup of $\mathbb{N}^r$, T the corresponding multiplicity tree and m_i for $i = 1, \dots, r$ the multiplicity sequences of its branches.*

We introduce the following integers

$$d(i) = \min \{j \in \mathbb{N} : m_i[j] = 1 \textit{ and the } i\textit{-th branch is not glued to other branches at level } j\},$$

for $i = 1, \dots, r$.

Then $\mathbf{c} = (c[1], \dots, c[r])$ is the conductor of S where

$$c[i] = \sum_{k=1}^{d(i)-1} m_i[k] \qquad \textit{for} \quad i = 1, \dots, r.$$

Proof. Denote by $N(T) = \left\{\mathbf{n}_i^j\right\}$ the set of the nodes of T. We call $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the non zero coordinate is in the i-th position. Now, from the definition of the integers $d(i)$, it follows that

- $\mathbf{n}_i^{d(i)} = e_i$ for all $i = 1, \dots, r$;
- $\mathbf{n}_i^{d(i)-1} \neq e_i$ for all $i = 1, \dots, r$.

We consider the subtree T' of T such that $N(T') = \left\{\mathbf{n}_i^{j(i)} : i = 1, \dots, r; j(i) = 1, \dots, d(i) - 1\right\}$. Then we have

- T' is rooted in $\mathbf{n}_1^1$ (it corresponds to an element of the associated Arf good semigroup);
- $e_i \notin N(T')$ for all $i = 1, \dots, r$;
- If T'' is such that $T' \subseteq T'' \subseteq T$ then $N(T'') \setminus N(T')$ consists only of nodes of the type e_i.

From the previous properties it is clear that the element corresponding to the subtree T' must be the conductor of the Arf semigroup associated to T. It is also trivial that the sum of all the elements of $N(T')$ is equal to $\mathbf{c} = (c[1], \dots, c[r])$ where

$$c[i] = \sum_{k=1}^{d(i)-1} m_i[k] \qquad \text{for} \quad i = 1, \dots, r.$$

□

Remark 11. Using the notations of Proposition 3.1.1, given an untwisted tree $T_E = (p_1, \ldots, p_{r-1})$, where $E = \{m_1, \ldots, m_r\}$, it is easy to show that

$$d(i) - 1 = \max(l(m_i), p_i, p_{i-1}) \text{ for } i = 2, \ldots, r-1$$
$$d(1) - 1 = \max(l(m_1), p_1) \text{ and } d(r) - 1 = \max(l(m_r), p_{r-1}).$$

Now, we focus on the case $r = 2$ and we determine a procedure to compute $\mathrm{Cond}(\mathbf{c})$ where $\mathbf{c}$ is a fixed arbitrary vector $(c[1], c[2])$. Suppose that $T_E = (p) \in \mathrm{Cond}(\mathbf{c})$ where $E = \{m_1, m_2\}$. From the previous remark $d(1)-1 = \max(l(m_1), p)$ and $d(2)-1 = \max(l(m_2), p)$. Furthermore, in the following, we will call *compatibility* between the multiplicity sequences m_1, m_2, denoted by $\mathrm{Comp}(m_1, m_2)$, the integer $k_E(1, 2)$, defined in Proposition 2.1.2, where $E = \{m_1, m_2\}$.

We have the following cases:

• Case $d(1) - 1 = l(m_1)$ and $d(2) - 1 = l(m_2)$.

We have $p \leq \min(l(m_1), l(m_2))$. Furthermore, we have $p \leq \mathrm{Comp}(m_1, m_2)$ because T is well defined. Since $T_E = (p) \in \mathrm{Cond}(\mathbf{c})$ we have:

$$c[1] = \sum_{k=1}^{d(1)-1} m_1[k] = \sum_{k=1}^{l(m_1)} m_1[k] \text{ and } c[2] = \sum_{k=1}^{d(2)-1} m_2[k] = \sum_{k=1}^{l(m_2)} m_2[k],$$

and from it we deduce that $m_1 \in \mathrm{Cond}(c[1])$ and $m_2 \in \mathrm{Cond}(c[2])$. So in this case T belongs to the following set:

$$S^1(\mathbf{c}) = \{T_E = (k) : E = \{m_1, m_2\}\, ; m_i \in \mathrm{Cond}(c[i]) \text{ and}$$
$$1 \leq k \leq \min(\mathrm{Comp}(m_1, m_2), l(m_1), l(m_2))\}\,.$$

Also, we can notice that $S^1(\mathbf{c}) \subseteq \mathrm{Cond}(\mathbf{c})$ (using the inverse implications).

• Case $d(1) - 1 = l(m_1)$ and $d(2) - 1 \neq l(m_2)$.

Hence $d(2) - 1 = p$ and $l(m_2) < p \leq \min(l(m_1), \mathrm{Comp}(m_1, m_2))$. Therefore:

$$c[1] = \sum_{k=1}^{l(m_1)} m_1[k]\, ,\, c[2] = \sum_{k=1}^{p} m_2[k] = \sum_{k=1}^{l(m_2)} m_2[k] + \sum_{k=l(m_2)+1}^{p} m_2[k] = \sum_{k=1}^{l(m_2)} m_2[k] + p - l(m_2),$$

and from this we can deduce $m_1 \in \mathrm{Cond}(c[1])$ and, denoted by $k_2 = c[2] - (p - l(m_2))$, $m_2 \in \mathrm{Cond}(k_2)$. Notice that $0 \leq k_2 < c[2]$.

Now, for all $0 \leq k < c[2]$ we define the set:

$$I_1(k) = \{T_E = (p) : E = \{m_1, m_2\}\, , m_1 \in \mathrm{Cond}(c[1]), m_2 \in \mathrm{Cond}(k) \text{ and}$$
$$p = l(m_2) + c[2] - k \leq \min(l(m_1), \mathrm{Comp}(m_1, m_2))\}\,.$$

Thus T belongs to

$$S_1^2(\mathrm{c}) = \bigcup_{k=0}^{c[2]-1} I_1(k).$$

With the inverse implication we can easily show that $S_1^2(\mathbf{c}) \subseteq \text{Cond}(\mathbf{c})$.
• Case $d(1) - 1 \neq l(m_1)$ e $d(2) - 1 = l(m_2)$.
We have $d(1) - 1 = p$ and $l(m_1) < p \leq \min(l(m_2), \text{Comp}(m_1, m_2))$. Hence:

$$c[1] = \sum_{k=1}^{p} m_1[k] = \sum_{k=1}^{l(m_1)} m_1[k] + \sum_{k=l(m_1)+1}^{p} m_1[k] = \sum_{k=1}^{l(m_1)} m_1[k] + p - l(m_1);\, c[2] = \sum_{k=1}^{l(m_2)} m_2[k],$$

and from this we obtain $m_2 \in \text{Cond}(c[2])$ and, setting $k_1 = c[1] - (p - l(m_1))$, we deduce $m_1 \in \text{Cond}(k_1)$. Notice that $0 \leq k_1 < c[1]$.
For all $0 \leq k < c[1]$ we define the set:

$$I_2(k) = \{T_E = (p) : E = \{m_1, m_2\}, m_1 \in \text{Cond}(k), m_2 \in \text{Cond}(c[2]) \text{ and } p = l(m_1) + c[1] - k \leq \min(l(m_2), \text{Comp}(m_1, m_2))\}.$$

Therefore T belongs to

$$S_2^2(\mathbf{c}) = \bigcup_{k=0}^{c[1]-1} I_2(k).$$

With the inverse implication we can easily show that $S_2^2(\mathbf{c}) \subseteq \text{Cond}(\mathbf{c})$.
• Case $d(1) - 1 \neq l(m_1)$ and $d(2) - 1 \neq l(m_2)$.
Then $d(2) - 1 = p$, $d(1) - 1 = p$ and we have $\max(l(m_1), l(m_2)) < p \leq \text{Comp}(m_1, m_2)$. It follows:

$$c[1] = \sum_{k=1}^{p} m_1[k] = \sum_{k=1}^{l(m_1)} m_1[k] + \sum_{k=l(m_1)+1}^{p} m_1[k] = \sum_{k=1}^{l(m_1)} m_1[k] + p - l(m_1)$$

$$c[2] = \sum_{k=1}^{p} m_2[k] = \sum_{k=1}^{l(m_2)} m_2[k] + \sum_{k=l(m_2)+1}^{p} m_2[k] = \sum_{k=1}^{l(m_2)} m_2[k] + p - l(m_2).$$

If we denote by $k_1 = c[1] - (p - l(m_1))$ and by $k_2 = c[2] - (p - l(m_2))$, we have $m_1 \in \text{Cond}(k_1)$ and $m_2 \in \text{Cond}(k_2)$.
Furthermore, notice that $0 \leq k_1 < c[1]$ and $0 \leq k_2 < c[2]$. Now, for all $0 \leq k_1 < c[1]$ and $0 \leq k_2 < c[2]$ we define the set:

$$I(k_1, k_2) = \{T_E = (p) : E = \{m_1, m_2\}, m_i \in \text{Cond}(k_i) \text{ for } i = 1, 2 \text{ and } p = l(m_1) + c[1] - k_1 = l(m_2) + c[2] - k_2 \leq \text{Comp}(m_1, m_2)\}.$$

So T belongs to

$$S^3(\mathbf{c}) = \bigcup_{0 \leq k_i < c[i]} I(k_1, k_2) \subseteq \text{Cond}(\mathbf{c}).$$

Even in this case we can show that $S^3(\mathbf{c}) \subseteq \text{Cond}(\mathbf{c})$.

We have studied all the possible cases so we proved

$$S^1(\mathbf{c}) \cup S^2_1(\mathbf{c}) \cup S^2_2(\mathbf{c}) \cup S^3(\mathbf{c}) = \text{Cond}(\mathbf{c}).$$

All the previous set can be computed by using the procedure given in the case $r = 1$ so we have found a procedure to compute $\text{Cond}(\mathbf{c})$ when $r = 2$.

Example 3.1.2. Let us compute $\text{Cond}([4, 5])$.

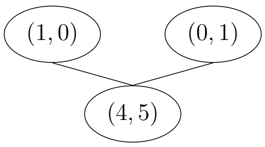
First of all we compute $S^1([4, 5])$. We need $\text{Cond}(4)$ and $\text{Cond}(5)$. They are:

$$\text{Cond}(4) = \{[4], [2, 2]\} \text{ and } \text{Cond}(5) = \{[5], [3, 2]\}.$$

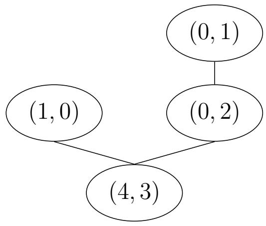
Hence when we compute $S^1([4, 5])$ we find:

- $E_1 = \{m_1 = [4], m_2 = [5]\}$. Thus $\text{Comp}(m_1, m_2) = 5$ and $\min(l(m_1), l(m_2)) = 1$. Then we have only the tree $T_1 = T_{E_1} = (1)$.
- $E_2 = \{m_1 = [4], m_2 = [3, 2]\}$. Thus $\text{Comp}(m_1, m_2) = 3$ and $\min(l(m_1), l(m_2)) = 1$. Then we have only the tree $T_2 = T_{E_2} = (1)$.
- $E_3 = \{m_1 = [2, 2], m_2 = [5]\}$. Thus $\text{Comp}(m_1, m_2) = 2$ and $\min(l(m_1), l(m_2)) = 1$. Then we have only the tree $T_3 = T_{E_3} = (1)$.
- $E_4 = \{m_1 = [2, 2], m_2 = [3, 2]\}$. We have $\text{Comp}(m_1, m_2) = 2$ and $\min(l(m_1), l(m_2)) = 2$. So we have the trees $T_4 = T_{E_4} = (1)$ and $T_5 = T_{E_4} = (2)$.

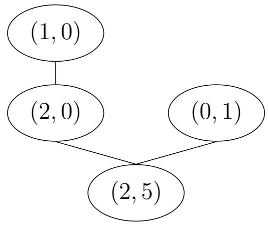
Hence $S^1([4, 5]) = \{T_1, T_2, T_3, T_4, T_5\}$.



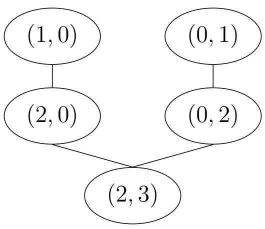
T_1



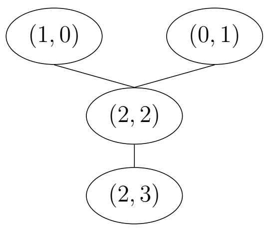
T_2



T_3



T_4



T_5

Now we compute $S^2_1([4, 5])$. The only value k such that $I_1(k) \neq \emptyset$ is $k = 4$ and we have:

- $c[1] = 4$, $k = 4$. If we consider $m_1 = [2,2] \in \mathrm{Cond}(4)$, $m_2 = [4] \in \mathrm{Cond}(4)$ and $E_5 = \{m_1, m_2\}$ we have

$$l(m_2) + c[2] - k = 2 \leq \min(l(m_1), \mathrm{Comp}(m_1, m_2)) = \min(2,2) = 2.$$

  Hence we have the tree $T_6 = T_{E_5} = (2)$.

Therefore $S_1^2([4,5]) = \{T_6\}$. Let us compute $S_2^2([4,5])$. The only value k such that $I_2(k) \neq \emptyset$ is $k = 3$:

- $k = 3$, $c[2] = 5$. If we consider $m_1 = [3] \in \mathrm{Cond}(3)$, $m_2 = [3,2] \in \mathrm{Cond}(5)$ and $E_6 = \{m_1, m_2\}$ we have

$$l(m_1) + c[1] - k = 1 + 4 - 3 = 2 \leq \min(l(m_2), \mathrm{Comp}(m_1, m_2) = \min(2,3) = 2.$$

  Hence we obtain the tree $T_7 = T_{E_6} = (2)$.

Therefore $S_2^2([4,5]) = \{T_7\}$.

We finally compute $S^3([4,5])$.

The only values of k_1 and k_2 such that $I(k_1, k_2) \neq \emptyset$ are the following:

- $k_1 = 2$, $k_2 = 3$. If we consider $m_1 = [2] \in \mathrm{Cond}(2)$, $m_2 = [3] \in \mathrm{Cond}(3)$ and $E_7 = \{m_1, m_2\}$ we have

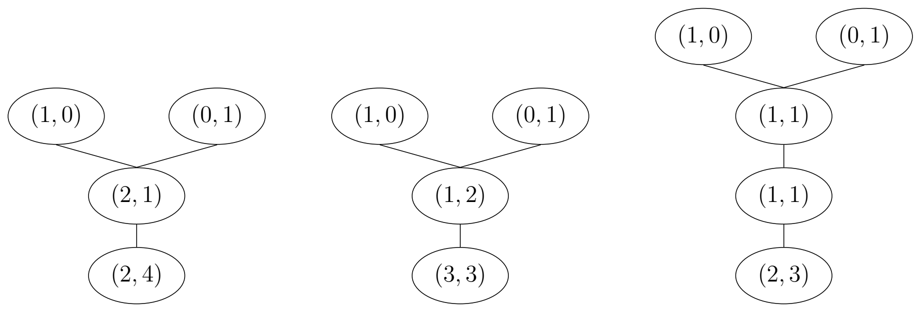
$$l(m_1) + c[1] - k_1 = 1 + 4 - 2 = 3 = 1 + 5 - 3 = l(m_2) + c[2] - k_2 \leq \mathrm{Comp}(m_1, m_2) = 3.$$

  Thus we have the tree $T_8 = T_{E_7} = (3)$.

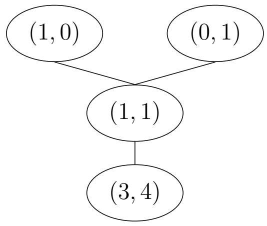
- $k_1 = 3$, $k_2 = 4$. If we consider $m_1 = [3] \in \mathrm{Cond}(3)$, $m_2 = [4] \in \mathrm{Cond}(4)$ and $E_8 = \{m_1, m_2\}$ we have

$$l(m_1) + c[1] - k_1 = 1 + 4 - 3 = 2 = 1 + 5 - 4 = l(m_2) + c[2] - k_2 \leq \mathrm{Comp}(m_1, m_2) = 4.$$

  Thus we get the tree $T_9 = T_{E_8} = (2)$. Hence $S^3([4,5]) = \{T_8, T_9\}$.



T_6 T_7 T_8



T_9

Summarizing, we have $\mathrm{Cond}([4,5]) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9\}$.

Example 3.1.3. Using the previous results it is easy to implement an algorithm that computes the number of Arf semigroups of $\mathbb{N}^2$ with a given conductor. Each entry of the following table is such a number, where the conductors range from $(1,1)$ to $(20,20)$.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	1	1	1	2	2	4	3	7	6	10	9	17	12	25	20	32	27	49	34	68
2	1	2	2	4	4	8	6	14	12	20	18	34	24	50	40	64	54	98	68	136
3	1	2	3	4	5	9	7	16	14	22	21	39	26	57	46	71	60	111	75	155
4	2	4	4	10	9	18	15	33	28	49	43	81	59	120	96	156	131	236	167	328
5	2	4	5	9	12	19	15	34	32	51	45	86	62	128	102	161	139	250	172	347
6	4	8	9	18	19	41	30	68	60	99	92	171	122	252	201	326	275	497	344	687
7	3	6	7	15	15	30	30	54	48	80	74	134	104	204	163	264	221	399	285	556
8	7	14	16	33	34	68	54	129	108	180	164	306	222	453	371	593	499	901	632	1251
9	6	12	14	28	32	60	48	108	108	160	147	271	202	404	330	522	459	809	566	1120
10	10	20	22	49	51	99	80	180	160	284	242	454	337	676	545	878	748	1336	961	1867
11	9	18	21	43	45	92	74	164	147	242	245	412	307	611	502	798	685	1215	868	1688
12	17	34	39	81	86	171	134	306	271	454	412	798	567	1148	927	1492	1273	2277	1608	3159
13	12	24	26	59	62	122	104	222	202	337	307	567	469	849	694	1115	961	1689	1224	2347
14	25	50	57	120	128	252	204	453	404	676	611	1148	849	1750	1383	2224	1897	3389	2403	4710
15	20	40	46	96	102	201	163	371	330	545	502	927	694	1383	1192	1805	1556	2753	1976	3822
16	32	64	71	156	161	326	264	593	522	878	798	1492	1115	2224	1805	2992	2493	4433	3174	6155
17	27	54	60	131	139	275	221	499	459	748	685	1273	961	1897	1556	2493	2244	3798	2734	5266
18	49	98	111	236	250	497	399	901	809	1336	1215	2277	1689	3389	2753	4433	3798	6867	4814	9394
19	34	68	75	167	172	344	285	632	566	961	868	1608	1224	2403	1976	3174	2734	4814	3634	6701
20	68	136	155	328	347	687	556	1251	1120	1867	1688	3159	2347	4710	3822	6155	5266	9394	6701	13219

3.1.3 Arf semigroups of $\mathbb{N}^r$ with a given conductor

In this section we study the general case. We want to develop a recursive procedure to calculate $\mathrm{Cond}(\mathbf{c})$ for $\mathbf{c} \in \mathbb{N}^r$, using the fact that we already know how to solve the base cases $r = 1$ and $r = 2$. In order to do that is very useful the following Lemma.

Lemma 3.1.4. *Consider $\boldsymbol{c} = (c[1], \ldots, c[r]) \in \mathbb{N}^r$, with $r \geq 3$ and suppose that the untwisted tree $T = T_E = (p_1, \ldots, p_{r-1}) \in$ Cond($\boldsymbol{c}$), where $E = \{m_1, \ldots, m_r\}$. If $t \in \{2, \ldots, r-1\}$, then we have that at least one of the following conditions must hold:*

- $T_1 = T_{E_1} = (p_1, \ldots, p_{t-1}) \in$ ***Cond***$((c[1], \ldots, c[t]))$ *with* $E_1 = \{m_1, \ldots, m_t\}$;
- $T_2 = T_{E_2} = (p_t, \ldots, p_{r-1}) \in$ ***Cond***$((c[t], \ldots, c[r]))$ *with* $E_2 = \{m_t, \ldots, m_r\}$.

Proof. We assume by contradiction that

- $T_1 \notin \text{Cond}((c[1], \ldots, c[t]))$;
- $T_2 \notin \text{Cond}((c[t], \ldots, c[r]))$.

Let us consider the following integers (which are clearly linked to the conductor):
$d_1(i) = \min\{j \in \mathbb{N} : m_i[j] = 1$ and the i-th branch in T_1 is not glued to any branches at level $j\}$, for $i = 1, \ldots, t$.
$d_2(i) = \min\{j \in \mathbb{N} : m_i[j] = 1$ and the i-th branch in T_2 is not glued to any branches at level $j\}$, for $i = t, \ldots, r$.
We have $d_1(l) = d(l)$, for all $l = 1, \ldots, t-1$, and $d_2(q) = d(q)$, for all $q = t+1, \ldots, r$. Furthermore, $d_1(t) \leq d(t)$ and $d_2(t) \leq d(t)$. In fact we have noticed that $d(t) - 1 = \max(l(m_t), p_{t-1}, p_t)$, while $d_1(t) - 1 = \max(l(m_t), p_{t-1})$ and $d_2(t) - 1 = \max(l(m_t), p_t)$.

From $T \in \text{Cond}(\mathbf{c})$ we deduce that

$$\sum_{k=1}^{d(i)-1} m_i[k] = c[i] \text{ for } i = 1, \ldots, r.$$

We denote respectively by $(c_1[1], \ldots, c_1[t])$ and by $(c_2[t], \ldots, c_2[r]))$ the conductors of T_1 and T_2.

We have:

$$c_1[l] = \sum_{k=1}^{d_1(l)-1} m_l[k] = \sum_{k=1}^{d(l)-1} m_l[k] = c[l] \text{ for } l = 1, \ldots, t-1$$

$$\text{and } c_2[q] = \sum_{k=1}^{d_2(q)-1} m_q[k] = \sum_{k=1}^{d(q)-1} m_q[k] = c[q], \text{ for } q = t+1, \ldots, r$$

and this implies, because $T_1 \notin \text{Cond}((c[1], \ldots, c[t]))$ and $T_2 \notin \text{Cond}((c[t], \ldots, c[r]))$, that

$$c_1[t] = \sum_{k=1}^{d_1(t)-1} m_t[k] \neq c[t] \text{ and } c_2[t] = \sum_{k=1}^{d_2(t)-1} m_t[k] \neq c[t],$$

and therefore we have $d_1(t) < d(t)$ and $d_2(t) < d(t)$.

From this it would follow

$$\begin{aligned} d(t) - 1 &= \max(l(m_t), p_{t-1}, p_t) = \max(\max(l(m_t), p_{t-1}), \max(l(m_t), p_t))) = \\ &= \max(d_1(t) - 1, d_2(t) - 1) < d(t) - 1, \end{aligned}$$

and we obtain a contradiction. $\square$

Now, using this Lemma, we can introduce an algorithm that solves our problem working inductively. Given $\mathbf{c} \in \mathbb{N}^r$, with $r \geq 3$, we want to compute Cond($\mathbf{c}$). We suppose that we are able to solve the problem for all $s < r$ and we develop a strategy for the r case.

Let us fix some notations. If $k = 2, \ldots, r-1$, we denote by $\mathbf{c}_k = (c[1], \ldots, c[k])$ and by $\mathbf{c}^k = (c[k+1], \ldots, c[r])$. Similarly, if $E = \{m_1, \ldots, m_r\}$, we denote by $E_k = \{m_1, \ldots, m_k\}$ and by $E^k = \{m_{k+1}, \ldots, m_r\}$. Furthermore, for $i = 1, \ldots, r-1$, we define the integers ${}^*p_i = \max(l(m_i), p_{i-1})$ and $p_i^* = \max(l(m_{i+1}), p_{i+1})$, where, by definition, we set $p_{r-1}^* = l(m_r)$ and ${}^*p_1 = l(m_1)$.

Fixed $\mathbf{c} \in \mathbb{N}^r$, we suppose to have a tree $T = T_E = (p_1, \ldots, p_{r-1}) \in$ Cond($\mathbf{c}$) with $E = \{m_1, \ldots, m_r\}$. Consider $t \in \{2, \ldots, r-1\}$. It follows from Lemma 3.1.4 that we only have two cases:

• Case $T_1 = T_{E_t} = (p_1, \ldots, p_{t-1}) \in$ Cond($\mathbf{c}_t$).

We clearly have $d_1(i) = d(i)$ for all $i = 1, \ldots, t-1$, while from $T_1 \in$ Cond($\mathbf{c}_t$) it follows that $d_1(t) = d(t)$. Hence:

$${}^*p_t = \max(l(m_t), p_{t-1}) = d_1(t) - 1 = d(t) - 1 = \max(p_t, p_{t-1}, l(m_t)) = \max({}^*p_t, p_t)$$

and we deduce that $p_t \leq {}^*p_t$.

We consider the tree $T_2 = T_{E^t} = (p_{t+1}, \ldots, p_{r-1})$, (if $t = r-1$ we have $T_2 = m_r$). We clearly have $d_2(i) = d(i)$ for all $i = t+2, \ldots, r$.

On the other hand $d_2(t+1) - 1 = \max(l(m_{t+1}), p_{t+1}) = p_t^*$ may be different from

$$d(t+1) - 1 = \max(l(m_{t+1}, p_{t+1}, p_t) = \max(p_t^*, p_t).$$

Hence we have the following two subcases:

► Subcase $d_2(t+1) = d(t+1)$.

In this case we have $T_2 \in$ Cond($\mathbf{c}^t$) and $p_t \leq p_t^*$ (if $t = r-1$ we have $T_2 = m_r \in$ Cond($c[r]$)). We also recall that we must have the compatibility condition $p_t \leq$ Comp(m_t, m_{t+1}).

Thus we have discovered that T belongs to the following set:

$$S_1^1(\mathbf{c}) = \{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\, ; T_{E_t} = (p_1, \ldots, p_{t-1}) \in \text{Cond}(\mathbf{c}_t);$$

$$T_{E^t} = (p_{t+1}, \ldots, p_{r-1}) \in \text{Cond}(\mathbf{c}^t) \text{ with } 1 \leq p_t \leq \min({}^*p_t, p_t^*, \text{Comp}(m_t, m_{t+1}))\}\,.$$

It is very easy to check that we also have $S_1^1(\mathbf{c}) \subseteq$ Cond($\mathbf{c}$). If $t = r-1$ the previous set has the following definition:

$$S_1^1(\mathbf{c}) = \{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\, ;\; T_{E_{r-1}} = (p_1, \ldots, p_{r-2}) \in \text{Cond}(\mathbf{c}_{r-1});$$

$$m_r \in \text{Cond}(c[r]); 1 \leq p_{r-1} \leq \min({}^*p_{r-1}, l(m_r), \text{Comp}(m_{r-1}, m_r))\}\,.$$

► Subcase $d_2(t+1) \neq d(t+1)$.

In this case we have

$$p_t = d(t+1) - 1 > d_2(t+1) - 1 = \max(l(m_{t+1}), p_{t+1}) = p_t^*.$$

Hence

$$c[t+1] = \sum_{k=1}^{d(t+1)-1} m_{t+1}[k] = \sum_{k=1}^{p_t} m_{t+1}[k] = \sum_{k=1}^{p_t^*} m_{t+1}[k] + p_t - p_t^*,$$

and from this it follows that $T_2 \in \text{Cond}((k[t+1], c[t+2], \ldots, c[r]))$, where

$$\sum_{k=1}^{p_t^*} m_{t+1}[k] = \sum_{k=1}^{d_2(t+1)-1} m_{t+1}[k] = k_{t+1} < c[t+1],$$

and we have $T_2 \in \text{Cond}(k_{t+1})$ in the case $t = r-1$. Thus we have $k_{t+1} = c[t+1] - (p_t - p_t^*)$. Notice that k_{t+1} cannot be equal to zero when $t \neq r-1$ (because $p_t^* \geq 1$), while it can be zero in the case $t = r-1$. Then, for all the $k_{t+1} \in \mathbb{N}$ such that $\min(r-1-t, 1) < k_{t+1} < c[t+1]$ we define the set:

$$I_1(k_{t+1}) = \{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\, ; T_{E_t} = (p_1, \ldots, p_{t-1}) \in \text{Cond}(\mathbf{c}_t);$$
$$T_{E^t} = (p_{t+1}, \ldots, p_{r-1}) \in \text{Cond}((k_{t+1}, c[t+2], \ldots, c[r]));$$
$$p_t = p_t^* + c[t+1] - k_{t+1} \leq \min({}^*p_t, \text{Comp}(m_t, m_{t+1}))\}\,.$$

Hence T belongs to the following set:

$$S_1^2(\mathbf{c}) = \bigcup_{k_{t+1}=\min(r-1-t,1)}^{c[t+1]-1} I_1(k_{t+1}),$$

and it is clear that $S_1^2(\mathbf{c}) \subseteq \text{Cond}(\mathbf{c})$.

If $t = r-1$ the previous set has the following definition:

$$I_1(k_r) = \big\{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\, ; T_{E_{r-1}} = (p_1, \ldots, p_{r-2}) \in \text{Cond}(\mathbf{c}_{r-1});$$
$$m_r \in \text{Cond}(k_r); p_{r-1} = l(m_r) + c[r] - k_r \leq \min({}^*p_{r-1}, \text{Comp}(m_{r-1}, m_r))\}\,.$$

• Case $T_2 = T_{E^{t-1}} = (p_t, \ldots, p_{r-1}) \in \text{Cond}(\mathbf{c}^{t-1})$.

We only have to adapt the considerations made in the previous case to this case. Thus we directly give the sets that arise without further justifications.

► If $t \neq 2$,

$$S_2^1(\mathbf{c}) = \big\{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\, ; T_{E^{t-1}} = (p_t, \ldots, p_{r-1}) \in \text{Cond}(\mathbf{c}^{t-1});$$

$$T_{E_{t-1}} = (p_1, \ldots, p_{t-2}) \in \text{Cond}(\mathbf{c}_{t-1}) \text{ con } 1 \leq p_{t-1} \leq \min({}^*p_{t-1}, p_{t-1}^*, \text{Comp}(m_t, m_{t-1}))\big\}\,.$$

▶ If $t = 2$,

$$S_2^1(\mathbf{c}) = \{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\,;\; T_{E^1} = (p_2, \ldots, p_{r-1}) \in \text{Cond}(\mathbf{c}^1);$$

$$m_1 \in \text{Cond}(c[1]); 1 \leq p_1 \leq \min(l(m_1), p_1^*, \text{Comp}(m_1, m_2))\}\,.$$

We have $S_2^1(\mathbf{c}) \subseteq \text{Cond}(\mathbf{c})$.

For all $k_{t-1} \in \mathbb{N}$ such that $\min(t-2, 1) \leq k_{t-1} < c[t-1]$ we consider:

▶ If $t \neq 2$,

$$I_2(k_{t-1}) = \left\{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\,; T_{E^{t-1}} = (p_t, \ldots, p_{r-1}) \in \text{Cond}(\mathbf{c}^{t-1});\right.$$

$$T_{E_{t-1}} = (p_1, \ldots, p_{t-2}) \in \text{Cond}((c[1], \ldots, c[t-2], k_{t-1}));$$

$$\left. p_{t-1} = {}^*p_{t-1} + c[t-1] - k_{t-1} \leq \min(p_{t-1}^*, \text{Comp}(m_{t-1}, m_t))\right\}\,.$$

▶ If $t = 2$,

$$I_2(k_1) = \left\{T_E = (p_1, \ldots, p_{r-1}) : E = \{m_1, \ldots, m_r\}\,; T_{E^1} = (p_2, \ldots, p_{r-1}) \in \text{Cond}(\mathbf{c}^1)\right.$$

$$\left. m_1 \in \text{Cond}(k_1); p_1 = l(m_1) + c[1] - k_1 \leq \min(p_1^*, \text{Comp}(m_1, m_2))\right\}\,.$$

We have that:

$$S_2^2(\mathbf{c}) = \bigcup_{k_{t-1}=\min(t-2,1)}^{c[t-1]-1} I_2(k_{t-1}) \subseteq \text{Cond}(\mathbf{c}).$$

The previous lemma ensures that we have considered all the possibilities. So we proved that

$$\text{Cond}(\mathbf{c}) \subseteq S_1^1(\mathbf{c}) \cup S_1^2(\mathbf{c}) \cup S_2^1(\mathbf{c}) \cup S_2^2(\mathbf{c}),$$

hence

$$S_1^1(\mathbf{c}) \cup S_1^2(\mathbf{c}) \cup S_2^1(\mathbf{c}) \cup S_2^2(\mathbf{c}) = \text{Cond}(\mathbf{c}).$$

Due to our induction hypothesis all the previous sets can be computed so we developed an algorithm which computes $\text{Cond}(\mathbf{c})$.

Now we have a way to compute all the untwisted multiplicity trees with a given conductor $\mathbf{c}$ for all the $\mathbf{c} \in \mathbb{N}^r$. Suppose that we want to find also the twisted multiplicity trees with conductor $\mathbf{c}$. We will call $\overline{\text{Cond}(\mathbf{c})}$ the set of all multiplicity trees (twisted or untwisted) associated to an Arf semigroup with conductor $\mathbf{c}$. Suppose that T is a twisted tree in $\overline{\text{Cond}(\mathbf{c})}$ with $\mathbf{c} \in \mathbb{N}^r$. Then there exists a permutation $\sigma \in S^r$, where S^r is the symmetric group, such that $\sigma(T)$ is untwisted and it clearly belongs to $\text{Cond}(\sigma(\mathbf{c}))$. From this it follows that:

$$\overline{\text{Cond}(\mathbf{c})} = \bigcup_{\sigma \in S^r} \left\{\sigma^{-1}(T) : T \in \text{Cond}(\sigma(\mathbf{c}))\right\}.$$

Example 3.1.5. Let us compute Cond$([3, 2, 4])$. In this case $r = 3$, therefore we have $t = 2$. First of all we compute $S_1^1([3, 2, 4])$. Because $t = r - 1$ the definition of this set is:

$$S_1^1([3, 2, 4]) = \{T_E = (p_1, p_2) : E = \{m_1, m_2, m_3\}\,;\ T_{E_2} = (p_1) \in \text{Cond}([3, 2]);$$
$$m_3 \in \text{Cond}(4); 1 \le p_2 \le \min(l(m_3), {}^*p_2, \text{Comp}(m_2, m_3))\}\,.$$

Then to do that we need the follwing sets:

- Cond$([3, 2]) = \{A_1, A_2\}$ where $A_1 = T_{F_1} = (1)$ and $A_2 = T_{F_2} = (2)$ with $F_1 = \{[3], [2]\}$ and $F_2 = \{[2], [\ ]\}$.
- Cond$(4) = \{[2, 2], [4]\}$.

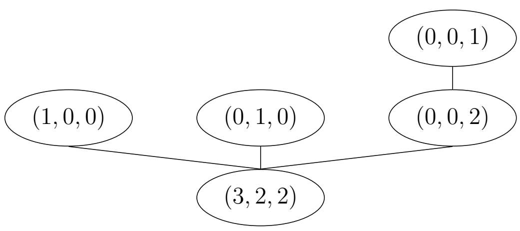
Hence we consider:

- $E_1 = \{m_1 = [3], m_2 = [2], m_3 = [2, 2]\}$ and we have $\min(\max(l(m_2), p_1), \text{Comp}(m_2, m_3), l(m_3)) = \min(1, 2, 2) = 1$. Thus we only have the tree $T_1 = T_{E_1} = (1, 1)$.
- $E_2 = \{m_1 = [3], m_2 = [2], m_3 = [4]\}$ and we have $\min(\max(l(m_2), p_1), \text{Comp}(m_2, m_3), l(m_3)) = \min(1, 3, 1) = 1$. Thus we only have the tree $T_2 = T_{E_2} = (1, 1)$.
- $E_3 = \{m_1 = [2], m_2 = [\ ], m_3 = [2, 2]\}$ and we have $\min(\max(l(m_2), p_1), \text{Comp}(m_2, m_3), l(m_3)) = \min(2, 2, 2) = 2$. Thus we have the trees $T_3 = T_{E_3} = (2, 1)$ and $T_4 = T_{E_3} = (2, 2)$.
- $E_4 = \{m_1 = [2], m_2 = [\ ], m_3 = [4]\}$ and we have $\min(\max(l(m_2), p_1), \text{Comp}(m_2, m_3), l(m_3)) = \min(2, 2, 1) = 1$. Thus we only have the tree $T_5 = T_{E_4} = (2, 1)$.

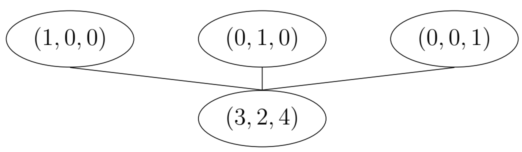
Hence $S_1^1([3, 2, 4]) = \{T_1, T_2, T_3, T_4, T_5\}$.

Now we compute $S_1^2([3, 2, 4])$. We find $k_3 = 3$ as the only value such that $I(k_3) \neq \emptyset$. In fact, if we consider A_2 and $m_3 = [3]$, we have:

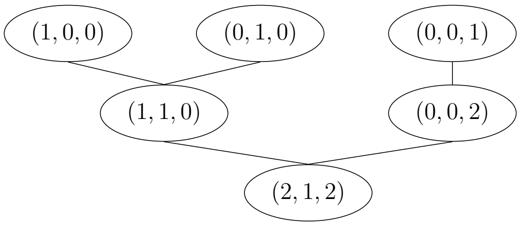
- $E_5 = \{m_1 = [2], m_2 = [\ ], m_3 = [3]\}$ and we have $l(m_3) + c[3] - k_3 = 2 \le \min(\max(l(m_2), p_1), \text{Comp}(m_2, m_3))) = \min(2, 2) = 2$. Thus we have the tree $T_6 = T_{E_5} = (2, 2)$.



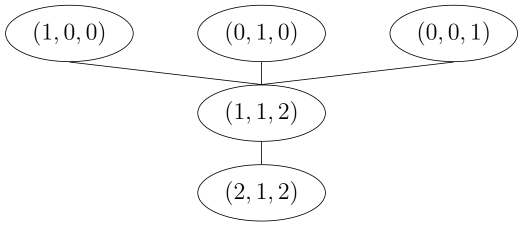
T_1



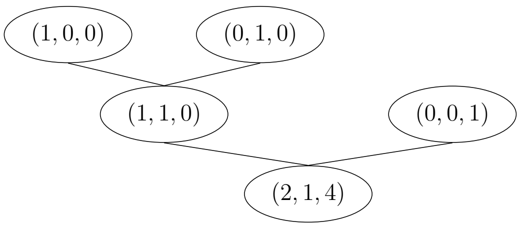
T_2



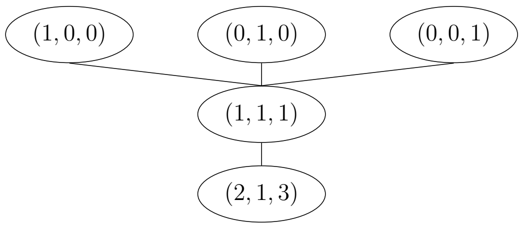
T_3



T_4



T_5



T_6

Now we compute $S_2^1([3,2,4])$. We are in the case $t = 2$ so its definition is:

$$S_2^1([3,2,4]) = \{T_E = (p_1, p_2) : E = \{m_1, m_2, m_3\}\,;\ T_{E^1} = (p_2) \in \text{Cond}([2,4]);$$

$$m_1 \in \text{Cond}(3); 1 \leq p_1 \leq \min(l(m_1), p_1^*, \text{Comp}(m_1, m_2))\}\,.$$

Then, to do that we need the following sets:

$$\text{Cond}([2,4]) = \{B_1, B_2, B_3, B_4\}\,,\ \text{where}$$

- $B_1 = T_{G_1} = (2)$ with $G_1 = \{[\ ], [2,2]\}\,;$
- $B_2 = T_{G_2} = (1)$ with $G_2 = \{[2], [2,2]\};$
- $B_3 = T_{G_3} = (2)$ with $G_3 = \{[\ ], [3]\}\,;$
- $B_4 = T_{G_4} = (1)$ with $G_4 = \{[2], [4]\}\,;$
- $\text{Cond}(3) = \{[3]\}\,.$

Hence we consider:

- $E_6 = \{m_1 = [3], m_2 = [\ ], m_3 = [2,2]\}$ and we have $\min(\max(p_2, l(m_2)), \text{Comp}(m_1, m_2), l(m_1)) = \min(2,2,1) = 1$. Thus we only have the tree $T_7 = T_{E_6} = (1,2)$.
- $E_1 = \{m_1 = [3], m_2 = [2], m_3 = [2,2]\}$ and we have $\min(\max(p_2, l(m_2)), \text{Comp}(m_1, m_2), l(m_1)) = \min(1,3,1) = 1$. Thus we only have the tree, already found in $S_1^1([3,2,4])$, $T_1 = T_{E_1} = (1,1)$.

- $E_7 = \{m_1 = [3], m_2 = [\ ], m_3 = [3]\}$ and we have $\min(\max(p_2, l(m_2)), \text{Comp}(m_1, m_2), l(m_1)) = \min(2, 2, 1) = 1$. Hence we have the tree $T_8 = T_{E_7} = (1, 2)$.

- $E_2 = \{m_1 = [3], m_2 = [2], m_3 = [4]\}$ and we have $\min(\max(p_2, l(m_2)), \text{Comp}(m_1, m_2), l(m_1)) = \min(1, 3, 1) = 1$. Thus we only have the tree, already found in $S_1^1([3, 2, 4])$, $T_2 = T_{E_2} = (1, 1)$.

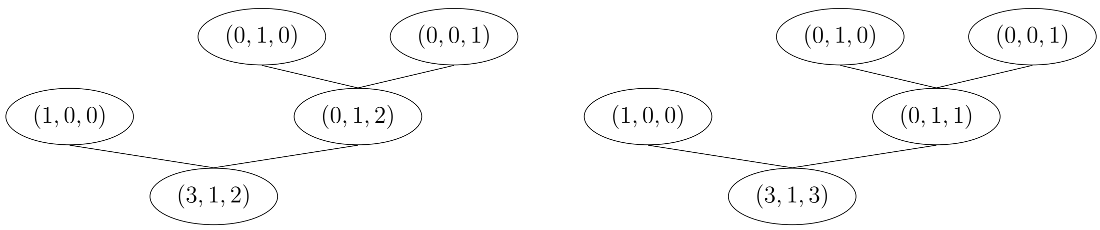
Hence $S_2^1([3, 2, 4]) = \{T_1, T_2, T_7, T_8\}$.

Now we compute $S_2^2([3, 2, 4])$. We find $k_1 = 2$ as the only value such that $I(k_1) \neq \emptyset$, and $I(2)$ contains two elements. In fact, if we consider B_1 and B_3 and $m_1 = [2]$, we have:

- $E_3 = \{m_1 = [2], m_2 = [\ ], m_3 = [2, 2]\}$ and we have
  $l(m_1) + c[1] - k_1 = 2 \leq \min(\max(p_2, l(m_2)), \text{Comp}(m_1, m_2))) = \min(2, 2) = 2$. Thus we only have the tree, already found in $S_1^1([3, 2, 4])$, $T_4 = T_{E_3} = (2, 2)$.

- $E_5 = \{m_1 = [2], m_2 = [\ ], m_3 = [3]\}$ and we have
  $l(m_1) + c[1] - k_1 = 2 \leq \min(\max(p_2, l(m_2)), \text{Comp}(m_1, m_2))) = \min(2, 2) = 2$. Thus we only have the tree, already found in $S_1^2([3, 2, 4])$, $T_6 = T_{E_5} = (2, 2)$.

Thus $S_2^1([3, 2, 4]) \cup S_2^2([3, 2, 4]) = \{T_1, T_2, T_4, T_6, T_7, T_8\}$.



T_7 $\qquad\qquad$ T_8

Hence, $\text{Cond}([3, 2, 4]) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$.

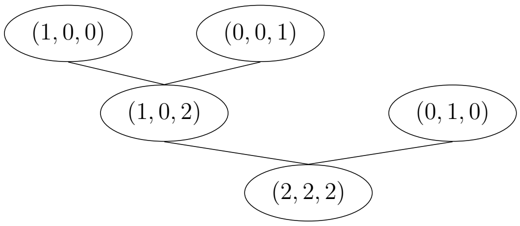
If we compute the set $\overline{\text{Cond}([3, 2, 4])}$, with the technique explained above, we find that:

$$\overline{\text{Cond}([3, 2, 4])} = \text{Cond}([3, 2, 4]) \bigcup \{T_9, T_{10}\}\ ,$$

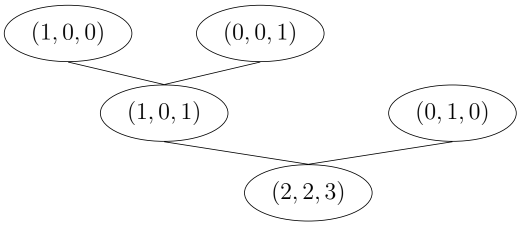
where

- $T_9 = M(T)_{E_8} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ where $E_8 = \{M_1 = [2], M_2 = [2], M_3 = [2, 2]\}$.

- $T_{10} = M(T)_{E_9} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ where $E_9 = \{M_1 = [2], M_2 = [2], M_3 = [3]\}$.



T_9



T_{10}

Example 3.1.6. It is easy to implement an algorithm that computes the number of untwisted Arf semigroups of $\mathbb{N}^3$ with a given conductor. In the following table we have the values obtained for some conductors.

$\mathbf{c}$	$\lvert\mathrm{Cond}(\mathbf{c})\rvert$	$\mathbf{c}$	$\lvert\mathrm{Cond}(\mathbf{c})\rvert$	$\mathbf{c}$	$\lvert\mathrm{Cond}(\mathbf{c})\rvert$	$\mathbf{c}$	$\lvert\mathrm{Cond}(\mathbf{c})\rvert$
$[1,1,1]$	1	$[8,8,8]$	2401	$[15,15,15]$	71736	$[7,8,9]$	843
$[2,2,2]$	4	$[9,9,9]$	1940	$[1,2,3]$	2	$[8,9,10]$	2901
$[3,3,3]$	9	$[10,10,10]$	8126	$[2,3,4]$	8	$[9,10,11]$	3913
$[4,4,4]$	50	$[11,11,11]$	6671	$[3,4,5]$	18	$[10,11,12]$	11178
$[5,5,5]$	72	$[12,12,12]$	37750	$[4,5,6]$	86	$[11,12,13]$	13942
$[6,6,6]$	425	$[13,13,13]$	18263	$[5,6,7]$	144	$[12,13,14]$	40278
$[7,7,7]$	294	$[14,14,14]$	123498	$[6,7,8]$	542	$[13,14,15]$	47675

Example 3.1.7. The following table contains the value of $|\overline{\mathrm{Cond}(\mathbf{c})}|$ for some values of $\mathbf{c}$.

$\mathbf{c}$	$\lvert\overline{\mathrm{Cond}(\mathbf{c})}\rvert$	$\mathbf{c}$	$\lvert\overline{\mathrm{Cond}(\mathbf{c})}\rvert$	$\mathbf{c}$	$\lvert\overline{\mathrm{Cond}(\mathbf{c})}\rvert$	$\mathbf{c}$	$\lvert\overline{\mathrm{Cond}(\mathbf{c})}\rvert$
$[1,1,1]$	1	$[7,7,7]$	406	$[1,2,3]$	2	$[7,8,9]$	1145
$[2,2,2]$	5	$[8,8,8]$	3217	$[2,3,4]$	10	$[8,9,10]$	3828
$[3,3,3]$	12	$[9,9,9]$	2650	$[3,4,5]$	26	$[9,10,11]$	5289
$[4,4,4]$	66	$[10,10,10]$	10992	$[4,5,6]$	110	$[10,11,12]$	14908
$[5,5,5]$	98	$[11,11,11]$	9131	$[5,6,7]$	192	$[11,12,13]$	19147
$[6,6,6]$	567	$[12,12,12]$	50903	$[6,7,8]$	701	$[12,13,14]$	53144

3.2 Finding Arf semigroups with a fixed genus

The aim of this section is to solve a similar problem to the one addressed in the previous one. The role of the conductor will be replaced by the genus. We firstly give a new procedure for the determination of all the Arf numerical semigroups with a fixed genus, slightly adapting the one given in the Subsection 3.1.1 for the conductor. Then, we give a way to compute the genus of an Arf good semigroup from its untwisted multiplicity tree and we give a procedure for the computation of the set of all the Arf good subsemigroups of $\mathbb{N}^r$ with a fixed genus n, that works by induction on the dimension r.

3.2.1 An algorithm for the Arf numerical semigroups with a given genus

The problem of determining the set of all the Arf numerical semigroups with fixed genus was also addressed and solved in [12], where the authors presented a recursive algorithm for the computation of such a set. Here we present a non-recursive procedure that appeared to be faster when implemented in GAP.

First of all we recall that the genus of a numerical semigroup S is the cardinality of $\mathbb{N} \setminus S$. If m is a multiplicity sequence we denote by $c(m)$ the conductor of the Arf semigroup $\mathrm{AS}(m)$ associated to m. It is easy to deduce from Proposition 1.3.8, that if m is a multiplicity sequence, then the genus of $\mathrm{AS}(m)$ is $c(m) - l(m) = \sum_{k=1}^{l(m)} (m_k - 1)$.

We denote by $\mathrm{Gen}(n)$ the set of the multiplicity sequences of the Arf numerical semigroups with genus n.

Our aim is to compute $\mathrm{Gen}(n)$ for all $n \in \mathbb{N}$. If $n = 0$ then $\mathrm{Gen}(n) = \{[\,]\}$. Thus we suppose $n \geq 1$. Denote by

$$U^n(i) = \{m \in \mathrm{Gen}(i) : m_1 + i - 1 \leq n\} \text{ for all } i = 1, \dots, n-1.$$

Now suppose that $m = [m_1, \dots, m_k] \in \mathrm{Gen}(n)$. If $k = 1$ then $m = [n+1]$, otherwise we have the following situation:

- $2 \leq m_1 \leq n$;
- $c([m_2, \dots, m_k]) = c([m_1, m_2, \dots, m_k]) - m_1$ and $l([m_2, \dots, m_k]) = l([m_1, \dots, m_k]) - 1$. So $c([m_2, \dots, m_k]) - l([m_2, \dots, m_k]) = n - m_1 + 1$, and $[m_2, \dots, m_k] \in \mathrm{Gen}(n - m_1 + 1)$;
- $m_1 \in \mathrm{AS}([m_2, \dots, m_k])$;
- $m_2 - m_1 \leq 0 \Rightarrow m_2 + (n - m_1 + 1) - 1 \leq n \Rightarrow [m_2, \dots, m_k] \in U^n(n - m_1 + 1)$.

So if we know $U^n(i)$ for $i = 1, \dots, n-1$, then we can compute $\mathrm{Gen}(n)$ in the following way:

$$\mathrm{Gen}(n) = \bigcup_{i=1}^{n-1} \{(n - i + 1) :: m | m \in U^n(i), n - i + 1 \in \mathrm{AS}(m)\} \cup \{[n+1]\}.$$

Thus we need a way to compute $U^n(i)$. Suppose that $m = [m_1, \dots, m_k] \in U^n(i)$. If $k = 1$, and $i + 1 + i - 1 = 2i \leq n$ then $m = [i+1]$, otherwise we have the following situation:

- $2 \leq m_1 \leq i$, and $[m_2, \dots, m_k] \in \mathrm{Gen}(i - m_1 + 1) = \mathrm{Gen}(q)$;
- $m_2 + q - 1 = m_2 + i - m_1 + 1 - 1 = m_2 - m_1 + i \leq i \leq n \Rightarrow [m_2, \dots, m_k] \in U^n(q)$;
- $m_1 \in \mathrm{AS}([m_2, \dots, m_k])$;
- $m_1 + i - 1 \leq n \Rightarrow 2m_1 \leq n - i + 1 + m_1 = n - (i + 1 - m_1) + 2 = n - q + 2 \Rightarrow m_1 \leq \left\lfloor \frac{n-q+2}{2} \right\rfloor$.

So each $U^n(i)$ can be constructed using $U^n(q)$ with $1 \leq q < i$. Thus we have the following algorithm for the computation of $\mathrm{Gen}(n)$.

```
input  : An integer n
output: The set Gen(n) of all the multiplicity sequences of Arf semigroups with genus
        n
Gen(n) ⟵ {[n + 1]}
for i ← 1 to n − 1 do
    if i ≤ ⌊n/2⌋ then
        U^n(i) ⟵ {[i + 1]}
    end
    else
        U^n(i) ⟵ ∅
    end
end
for i ← 1 to n − 1 do
    for m ∈ U^n(i) do
        if n − i + 1 ∈ AS(m) then
            Gen(n) ⟵ Gen(n) ∪ {(n − i + 1) :: m}
        end
        for k ∈ AS(m) ∩ {2, . . . , ⌊(n−i+2)/2⌋} do
            U^n(i + k − 1) ⟵ U^n(i + k − 1) ∪ {(k) :: m}
        end
    end
end
Gen(n)
```

Algorithm 2:

3.2.2 Arf semigroups of $\mathbb{N}^r$ with given genus

The aim of this section is to find a way to determine all the Arf good semigroups of $\mathbb{N}^r$ with a given genus. For the Remark 4, it is not restrictive to focus only on the Arf semigroups associated to untwisted trees. Recall that the genus of a good semigroup can be computed as:

$$g(S) = d(\mathbb{N}^r \setminus C) - d(S \setminus C).$$

Since $d(\mathbb{N}^r \setminus C)$ is the length of a saturated chain in $\mathbb{N}^r$ from the vector $\mathbf{0} \in \mathbb{N}^r$ to the conductor $C(S) = (c[1], \ldots, c[r])$, it is easy to show that

$$d(\mathbb{N}^r \setminus C) = \sum_{k=1}^{r} c[k].$$

On the other hand, $d(S \setminus C)$ is the length of a saturated chain in S from $\mathbf{0} \in S$ to $\delta \in S$.

Remark 12. Because the conductor δ is an element of the Arf semigroup $S(T)$, it can be expressed as a sum of nodes in a subtree T' of T. From Proposition 3.1.1, it easily follows that T' is the subtree consisting of the nodes of T that are different from the unit vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

For the Arf good local semigroups with untwisted multiplicity tree we have the following theorem:

Theorem 3.2.1. *Suppose that $T = T_E = (p_1, \ldots, p_{r-1})$ is an untwisted multiplicity tree of an Arf semigroup where $E = \{m_1, \ldots, m_r\}$ is a collection of multiplicity sequences.*
Then

$$g(S(T)) = \sum_{k=1}^{r} g(AS(m_k)) + \sum_{k=1}^{r-1} p_k,$$

where $S(T)$ is the Arf semigroup associated to the tree T and $AS(m_k)$ is the Arf numerical semigroup associated to the multiplicity sequence m_k.

Proof. Denoted by $\delta = (c[1], \ldots, c[r])$ the conductor of $S(T)$, and by $C = \delta + \mathbb{N}^r$. We know that

$$g(S(T)) = d(\mathbb{N}^r \setminus C) - d(S \setminus C).$$

We have

$$d(\mathbb{N}^r \setminus C) = \sum_{k=1}^{r} c[k] = \sum_{k=1}^{\max(l(m_1),p_1)} m_1[k] + \ldots + \sum_{k=1}^{\max(l(m_i),p_i,p_{i-1})} m_i[k] + \ldots + \sum_{k=1}^{\max(l(m_r),p_{r-1})} m_r[k] =$$

$$= \sum_{k=1}^{l(m_1)} m_1[k] + \max(l(m_1), p_1) - l(m_1) + \ldots + \sum_{k=1}^{l(m_i)} m_i[k] + \max(l(m_i), p_i, p_{i-1}) - l(m_i) + \ldots$$

$$\ldots + \sum_{k=1}^{l(m_r)} m_r[k] + \max(l(M_r), p_{r-1}) - l(m_r) = \sum_{j=1}^{r} \left(\sum_{k=1}^{l(m_j)} m_j[k] - l(m_j) \right) + \max(l(m_1), p_1) +$$

$$+ \sum_{k=2}^{r-1} \max(l(m_k), p_k, p_{k-1}) + \max(l(m_r), p_{r-1}),$$

where we are using again Proposition 3.1.1.

Now we want to compute $d(S \setminus C)$. We need a saturated chain in $S(T)$ from $\mathbf{0}$ to δ. Suppose that we have

$$\mathbf{0} = a_0 < a_1 < \cdots < a_l = \delta,$$

a saturated chain in $S(T)$. We clearly have $a_1 = (m_1[1], \ldots, m_r[1])$, that is the multiplicity vector of $S(T)$. Let us consider $a_q \in S(T)$, with $q = 1, \ldots, l-1$. From the properties of the multiplicity tree of an Arf semigroup, there exists a subtree T' of T, rooted in the node corresponding to a_1, such that a_q is the sum of all the nodes belonging to T'. As usual we denote by $\mathbf{n}_i^j$ the node of T that is in the i-th branch and on the j-th level. We denote, given a subtree T' of T, by $\mathrm{N}(T')$ the set of nodes that appears in T'.

Now, it is clear that, in order to have a saturated chain, a_{q+1} must be the sum of all the nodes belonging to a subtree T'' of T such that:

- $T' \subseteq T''$;
- $N(T'') \setminus N(T') = \{\mathbf{n}_i^j\}$;
- $\mathbf{n}_i^{j-1} \in N(T')$;
- $\mathbf{n}_i^j \neq e_i$, where e_i is the i-th canonical vector of $\mathbb{N}^r$ (by Remark 12 since $a_q \neq \delta$).

From the previous remark, it easily follows that

$$d(S \setminus C) = |\left\{\mathbf{n}_i^j \in N(T) : \mathbf{n}_i^j \neq e_i\right\}|,$$

and we need to compute this cardinality. Taking in account the expressions for $c[i]$, it follows that there are

- $\max(l(m_1), p_1)$ nodes along the first branch that are different from e_1;
- $\max(l(m_i), p_i, p_{i-1})$ nodes along the i-th branch different from e_i, for $2 \leq i \leq r-1$;
- $\max(l(m_r), p_{r-1})$ nodes along the last branch that are different from e_r.

Now from $T = T_E = (p_1, \ldots, p_{r-1})$ we deduce that the i-th and $i+1$-th branches have p_i nodes in common for each $i = 1, \ldots, r-1$. Therefore we can conclude:

$$d(S \setminus C) = \max(l(m_1), p_1) + \sum_{k=2}^{r-1} \max(l(m_k), p_k, p_{k-1}) + \max(l(m_r), p_{r-1}) - \sum_{k=1}^{r-1} p_k.$$

Finally we have:

$$g(S(T)) = d(\mathbb{N}^r \setminus C) - d(S \setminus C) = \sum_{j=1}^{r} \left(\sum_{k=1}^{l(m_j)} m_j[k] - l(m_j) \right) + \sum_{k=1}^{r-1} p_k,$$

and, because $\sum_{k=1}^{l(m_j)} m_j[k] - l(m_j) = g(\mathrm{AS}(m_j))$, we have:

$$g(S(T)) = \sum_{k=1}^{r} g(\mathrm{AS}(m_k)) + \sum_{k=1}^{r-1} p_k,$$

and the proof is complete. $\square$

Now we denote by $\mathrm{Gen}(r, n)$ the set of all the untwisted multiplicity tree associated to Arf good semigroups in $\mathbb{N}^r$ with genus n. Given a $n \in \mathbb{N}$ we want to find a way to compute the set $\mathrm{Gen}(r, n)$. We do that using recursion on r. From the previous section we know how to compute $\mathrm{Gen}(1, n)$, so the base case is done. First of all, we notice that we need $n \geq r - 1$. In fact an untwisted Arf semigroup S of $\mathbb{N}^r$ can be described by a tree $T = T_E = (p_1, \ldots, p_{r-1})$ with $E = \{m_1, \ldots, m_r\}$, and we have just showed that

$$g(S(T)) = \sum_{k=1}^{r} g(\mathrm{AS}(m_k)) + \sum_{k=1}^{r-1} p_k,$$

where $g(\mathrm{AS}(m_k)) \geq 0$ and $p_k \geq 1$ for all the k. Then $g(S(T)) \geq r - 1$.

We fix a $r \geq 2$ and $n \geq r - 1$ and suppose that $T = T_E = (p_1, \ldots, p_{r-1})$ is a multiplicity tree in $\mathrm{Gen}(r, n)$, where $E = \{m_1, \ldots, m_r\}$ is a collection of r multiplicity sequences.

Consider $t < r$. We have:

$$n = \sum_{j=1}^{r} g(\mathrm{AS}(m_j)) + \sum_{j=1}^{r-1} p_j = \left(\sum_{j=1}^{t} g(\mathrm{AS}(m_j)) + \sum_{j=1}^{t-1} p_j \right) + p_t + \left(\sum_{j=t+1}^{r} g(\mathrm{AS}(m_j)) + \sum_{j=t+1}^{r-1} p_j \right),$$

therefore if we denote by

$$k_1 = \sum_{j=1}^{t} g(\mathrm{AS}(m_j)) + \sum_{j=1}^{t-1} p_j \quad \text{and} \quad k_2 = \sum_{j=t+1}^{r} g(\mathrm{AS}(m_j)) + \sum_{j=t+1}^{r-1} p_j,$$

we have:

$$n - p_t = k_1 + k_2.$$

Now, we have:

- The tree $T^1 = T_{E_t} = (p_1, \dots, p_{t-1})$, with $E_t = \{m_1, \dots, m_t\}$, belongs to Gen$(t, k_1)$ $(k_1 \geq t-1)$;
- The tree $T^2 = T_{E^t} = (p_{t+1}, \dots, p_{r-1})$, with $E^t = \{m_{t+1}, \dots, m_r\}$, belongs to Gen$(r-t, n-p_t-k_1)$;
- $1 \leq p_t \leq$ Comp(m_t, m_{t+1});
- $k_1 + k_2 \geq r-2 \Rightarrow 1 \leq p_t \leq n-r+2$;
- $k_2 \geq r-1-t \Rightarrow k_1 \leq n-p_t-r+1+t$.

Now, for each $1 \leq p \leq n-r+2$ and $t-1 \leq k(p) \leq n-p-r+1+t$ we define the set

$$I_r^n(t,p,k(p)) = \{T_E = (p_1, \dots, p_{r-1}) : E = \{m_1, \dots, m_r\}, T_{E_t} = (p_1, \dots, p_{t-1}) \in$$
$$\in \text{Gen}(t, k(p)), T_{E^t} = (p_{t+1}, \dots, p_{r-1}) \in \text{Gen}(r-t, n-p-k(p)),$$
$$\text{and } p_t = p \leq \text{Comp}(m_t, m_{t+1})\}.$$

So we can deduce that T belongs to the following set:

$$S(r,n) = \bigcup_{p=1}^{n-r+2} \left(\bigcup_{k(p)=t-1}^{n-p-r+t+1} I_r^n(t,p,k(p)) \right).$$

With the inverse implications, it is very easy to show that $S(r,n) \subseteq$ Gen(r,n). Then we have Gen$(r,n) = S(r,n)$.

Notice that the computation of Gen(r,n) involves the computation of Gen(t,k) and Gen$(r-t,k)$, then using recursion and the base case we can solve our problem.

Remark 13. If we have $T = T_E = (p_1, \dots, p_{r-1})$ with $E = \{m_1, \dots, m_r\}$, we will denote by $T^{-1} = T_{E^{-1}} = (p_{r-1}, \dots, p_1)$ where $E^{-1} = \{m_r, \dots, m_1\}$.

It is clear that if $T \in$ Gen(r,n) then $T^{-1} \in$ Gen(r,n) too. If U is a set of multiplicity trees, we denote by $U^{-1} = \{T^{-1} : T \in U\}$.

We have the following proposition:

Proposition 3.2.2. *If $t \leq \frac{r}{2}$ and $\lambda(p) \geq \frac{n-p-1}{2}$ then*

$$\textit{Gen}(r,n) = \bigcup_{p=1}^{n-r+2} \left(\bigcup_{k(p)=t-1}^{\lambda(p)} \left(I_r^n(t,p,k(p)) \cup (I_r^n(t,p,k(p)))^{-1} \right) \right).$$

Proof. Consider $t \leq \frac{r}{2}$ and $\lambda(p) \geq \frac{n-p-1}{2}$, we show that for any $p = 1, \dots, n-r+2$, we have

$$\bigcup_{k(p)=\lambda(p)+1}^{n-p-r+t+1} I_r^n(t,p,k(p)) \subseteq \left(\bigcup_{k(p)=t-1}^{\lambda(p)} I_r^n(t,p,k(p)) \right)^{-1}.$$

Consider $T \in \bigcup_{k(p)=\lambda(p)+1}^{n-p-r+t+1} I_r^n(t,p,k(p))$, then

$T = T_E = (p_1, \ldots, p_{r-1})$ with $E = \{m_1, \ldots, m_r\}$ and $T_{E^t} = (p_{t+1}, \ldots, p_{r-1}) \in \mathrm{Gen}(r-t, n-p-\overline{k})$,

where $\lambda(p) + 1 \leq \overline{k} \leq n - p - r + t + 1$.

We want to show that

$$T \in \left(\bigcup_{k(p)=t-1}^{\lambda(p)} I_r^n(t,p,k(p)) \right)^{-1} \iff T^{-1} \in \bigcup_{k(p)=t-1}^{\lambda(p)} I_r^n(t,p,k(p)).$$

Let us consider the subtree T^1 of T^{-1} with $T^1 = T_{(E^{-1})_t} = (p_{r-1}, \ldots, p_{r-t+1})$, where $(E^{-1})_t = \{m_r, \ldots, m_{r-t+1}\}$.

Now we have:

$$t \leq \frac{r}{2} \Rightarrow 2t \leq r \Rightarrow t \leq r - t \Rightarrow t + 1 \leq r - t + 1,$$

and from this and from $T_{E^t} = (p_{t+1}, \ldots, p_{r-1}) \in \mathrm{Gen}(r - t, n - p - \overline{k})$ we can deduce that

$$T^1 = T_{(E^{-1})_t} = (p_{r-1}, \ldots, p_{r-t+1}) \in \mathrm{Gen}(t, x),$$

where $t - 1 \leq x \leq n - p - \overline{k}$.

Thus, from $\lambda(p) + 1 \leq \overline{k}$ and from $\lambda(p) \geq \frac{n-p-1}{2}$ we deduce:

$$-\overline{k} \leq -\lambda(p) - 1 \text{ and } n - p - 1 \leq 2\lambda(p) \Rightarrow x \leq n - p - \lambda(p) - 1 \leq \lambda(p).$$

Therefore, from $T^{-1} \in \mathrm{Gen}(r, n)$ and from the previous inequality we have

$$T^{-1} \in I_r^n(t,p,x) \subseteq \bigcup_{k(p)=t-1}^{\lambda(p)} I_r^n(t,p,k(p)),$$

and the claim is proved. Therefore we have:

$$\mathrm{Gen}(r,n) \supseteq \bigcup_{p=1}^{n-r+2} \left(\bigcup_{k(p)=t-1}^{\lambda(p)} \left(I_r^n(t,p,k(p)) \cup \left(I_r^n(t,p,k(p))\right)^{-1} \right) \right) \supseteq$$

$$\supseteq \bigcup_{p=1}^{n-r+2} \left(\bigcup_{k(p)=t-1}^{n-p-r+t+1} I_r^n(t,p,k(p)) \right) = \mathrm{Gen}(r,n),$$

and the proof is complete. $\square$

The previous proposition suggests us an easier way to compute $\text{Gen}(r,n)$. In fact we have to consider a smaller amount of sets of the type $I_r^n(t,p,k(p))$, completing the computation with sets of the type $(I_r^n(t,p,k(p)))^{-1}$ that are very easy to obtain once we have $I_r^n(t,p,k(p))$. To speed up the process is also useful to set $\lambda(p) = \left\lceil \frac{n-p-1}{2} \right\rceil$ and $t = \left\lfloor \frac{r}{2} \right\rfloor$.

Denote by $\overline{\text{Gen}(r,n)}$ the set of all the multiplicity trees (twisted and untwisted) of the Arf semigroups in $\mathbb{N}^r$ with genus n. We can already compute all the untwisted ones. Suppose that T is a twisted tree in $\overline{\text{Gen}(r,n)}$. Then there exists a permutation $\sigma \in S^r$ such that $\sigma(T)$ is untwisted. From the formula of the genus it is very easy to see that $\sigma(T) \in \text{Gen}(r,n)$. Thus we have:

$$\overline{\text{Gen}(r,n)} = \bigcup_{\sigma \in S^r} \left\{ \sigma^{-1}(T) : T \in \text{Gen}(r,n) \right\}.$$

Example 3.2.3. We compute $\text{Gen}(2,3)$. In this case we have $t=1$.

We have to consider the sets $I_2^3(1,p,k(p))$ and $(I_2^3(1,p,k(p)))^{-1}$ for each $1 \le p \le 3$ and $0 \le k(p) \le \left\lceil \frac{3-p-1}{2} \right\rceil$.

Case: $p = 1 \Rightarrow 0 \le k(1) \le 1$.

- $k(1) = 0$. To compute $I_2^3(1,1,0)$ we need $m_1 \in \text{Gen}(1,0)$ and $m_2 \in \text{Gen}(1,2)$. They are

  $$\text{Gen}(1,0) = \{[\,]\} \text{ and } \text{Gen}(1,2) = \{[3],[2,2]\}.$$

  Thus we can consider

  - $E_1 = \{m_1 = [\,], m_2 = [3]\}$.
    In this case $1 = p \le \text{Comp}(m_1,m_2) = 2$ then the tree $T_1 = T_{E_1} = (1)$ belongs to $\text{Gen}(2,3)$.
  - $E_2 = \{m_1 = [\,], m_2 = [2,2]\}$.
    In this case $1 = p \le \text{Comp}(m_1,m_2) = 3$ then the tree $T_2 = T_{E_2} = (1)$ belongs to $\text{Gen}(2,3)$.

  Therefore $I_2^3(1,1,0) = \{T_1, T_2\}$. We have now to compute $(I_2^3(1,1,0))^{-1}$

  - $T_3 = T_1^{-1} = T_{E_3} = (1) \in \text{Gen}(2,3)$, with $E_3 = E_1^{-1} = \{[3],[\,]\}$.
  - $T_4 = T_2^{-1} = T_{E_4} = (1) \in \text{Gen}(2,3)$, with $E_4 = E_2^{-1} = \{[2,2],[\,]\}$.

  Therefore $(I_2^3(1,1,0))^{-1} = \{T_3, T_4\}$.

- $k(1) = 1$. To compute $I_2^3(1,1,1)$ we need $m_1 \in \text{Gen}(1,1)$ and $m_2 \in \text{Gen}(1,1)$. We have

  $$\text{Gen}(1,1) = \{[2]\}.$$

  Thus we only have to consider

– $E_5 = \{m_1 = [2], m_2 = [2]\}$.
In this case $1 = p \leq \mathrm{Comp}(m_1, m_2) = +\infty$ then the tree $T_5 = T_{E_5} = (1)$ belongs to $\mathrm{Gen}(2, 3)$.

Therefore $I_2^3(1, 1, 1) = \{T_5\}$. In this case $I_2^3(1, 1, 1) = (I_2^3(1, 1, 1))^{-1}$.

Case: $p = 2 \Rightarrow k(2) = 0$.

- $k(2) = 0$. To compute $I_2^3(1, 2, 0)$ we need $m_1 \in \mathrm{Gen}(1, 0)$ and $m_2 \in \mathrm{Gen}(1, 1)$.

  Thus we only have to consider

  – $E_6 = \{m_1 = [\,], m_2 = [2]\}$.
  In this case $2 = p \leq \mathrm{Comp}(m_1, m_2) = 2$ then the tree $T_6 = T_{E_6} = (2)$ belongs to $\mathrm{Gen}(2, 3)$.

  Therefore $I_2^3(1, 2, 0) = \{T_6\}$. We have now to compute $(I_2^3(1, 2, 0))^{-1}$

  – $T_7 = T_6^{-1} = T_{E_7} = (2) \in \mathrm{Gen}(2, 3)$, with $E_7 = E_6^{-1} = \{[2], [\,]\}$.

  Therefore $(I_2^3(1, 2, 0))^{-1} = \{T_7\}$.

Case: $p = 3 \Rightarrow k(3) = 0$.

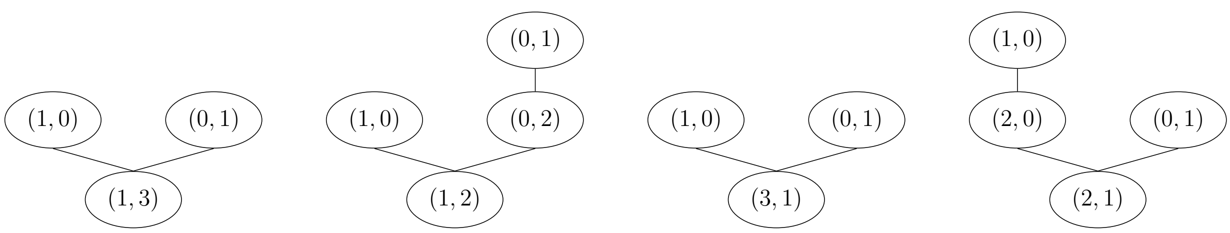
- $k(3) = 0$.

  To compute $I_2^3(1, 3, 0)$ we need $m_1 \in \mathrm{Gen}(1, 0)$ and $m_2 \in \mathrm{Gen}(1, 0)$.

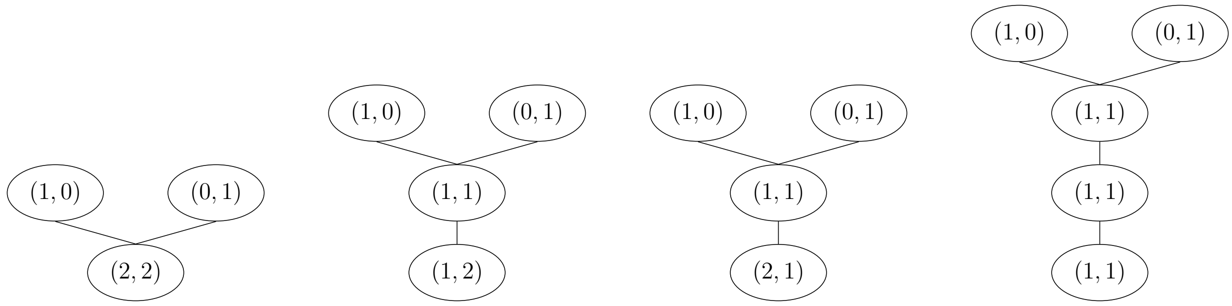
  Thus we only have to consider

  – $E_8 = \{m_1 = [\,], m_2 = [\,]\}$.
  In this case $3 = p \leq \mathrm{Comp}(m_1, m_2) = +\infty$ then the tree $T_8 = T_{E_8} = (3)$ belongs to $\mathrm{Gen}(2, 3)$.

  Therefore $I_2^3(1, 3, 0) = \{T_8\}$. In this case $I_2^3(1, 3, 0) = (I_2^3(1, 3, 0))^{-1}$. Thus $\mathrm{Gen}(2, 3) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$. We obviously have $\mathrm{Gen}(2, 3) = \overline{\mathrm{Gen}(2, 3)}$ because we have only two branches.



T_1 T_2 T_3 T_4



T_5 T_6 T_7 T_8

In the following table we report the cardinality of $\mathrm{Gen}(2, n)$ for n up to 32.

n	$\lvert\mathrm{Gen}(2, n)\rvert$	n	$\lvert\mathrm{Gen}(2, n)\rvert$	n	$\lvert\mathrm{Gen}(2, n)\rvert$	n	$\lvert\mathrm{Gen}(2, n)\rvert$
1	1	9	251	17	4386	25	35203
2	3	10	385	18	5874	26	44209
3	8	11	577	19	7773	27	55175
4	16	12	837	20	10195	28	68493
5	32	13	1207	21	13270	29	84540
6	56	14	1701	22	17138	30	103898
7	99	15	2361	23	21922	31	127031
8	157	16	3239	24	27882	32	154681

Using the previous results, it is easy to implement an algorithm that computes the number of untwisted Arf semigroups of $\mathbb{N}^r$ with a given genus n. Each entry of the following table is such a number, for $1 \leq r \leq 16$ and $0 \leq n \leq 15$.

$r \setminus n$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	1	1	2	3	4	6	8	10	13	17	21	26	31	36	47	55
2	0	1	3	8	16	32	56	99	157	251	385	577	837	1207	1701	2361
3	0	0	1	5	18	49	120	263	543	1048	1943	3458	5957	9957	16246	25896
4	0	0	0	1	7	32	110	324	846	2032	4544	9620	19420	37686	70618	128399
5	0	0	0	0	1	9	50	207	716	2169	5958	15119	35994	81196	175001	362501
6	0	0	0	0	0	1	11	72	348	1384	4772	14769	41919	110859	276257	654422
7	0	0	0	0	0	0	1	13	98	541	2432	9403	32385	101658	295681	806530
8	0	0	0	0	0	0	0	1	15	128	794	3980	17050	64678	222474	705806
9	0	0	0	0	0	0	0	0	1	17	162	1115	6164	28973	120016	448873
10	0	0	0	0	0	0	0	0	0	1	19	200	1512	9136	46736	209871
11	0	0	0	0	0	0	0	0	0	0	1	21	242	1993	13064	72239
12	0	0	0	0	0	0	0	0	0	0	0	1	23	288	2566	18132
13	0	0	0	0	0	0	0	0	0	0	0	0	1	25	338	3239
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	27	392
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	29
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

From the previous table, we can also deduce, considering the sum of all the entries in the corresponding column, the number of all the local untwisted Arf semigroups with a given genus n (in all the possible dimensions). We call

$$\mathrm{NG}(n) = |\,\{S \text{Arf semigroup} : g(S) = n\}\,|,$$

such a number.

Thus we have

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
NG(n)	1	2	6	17	46	129	356	989	2737	7588	21031	58289	161535	447693	1240773	3438746

Example 3.2.4. In the following table we report the cardinality of the sets $\overline{\mathrm{Gen}(r,n)}$ for $1 \leq r \leq 9$ and $0 \leq n \leq 8$.

$r \setminus n$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	1	2	3	4	6	8	10	13
2	0	1	3	8	16	32	56	99	157
3	0	0	1	6	22	61	151	334	693
4	0	0	0	1	10	51	189	576	1555
5	0	0	0	0	1	15	105	505	1906
6	0	0	0	0	0	1	21	197	1208
7	0	0	0	0	0	0	1	28	343
8	0	0	0	0	0	0	0	1	36
9	0	0	0	0	0	0	0	0	1

Chapter 4

An algorithm for the computation of the Arf closure of an algebroid curve

In this Chapter we present a procedure for the computation of the Arf closure of an algebroid curve. In Section 4.1 we explain how to generalize the procedure presented for the algebroid branches to this more general case determining all the required tools for the computation. In Section 4.2 and Section 4.3 we give an algorithm that lets us recover all these information starting from a parametrization of the given algebroid curve R, and we show how to deduce from them a presentation for the Arf closure ${}^*R$. Finally in Section 4.4, we improve the efficiency of the algorithm by finding a bound for the truncation of all the power series arising during the computations, generalizing the ideas presented in [2] for the algebroid branches.

4.1 The computation of the Arf closure of an algebroid curve

Let R be an algebroid curve. We want to show how the Lipman sequence R_j , of the successive blowups of R, can be used to compute and to give a presentation for the Arf closure ${}^*R$ of R. The strategy is to adapt the construction presented in Section 1.3 for the algebroid branches to this more general case. We build the Arf closure by using the following inductive process on the number of branches r.

- Base case: $r = 1$ we already know how to construct it.
- Inductive step. We suppose that we can solve the problem for $m < r$ and we give a solution for r.

  If R_j is not local then, as we saw in Subsection 1.4.1, there exists a partition $\mathfrak{P}(R_j) = \{P_{j,1}, \dots, P_{j,t}\}$, with
  $$P_{j,i} = \left\{q_{i,1}, \dots, q_{i,k(i)}\right\},$$
  such that
  $$R_j = \pi_{P_{j,1}}(R_j) \times \cdots \times \pi_{P_{j,t}}(R_j),$$

where $\pi_{P_{j,i}}(R_j)$ is a local ring isomorphic to a subring of $\mathbb{K}[[t_{q_{i,1}}]] \times \cdots \times \mathbb{K}[[t_{q_{i,k(i)}}]]$.

In this case, we have:

$$^*R_j = {}^*(\pi_{P_{j,1}}(R_j)) \times \cdots \times {}^*(\pi_{P_{j,t}}(R_j)),$$

and, for the inductive step, we have a way to compute each $^*(\pi_{P_{j,i}}(R_j))$, since $k(i) < r$ for all i.

If R_j is a local subring of $\overline{R} = \prod_{i=1}^{r} \mathbb{K}[[t_i]]$, then using the same idea of Arf (cf.[1, p.267]) it is easy to see that

$$^*R_j = \mathbb{K}(1, \ldots, 1) + x_j \cdot {}^*(R_{j+1}),$$

where x_j is an element of minimal value in R_j.

If R_{j+1} is local in $\overline{R}$ we can compute $^*R_{j+1}$ in the same way using $R_{j+2} = \mathrm{Bl}(R_{j+1})$ and an element of minimal value x_{j+1} in R_{j+1}. But we know that there exist an N such that R_N is not local (in fact the blow-up sequence has to stabilize into $\overline{R} = \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$) and therefore we are able to compute $^*R_N$ as we have already seen in the non-local case.

Then, if we suppose that N is the first integer such that R_N is not local, and we start from the local ring $R = R_1$, we obtain:

$$\begin{array}{rcl} ^*R_1 & = & \mathbb{K}(1, \ldots, 1) + x_1 \cdot {}^*R_2 \\ ^*R_2 & = & \mathbb{K}(1, \ldots, 1) + x_2 \cdot {}^*R_3 \\ \cdots & & \cdots \\ ^*R_{N-1} & = & \mathbb{K}(1, \ldots, 1) + x_{N-1} \cdot {}^*R_N, \end{array}$$

and from this it follows that

$$^*R_1 = \mathbb{K}(1, \ldots, 1) + \mathbb{K}x_1 + \mathbb{K}x_1x_2 + \ldots + x_1 \ldots x_{N-1} \cdot {}^*R_N.$$

where x_i is an element of minimal valuation of R_i.

From this procedure we see that it is important to compute the blow-up sequence R_i until $R_m = \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$ to understand how to compute $^*R$. In the following section, we will present an algorithm that gives us a way to compute this sequence along its multiplicity tree starting from a parametrization of the ring R.

4.2 The algorithm in the two-branches case

In this section we give an algorithm for the computation of the Arf closure of an algebroid curve with two branches having the following parametrization:

$$R = \mathbb{K}[[(\phi_1(t), \psi_1(u)), \ldots, (\phi_n(t), \psi_n(u))]].$$

First of all we fix some notations. We will always assume that a parametrization does not contain an element $y = (\phi(t), \psi(u))$ such that $\text{ord}(\phi(t)) = \text{ord}(\psi(u)) = 0$ and with $\phi(0) = \psi(0)$. If, in the following constructions, we produce a parametrization that contains such an element, we always convert it to $\overline{y} = y - (\phi(0), \psi(0))$ (it is possible to do that because $(\phi(0), \psi(0))$ is a multiple of the unit vector). For each $q \geq 0$ we will denote by

$$R_q = \mathbb{K}\left[\left[\left(\phi_1^{(q)}(t), \psi_1^{(q)}(u)\right), \ldots, \left(\phi_{n(q)}^{(q)}(t), \psi_{n(q)}^{(q)}(u)\right)\right]\right],$$

the parametrization of the q-th blow-up of R (we put by definition $R_1 = R$).

The following lemma will help us to understand when a R_q is not local from its parametrization.

Lemma 4.2.1. *Consider*

$$R = \mathbb{K}[[(\phi_1(t), \psi_1(u)), \ldots, (\phi_n(t), \psi_n(u))]].$$

We have that

$$R = \mathbb{K}[[\phi_1(t), \ldots, \phi_n(t)]] \times \mathbb{K}[[\psi_1(u), \ldots, \psi_n(u)]]$$

if and only if at least one of the following two conditions holds:

- *There exists $(\phi_i(t), \psi_i(u))$ in the parametrization such that*

$$\textit{ord}(\phi_i(t)) \cdot \textit{ord}(\psi_i(t)) = 0 \textit{ and } \textit{ord}(\phi_i(t))^2 + \textit{ord}(\psi_i(t))^2 \neq 0;$$

- *There exists $y = (\phi_i(t), \psi_i(u))$ in the parametrization such that*

$$\nu(y) = (0,0) \textit{ and } \phi_i(0) \neq \psi_i(0).$$

Proof. ($\Leftarrow$). Let us suppose that the first condition holds. Without loss of generality, we can suppose that the element $y = (\phi_1(t), \psi_1(u))$ in the parametrization is such that $\text{ord}(\phi_1(t)) = 0$ and $\text{ord}(\psi_1(u)) \neq 0$. Then we have $\phi_1(0) \neq 0$. Therefore $\phi_1(t)$ is invertible in $\mathbb{K}[[\phi_1(t)]]$ because its inverse is

$$(\phi_1(t))^{-1} = (\phi_1(0))^{-1} \cdot \sum_{i=0}^{+\infty} (-1)^i \left(\frac{\phi_1(t) - \phi_1(0)}{\phi_1(0)}\right)^i.$$

Thus in $\mathbb{K}[[y]] \subseteq R$ there exists an element of the form $z = ((\phi_1(t))^{-1}, g(u))$. Then we have

$$R \ni y \cdot z = (1, \psi_1(u) \cdot g(u)) = (1, h(u)),$$

where $\text{ord}(h(u)) > 0$. But $(1,1) \in R$ so $(1, h(u)) - (1,1) = (0, -1 + h(u))$ belongs to R. Now, $h(u) \in \mathbb{K}[[\psi_1(u)]]$ and therefore $-1 + h(u)$ is invertible in this ring. From this it follows again that there exist an element of the type $(l(t), (-1 + h(u))^{-1}) \in R$ and we have:

$$R \ni (0, -1 + h(u)) \cdot \left(l(t), (-1 + h(u))^{-1}\right) = (0, 1) \Rightarrow (1, 1) - (0, 1) = (1, 0) \in R.$$

Finally we obtain that

$$\mathbb{K}[[\phi_1(t), \ldots, \phi_n(t)]] \times \{0\} = (1, 0) \cdot R \subseteq R,$$
$$\{0\} \times \mathbb{K}[[\psi_1(u), \ldots, \psi_n(u)]] = (0, 1) \cdot R \subseteq R,$$

therefore we have $\mathbb{K}[[\phi_1(t), \ldots, \phi_n(t)]] \times \mathbb{K}[[\psi_1(u), \ldots, \psi_n(u)]] \subseteq R$ and because the inverse containment is trivial we have our thesis. Suppose now that the second condition holds. Let us consider $y = (\phi_i(t), \psi_i(u))$ in the parametrization such that

$$\nu(y) = (0, 0) \text{ and } \phi_i(0) \neq \psi_i(0).$$

Thus if we consider $(\phi_i(0), \phi_i(0)) \in R$ we have that $y - (\phi_i(0), \phi_i(0)) \in R$ is an element that fulfils the first condition and we can use the same arguments of the first part of the proof.
($\Rightarrow$). It is trivial, in fact if we suppose by contradiction that in the parametrization there are no elements that fulfil the condition of the theorem, then it would easily follow that in R we cannot find an element $(\phi(t), \psi(u))$ such that $\phi(t)$ is invertible and $\psi(u)$ is not invertible and this is absurd for the hypotheses on R. $\square$

If R_q is local in $\mathbb{K}[[t]] \times \mathbb{K}[[u]]$, we can consider the multiplicity vector $\text{mult}(R_q)$, that can be computed as

$$\text{mult}(R_q) = \left(\min\left\{\text{ord}(\phi_i^{(q)}(t)), i = 1, \ldots, n(q)\right\}, \min\left\{\text{ord}(\psi_i^{(q)}(u)), i = 1, \ldots, n(q)\right\}\right),$$

and denote by x_{R_q} an element of R_q with valuation $\text{mult}(R_q)$.

Remark 14. For the choice of the element x_{R_q} we can always consider either one of the $\left(\phi_i^{(q)}(t), \psi_i^{(q)}(u)\right)$ or the sum of two of them. To see it we denote by $y_i = \left(\phi_i^{(q)}(t), \psi_i^{(q)}(u)\right)$ for $i = 1, \ldots, n(q)$. If there exists y_i in the parametrization such that $\text{mult}(R_q) = \nu(y_i)$ we can set $x_{R_q} = y_i$. Otherwise, for the definition of $\text{mult}(R_q)$ there must exist i, j with $i \neq j$ such that

$$\left(\text{ord}(\phi_i^{(q)}(t)), \text{ord}(\psi_j^{(q)}(u))\right) = \text{mult}(R_q),$$

then $y_i + y_j$ is a good choice for x_{R_q} (in this case order cancellations cannot happen).

Remark 15. If we have a ring S such that

$$S = \mathbb{K}[[\phi_1(t), \ldots, \phi_n(t)]] \times \mathbb{K}[[\psi_1(u), \ldots, \psi_n(u)]],$$

then S is not local and, following the notations of the Section 1.4.1, we have that

$$\text{mult}^*(S) = \{(m_{1,1}, 0), (0, m_{2,1})\},$$

where $m_{1,1}$ is the multiplicity of the algebroid branch associated to $S_1 = \mathbb{K}[[\phi_1(t), \ldots, \phi_n(t)]]$ and $m_{2,1}$ is the multiplicity of the algebroid branch associated to $S_2 = \mathbb{K}[[\psi_1(u), \ldots, \psi_n(u)]]$. It is easy to show that we have

- $m_{1,1} = \min\{\text{ord}(\phi_i(t) - \phi_i(0)) : i = 1, \dots, n\}$;
- $m_{2,1} = \min\{\text{ord}(\psi_i(u) - \psi_i(0)) : i = 1, \dots, n\}$.

Then we can denote by x_S^1 an element of S_1 with order $m_{1,1}$ and by x_S^2 an element of S_2 with order $m_{2,1}$. It is clear that there exist i, j such that $x_S^1 = \phi_i(t) - \phi_i(0)$ and $x_S^2 = \psi_j(u) - \psi_j(0)$.

Now we want to develop an algorithm for the computation of the Arf closure ${}^*R$ of R. As we have seen in the previous section, we need to compute the blow-up chain R_m of R in order to find the multiplicity tree of ${}^*R$. In particular we have to find an integer N such that $R_N = \mathbb{K}[[t]] \times \mathbb{K}[[u]]$. From the properties of the ring of formal power series this is equivalent to find an N such that R_N is not local and such that

$$\text{mult}^*(R_N) = \{(1,0), (0,1)\}\,.$$

We can consider the following algorithm.

input : $R = \mathbb{K}[[(\phi_1(t), \psi_1(u)), \dots, (\phi_n(t), \psi_n(u))]]$
output: The sequence R_q of blow-ups of R until $R_q = \mathbb{K}[[t]] \times \mathbb{K}[[u]]$

$m \longleftarrow 1$
$R_1 \longleftarrow R$
while *mult*$^*(R_q) \neq \{(1,0), (0,1)\}$ **do**
  if R_q *is local* **then**
    $q \longleftarrow q + 1$
    $R_q \longleftarrow [(x_{R_{q-1}})^{-1} R_{q-1}]$
  end
  if $R_q = R_q^1 \times R_q^2$ *is not local* **then**
    $q \longleftarrow q + 1$
    $R_q \longleftarrow \left[(x^1_{R_{q-1}})^{-1} R^1_{q-1}\right] \times \left[(x^2_{R_{q-1}})^{-1} R^2_{q-1}\right]$
  end
end
return $R_1, R_2, \dots, R_q$

Algorithm 3:

The algorithm produces the blow-up chain because we know that in the local case we have $R_q = \left[(x_{R_{q-1}})^{-1} R_{q-1}\right]$ and we know that in this case a parametrization for R_q is given by

$$R_q = \mathbb{K}\left[\left[\frac{\left(\phi_1^{(q-1)}(t), \psi_1^{(q-1)}(u)\right)}{x_{R_{q-1}}}, \dots, \frac{\left(\phi_n^{(q-1)}(t), \psi_n^{(q-1)}(u)\right)}{x_{R_{q-1}}}, x_{R_{q-1}}\right]\right].$$

On the other hand, if R_{q-1} is not local we have that

$$R_{q-1} = \mathbb{K}[[\phi_1^{(q-1)}(t), \dots, \phi_{n(q-1)}^{(q-1)}(t)]] \times \mathbb{K}[[\psi_1^{(q-1)}(u), \dots, \psi_{n(q-1)}^{(q-1)}(u)]],$$

therefore in order to find R_q we can apply the algorithm for the branch case to each component of the Cartesian product finding $R_q = \left[(x^1_{R_{q-1}})^{-1} R^1_{q-1}\right] \times \left[(x^2_{R_{q-1}})^{-1} R^2_{q-1}\right]$ which can be computed as

$$R_q = \mathbb{K}\left[\left[\frac{\phi_1^{(q-1)}(t)}{x^1_{R_{q-1}}}, \dots, \frac{\phi_{n(q-1)}^{(q-1)}(t)}{x^1_{R_{q-1}}}, x^1_{R_{q-1}}\right]\right] \times \mathbb{K}\left[\left[\frac{\psi_1^{(q-1)}(u)}{x^2_{R_{q-1}}}, \dots, \frac{\psi_{n(q-1)}^{(q-1)}(u)}{x^2_{R_{q-1}}}, x^2_{R_{q-1}}\right]\right].$$

So, because at each step we know a parametrization for the q-th blow-up we have a way to compute the $q+1$-th one and we can stop when we reach $\text{mult}^*(R_q) = \{(1,0),(0,1)\}$.

Remark 16. In the previous algorithm,we divide by an element of minimal valuation, considering element of the type $\frac{(\phi(t), \psi(u))}{x}$. It is convenient to work with such an element as a fraction (cancelling if possible the common factors between the numerator and the denominator) . In this way we can still express it by a finite set of information avoiding the problem of expanding it in power series.

When the algorithm stops, we are able to build the multiplicity tree T of ${}^*R$. It will be a multiplicity tree of an Arf semigroup of $\mathbb{N}^2$, therefore it can be represented by a collection $E = \{m_1, m_2\}$ of two multiplicity sequences and an integer p_1, where p_1 is the highest level where the two branches of T are still glued. To find p_1 we have to check the first h such that, in our algorithm, we obtain that R_h is not local. Then we have $p_1 = h - 1$.

Furthermore, if $R_1 = R, R_2, \dots, R_q$ is the output of the algorithm we have that:

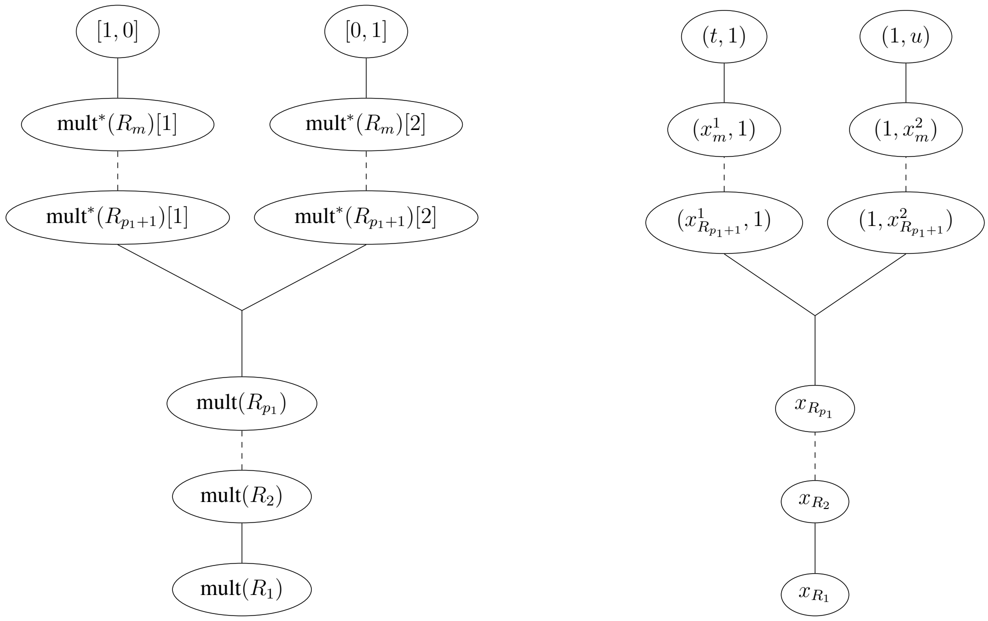
$$m_{1,i} = \text{mult}(R_i)[1] \text{ for } i = 1, \dots, p_1 \text{ and } m_{1,i} = (\text{mult}^*(R_i)[1])[1] \text{ for } i = p_1 + 1, \dots, N;$$

$$m_{2,i} = \text{mult}(R_i)[2] \text{ for } i = 1, \dots, p_1 \text{ and } m_{2,i} = (\text{mult}^*(R_i)[2])[2] \text{ for } i = p_1 + 1, \dots, N.$$

Remark 17. The multiplicity sequences m_1 and m_2 can be also found by using the algorithm for the branch case, applied to the rings

$$R^1 = \mathbb{K}[[\phi_1(t), \dots, \phi_n(t)]] \text{ and } R^2 = \mathbb{K}[[\psi_1(u), \dots, \psi_n(u)]].$$

In the following image we have the multiplicity tree and the minimal tree of R.



Notice that the algorithm computes all the tools needed to construct the previous two trees. If the tree T of ${}^*R$ is represented by the matrix $M(T)_E = \begin{pmatrix} 0 & p_1 \\ 0 & 0 \end{pmatrix}$ with $E = \{m_1, m_2\}$, the conductor of the associated Arf semigroup is $c = (c[1], c[2])$ with

$$c[i] = \sum_{k=1}^{\max(l(m_i),p_1)} m_{i,k}.$$

Thus we have that $(t^{c[1]}, u^{c[2]}) \cdot (\mathbb{K}[t] \times \mathbb{K}[u]) \subseteq {}^*R$.

Discussion 4.2.2. Now we want to find a method to compute a presentation of the Arf closure. In the previous section, we have seen how to construct it recursively. In the two-branches case we have that:

$$\begin{aligned}
&{}^*R_i = \mathbb{K}(1,1) + x_{R_i}{}^*R_{i+1} \quad \text{for } i = 1, \dots, p_1\\
&{}^*R_{p_1+1} = {}^*R^1_{p_1+1} \times {}^*R^2_{p_1+1}\\
&R^1_i = \mathbb{K}[[t]] \quad \text{for } i > \max\{l(m_1), p_1\}\\
&R^2_i = \mathbb{K}[[u]] \quad \text{for } i > \max\{l(m_2), p_1\}.
\end{aligned}$$

If $\max\{l(m_j), p_1\} > p_1 \quad {}^*R^j_i = \mathbb{K} + x^j_{R_i}{}^*R^j_{i+1} \qquad \text{for } i = p_1+1, \dots, \max(l(m_j), p_1);\ j = 1, 2.$

If we denote by $d_j = \max(l(m_j), p_1)$, by substituting the expression in the reverse order we find that:

$$^*R^1_{d_1} = \mathbb{K} + x^1_{R_{d_1}}\mathbb{K}[[t]]; \qquad ^*R^2_{d_2} = \mathbb{K} + x^2_{R_{d_2}}\mathbb{K}[[u]];$$
$$^*R^1_{d_1-1} = \mathbb{K} + x^1_{R_{d_1-1}}\mathbb{K} + x^1_{R_{d_1-1}}x^1_{R_{d_1}}\mathbb{K}[[t]]; \qquad ^*R^2_{d_2-1} = \mathbb{K} + x^2_{R_{d_2-1}}\mathbb{K} + x^2_{R_{d_2-1}}x^2_{R_{d_2}}\mathbb{K}[[u]];$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$^*R^1_{p_1+1} = \mathbb{K} + x^1_{R_{p_1+1}}\mathbb{K} + x^1_{R_{p_1+1}}x^1_{R_{p_1+2}}\mathbb{K} + \dots + x^1_{R_{p_1+1}}x^1_{R_{p_1+2}}\dots x^1_{R_{d_1}}\mathbb{K}[[t]];$$
$$^*R^2_{p_1+1} = \mathbb{K} + x^2_{R_{p_1+1}}\mathbb{K} + x^2_{R_{p_1+1}}x^2_{R_{p_1+2}}\mathbb{K} + \dots + x^2_{R_{p_1+1}}x^2_{R_{p_1+2}}\dots x^2_{R_{d_2}}\mathbb{K}[[u]];$$

and

$$^*R_{p_1+1} = {^*R^1_{p_1+1}} \times {^*R^2_{p_1+1}};$$
$$^*R_{p_1} = \mathbb{K}(1,1) + x_{R_{p_1}}{^*R^1_{p_1+1}} \times {^*R^2_{p_1+1}};$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$^*R = \mathbb{K}(1,1) + x_{R_1}\mathbb{K} + \dots + x_{R_{p_1}}x_{R_{p_1-1}}\dots x_{R_1}({^*R^1_{p_1+1}} \times {^*R^2_{p_1+1}}).$$

Finally, comparing the last two relations, we obtain

$$^*R = \mathbb{K}(1,1) + x_{R_1}\mathbb{K} + \dots +$$
$$+ x_{R_{p_1}}x_{R_{p_1-1}}\dots x_{R_1}\left[(\mathbb{K} + \dots + x^1_{R_{p_1+1}}\dots x^1_{R_{d_1}}\mathbb{K}[[t]]) \times (\mathbb{K} + \dots + x^2_{R_{p_1+1}}\dots x^2_{R_{d_2}}\mathbb{K}[[u]])\right].$$

Developing the Cartesian product, we find:

$$^*R = \mathbb{K}(1,1) + x_{R_1}\mathbb{K} + \dots + x_{R_{p_1}}\dots x_{R_1}\mathbb{K} + x_{R_{p_1}}\dots x_{R_1}(1, x^2_{R_{p_1+1}})\mathbb{K} + \dots +$$
$$+ x_{R_{p_1}}\dots x_{R_1}(1, x^2_{R_{p_1+1}}\dots x^2_{R_{d_2}})(\mathbb{K} \times \mathbb{K}[[u]]) + x_{R_{p_1}}\dots x_{R_1}(x^1_{R_{p_1+1}}, 1)\mathbb{K} + \dots +$$
$$+ x_{R_{p_1}}\dots x_{R_1}(x^1_{R_{p_1+1}}, x^2_{R_{p_1+1}}\dots x^2_{R_{d_2}})(\mathbb{K} \times \mathbb{K}[[u]]) + \dots +$$
$$+ x_{R_{p_1}}\dots x_{R_1}(x^1_{R_{p_1+1}}\dots x^1_{R_{d_1}}, 1)(\mathbb{K}[[t]] \times \mathbb{K}) + \dots + (t^{c[1]}, u^{c[2]}) \cdot (\mathbb{K}[[t]] \times \mathbb{K}[[u]]),$$

because

$$x_{R_{p_1}}\dots x_{R_1}(x^1_{R_{p_1+1}}\dots x^1_{R_{d_1}}, x^2_{R_{p_1+1}}\dots x^2_{R_{d_2}}) \cdot (\mathbb{K}[[t]] \times \mathbb{K}[[u]]) = (t^{c[1]}, u^{c[2]}) \cdot (\mathbb{K}[[t]] \times \mathbb{K}[[u]]).$$

Notice that the elements with valuation greater than the conductor can be erased. We observe that the elements in the expression have all different valuation and each of them has valuation corresponding to an element in $\nu(R)$ that is not greater than the conductor.
The elements with valuation not smaller than the conductor have to belong to the set $(t^{c[1]}, y) \cdot (\mathbb{K}[[t]] \times \mathbb{K})$ with $ord(y) < c[2]$ or $(z, u^{c[2]}) \cdot (\mathbb{K} \times \mathbb{K}[[u]])$ with $ord(z) < c[1]$.
Each element of the set $(t^{c[1]}, y) \cdot (\mathbb{K}[[t]] \times \mathbb{K})$ can be written as a sum of an element in $(0, y)\mathbb{K}$ and an element of $(t^{c[1]}, u^{c[2]}) \cdot (\mathbb{K}[[t]] \times \mathbb{K}[[u]])$. Similarly each element of the set $(z, u^{c[2]}) \cdot (\mathbb{K} \times \mathbb{K}[[u]])$ can be written as a sum of an element in $(z, 0)\mathbb{K}$ and an element of $(t^{c[1]}, u^{c[2]}) \cdot$

$(\mathbb{K}[[t]] \times \mathbb{K}[[u]])$.
If we define

$$Y^0 = \{(y,z) \in {}^*R : v((y,z)) < c\};$$
$$Y^1 = \{(0,y) \in {}^*R : ord(y) < c[2]\};$$
$$Y^2 = \{(z,0) \in {}^*R : ord(z) < c[1]\};$$
$$Y := Y_0 \cup Y_1 \cup Y_2,$$

we have a presentation of the type

$${}^*R = \mathbb{K}(1,1) + \mathbb{K}y_1 + \cdots + \mathbb{K}y_k + (t^{c[1]}, u^{c[2]}) \cdot (\mathbb{K}[[t]] \times \mathbb{K}[[u]]),$$

where the elements y_i belong to Y and we have one and only one representative for each valuation not greater than the conductor. In other words

$$\nu(y_i + (t^{c[1]}, u^{c[2]})) \in \text{Small}(\nu({}^*R)) \text{ for all } i.$$

Recall that from the properties of the multiplicity tree of an Arf semigroup, it follows that an element v of Small$(\nu({}^*R))$ can be obtained as the sum of the nodes of a subtree of $T(R)$ rooted in mult(R) and contained in the subtree that gives the conductor.
Then it is easy to find an element y with valuation $v \in$ Small$(\nu({}^*R))$. It suffices to consider the corresponding subtree in the minimal tree of Small$(\nu({}^*R))$ and multiply all its nodes. We suppose that $s_1, \dots, s_k$ are the elements of ${}^*R$ such that

$$\{\nu(s_1), \dots, \nu(s_k), c\} = \text{Small}(\nu({}^*R)),$$

if we consider the elements $\overline{s_1}, \dots, \overline{s_k}$, obtained by truncating the monomials of degree bigger that the corresponding component of the conductor, it is easy to see that they are the elements y_i that we were looking for.

Example 4.2.3. Consider

$$R = R_1 = \mathbb{K}[[(t^5 + t^{10}, u^7), (t^8, u^{11} + u^{13})]].$$

We have mult$(R_1) = (5,7)$. We can choose $x_{R_1} = (t^5 + t^{10}, u^7)$ as an element of minimal value in R_1. Therefore we have

$$R_2 = \mathbb{K}\left[\left[x_{R_1} = (t^5 + t^{10}, u^7), \frac{(t^8, u^{11} + u^{13})}{x_{R_1}}\right]\right] = \mathbb{K}\left[\left[(t^5 + t^{10}, u^7), \left(\frac{t^3}{1+t^5}, u^4 + u^6\right)\right]\right].$$

R_2 is still local and we have mult$(R_2) = (3,4)$. We can choose $x_{R_2} = \left(\frac{t^3}{1+t^5}, u^4 + u^6\right)$. Thus we have

$$R_3 = \mathbb{K}\left[\left[x_{R_2} = \left(\frac{t^3}{1+t^5}, u^4 + u^6\right), \frac{(t^5 + t^{10}, u^7)}{x_{R_2}}\right]\right] =$$

$$= \mathbb{K}\left[\left[\left(\frac{t^3}{1+t^5}, u^4+u^6\right), \left(t^2(1+t^5)^2, \frac{u^3}{1+u^2}\right)\right]\right].$$

R_3 is still local and we have $\mathrm{mult}(R_3) = (2,3)$. We can choose $x_{R_3} = \left(t^2(1+t^5)^2, \frac{u^3}{1+u^2}\right)$. Thus we have

$$R_4 = \mathbb{K}\left[\left[x_{R_3} = \left(t^2(1+t^5)^2, \frac{u^3}{1+u^2}\right), \left(\frac{t}{(1+t^5)^3}, u(1+u^2)^2\right)\right]\right].$$

R_4 is still local and we have $\mathrm{mult}(R_4) = (1,1)$. We can choose $x_{R_4} = \left(\frac{t}{(1+t^5)^3}, u(1+u^2)^2\right)$. Thus we have

$$R_5 = \mathbb{K}\left[\left[x_{R_4} = \left(\frac{t}{(1+t^5)^3}, u(1+u^2)^2\right), \left(t(1+t^5)^5, \frac{u^2}{(1+u^2)^3}\right)\right]\right].$$

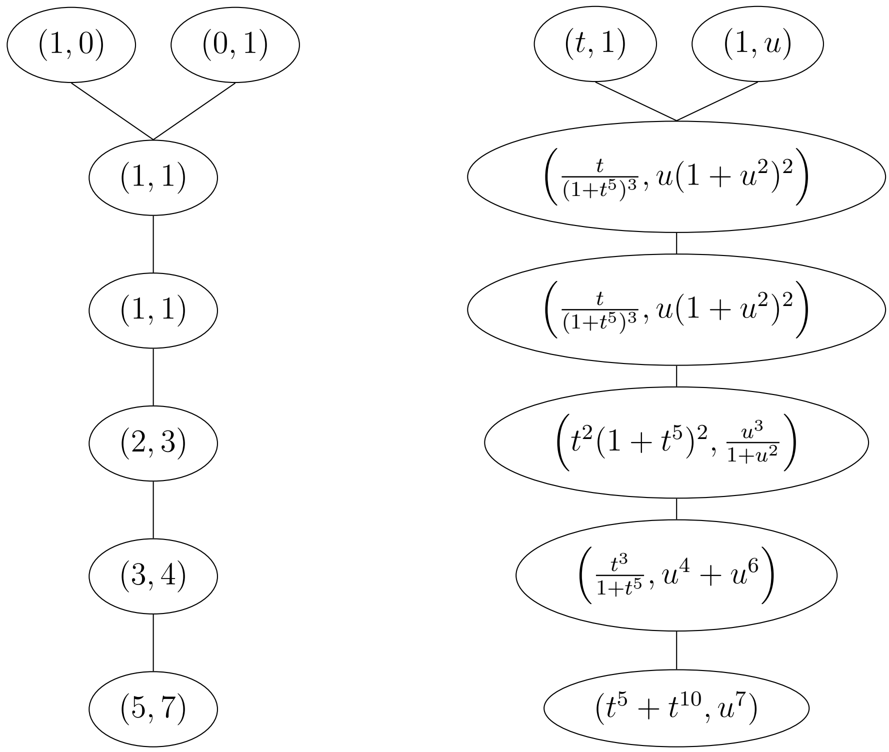
R_5 is still local and we have $\mathrm{mult}(R_5) = (1,1)$. We can choose again $x_{R_5} = \left(\frac{t}{(1+t^5)^3}, u(1+u^2)^2\right)$. Thus we have

$$R_6 = \mathbb{K}\left[\left[x_{R_5} = \left(\frac{t}{(1+t^5)^3}, u(1+u^2)^2\right), \left((1+t^5)^8, \frac{u}{(1+u^2)^5}\right)\right]\right].$$

This time, for the Lemma 4.2.1, we have that R_6 is not local because we have the element $\left((1+t^5)^8, \frac{u}{(1+u^2)^5}\right)$ with valuation $(0,1)$. We can write:

$$R_6 = \mathbb{K}\left[\left[\frac{t}{(1+t^5)^3}, (1+t^5)^8\right]\right] \times \mathbb{K}\left[\left[u(1+u^2)^2, \frac{u}{(1+u^2)^5}\right]\right] = K[[t]] \times K[[u]].$$

Thus we have $\mathrm{mult}^*(R_6) = \{(1,0),(0,1)\}$, and we can stop the algorithm. Then the multiplicity tree of ${}^*R$ and the minimal tree are:



The multiplicity tree T is $M(T)_E = \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$ where $E = \{m_1 = [5,3,2], m_2 = [7,4,3]\}$. We can easily see that conductor c of $\nu(^*R)$ is $c = (12,16)$. We can also compute $\mathrm{Small}(\nu(^*R))$ finding that

$$\mathrm{Small}(\nu(^*R)) = \{(5,7),(8,11),(10,14),(11,15),(12,16)\}\,.$$

Considering the expression of the elements of $\mathrm{Small}(\nu(^*R))$ as a sum of nodes in a subtree of T we can produce the following elements of $^*R$ as a product of the corresponding nodes on the minimal tree of $^*R$:

$$\left\{(t^5+t^{10},u^7),(t^8,u^{11}+u^{13}),(t^{10}(1+t^5)^2,u^{14}),\left(\frac{t^{11}}{1+t^5},u^{15}(1+u^2)^2\right),(t^{12},u^{16})\right\}.$$

Finally we have

$$^*R = \mathbb{K}(1,1)+\mathbb{K}(t^5+t^{10},u^7)+\mathbb{K}(t^8,u^{11}+u^{13})+\mathbb{K}(t^{10}(1+t^5)^2,u^{14})+\mathbb{K}\left(\frac{t^{11}}{1+t^5},u^{15}(1+u^2)^2\right)+$$

$$+(t^{12},u^{16})\left(\mathbb{K}[[t]]\times\mathbb{K}[[u]]\right) = \mathbb{K}(1,1)+\mathbb{K}(t^5+t^{10},u^7)+\mathbb{K}(t^8,u^{11}+u^{13})+\mathbb{K}(t^{10},u^{14})+\mathbb{K}\left(t^{11},u^{15}\right)+$$

$$+(t^{12},u^{16})\left(\mathbb{K}[[t]]\times\mathbb{K}[[u]]\right).$$

Notice that the fact that we know the conductor of $^*R$ allows us to simplify some of the elements corresponding to the small elements by truncating the terms that have order greater than the conductor.

4.3 The algorithm in the general case

In this section we explain how to generalize the algorithm presented in the previous one to algebroid curve with more than two branches. First of all we fix the notations. We want to find the Arf closure of the ring $R \subseteq \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$ with the following parametrization

$$R = R_1 = \mathbb{K}[[(\phi_{11}(t_1), \ldots, \phi_{1r}(t_r)), \ldots, (\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))]].$$

Similarly to the previous section, we will always replace an element of the parametrization $y = (\phi_{j1}(t_1), \ldots, \phi_{jr}(t_r))$ such that

$$\mathrm{ord}(\phi_{ji}(t_i)) = 0 \text{ and with } \phi_{j1}(0) = \phi_{ji}(0) \text{ for all } i = 1, \ldots, r,$$

with the element $\overline{y} = y - \phi_{j1}(0) \cdot (1, \ldots, 1)$.

To compute the Arf closure ${}^*R$, we have to find the sequence of blow-ups R_q of R. We will give an inductive algorithm for the computation of R_q.

We will denote by

$$R_q = \mathbb{K}\left[\left[\left(\phi_{11}^{(q)}(t_1), \ldots, \phi_{1r}^{(q)}(t_r)\right), \ldots, \left(\phi_{k(q)1}^{(q)}(t_1), \ldots, \phi_{k(q)r}^{(q)}(t_r)\right)\right]\right].$$

The rings R_q are semilocal subrings of $\mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$. We know, as explained in Section 1.4.1, that for a semilocal ring S there exists a partition $\mathfrak{P}(S) = \{P_1, \ldots, P_t\}$ of $\{1, \ldots, r\}$, with

$$P_i = \left\{q_{i,1}, \ldots, q_{i,k(i)}\right\},$$

such that $S \cong \prod_{i=1}^{t} \pi_{P_i}(S)$. Now we explain how to determine the partition $\mathfrak{P}(S)$.

If $i, j \in \{1, \ldots, r\}$ with $i \neq j$ we denote by $\pi_{i,j}$ the projection

$$\pi_{i,j} : \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]] \to \mathbb{K}[[t_i]] \times \mathbb{K}[[t_j]].$$

We have the following obvious Lemma:

Lemma 4.3.1. *Consider $S \subseteq \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$, semilocal ring. We define the equivalence relation $\sim$ on $\{1, \ldots, r\}$, such that $i \sim j$ if $i = j$ or if $\pi_{i,j}(S)$ is local in $\mathbb{K}[[t_i]] \times \mathbb{K}[[t_j]]$. Then $\mathfrak{P}(S)$ is the partition of $\{1, \ldots, r\}$ into equivalence classes with respect to $\sim$.*

If

$$S = \mathbb{K}[[(\phi_{11}(t_1), \ldots, \phi_{1r}(t_r)), \ldots, (\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))]],$$

then

$$\pi_{i,j}(S) = \mathbb{K}[[(\phi_{1i}(t_i), \phi_{1j}(t_j)), \ldots, (\phi_{ki}(t_i), \phi_{kj}(t_j))]];$$

since in the two branches case we know how to understand if a ring is local from its parametrization, we have the following algorithm to compute $\mathfrak{P}(S)$:

input : $S = \mathbb{K}[[(\phi_{11}(t_1), \ldots, \phi_{1r}(t_r)), \ldots, (\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))]]$
output: The partition $\mathfrak{P}(S)$

$N \longleftarrow \{1, \ldots, r\}$
for $i \in N$ **do**
  $P_i \longleftarrow \{i\}$
  for $j \in N_{>i}$ **do**
    if $\pi_{i,j}(S)$ *is local* **then**
      $P_i \longleftarrow P_i \cup \{j\}$
      $N \longleftarrow N \setminus \{j\}$
    end
  end
end
return $\mathfrak{P}(S) = \{P_1, P_{i_2}, \ldots, P_{i_t}\}$

Algorithm 4:

Now we can give an algorithm for computing the blow-up sequence of R. We will do it by working on induction on the number r of branches. We need to show a procedure to compute R_{q+1} from R_q.

- Base: $r = 2$.

  For $r = 2$ we have already seen, in the previous section, how to compute the R_q.

- Inductive step.

  We suppose that we are able to solve the problem for rings with less than q branches and we give a procedure for rings with exactly q branches.

  We have two cases:

  If R_q is local in $\overline{R}$ we consider x_{R_q}, the element of R_q such that $\nu(x_{R_q}) = \text{mult}(R_q)$ (we can find it as a linear combinations of the elements of the parametrization of R_q).

  Then we know that

$$R_{q+1} = \mathbb{K}\left[\left[x_{R_q}, \frac{\left(\phi_{11}^{(q)}(t_1), \ldots, \phi_{1r}^{(q)}(t_r)\right)}{x_{R_q}}, \ldots, \frac{\left(\phi_{k(q)1}^{(q)}(t_1), \ldots, \phi_{k(q)r}^{(q)}(t_r)\right)}{x_{R_q}}\right]\right].$$

  If R_q is not local then we have that there exist a partition $\mathfrak{P}(R_q) = \{P_{q,1}, \ldots, P_{q,t}\}$ such that

$$R_q = \prod_{i=1}^{t} \pi_{P_{q,i}}(R_q).$$

Notice that $\pi_{P_{q,i}}(R_q)$ can be computed from the parametrization of R_q and it is isomorphic to a local ring with less than r branches. Then for the inductive step we know how to compute the blowup $\mathrm{Bl}(\pi_{P_{q,i}}(R_q))$ and we have that:

$$R_{q+1} = \prod_{i=1}^{t} \mathrm{Bl}(\pi_{P_{j,i}}(R_q)).$$

Remark 18. It is clear that, with our definitions, we have

$$S = \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]] \iff \mathrm{mult}^*(S) = \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}.$$

So we have a procedure to find the first N such that $R_N = \mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]$. From this procedure we can find the sequence

$$\mathrm{multset}(R_1), \mathrm{multset}(R_2), \dots, \mathrm{multset}(R_N),$$

from which we can build the multiplicity tree of ${}^*R$ up to level N.

Once we know the multiplicity tree T and the minimal tree we are able to give an expression for the Arf closure ${}^*R$ using the strategy presented in the previous section. In fact we can compute the conductor c of the semigroup of values of the Arf closure and then use the correspondence between the small elements of the Arf semigroup $\nu({}^*R)$ and the elements of ${}^*R$ to find $\{s_1, \dots, s_l = c\} \subseteq {}^*R$ such that:

$${}^*R = \mathbb{K}(1,\dots,1) + \mathbb{K}s_1 + \ldots + \mathbb{K}s_{l-1} + (t_1^{c[1]}, \dots, t_r^{c[r]})\,(\mathbb{K}[[t_1]] \times \cdots \times \mathbb{K}[[t_r]]).$$

Example 4.3.2. We want to compute the Arf closure of the following ring

$$R = R_1 = \mathbb{K}[[(t^5 - t^8, u^2 + u^6, v^3, w^2 + w^9), (t^6, u^2 + u^7 + u^{10}, v^7 - v^9, w^2 + w^7)]].$$

In order to simplify the notation we will denote by R_i^j the local ring isomorphic to $\pi_{P_j}(R_i)$, and we denote by $x_{R_i^j}$ an element of minimal valuation in R_i^j, and by x_i^j the corresponding element of minimal valuation in $\pi_{P_j}(R_i)$. With an abuse of notation we also write

$$R_i = \prod_{j=1}^{t} \pi_{P_j}(R_i) \cong \prod_{j=1}^{t} R_i^j.$$

It is easy to verify that $\pi_{1,2}(R), \pi_{1,3}(R)$ and $\pi_{1,4}(R)$ are all local. Then, for Lemma 4.3.1, it follows that $\mathfrak{P}(R) = \{\{1,2,3,4\}\}$, therefore R is local.

We have that $\mathrm{mult}(R_1) = (5,2,3,2)$. As the minimal element x_{R_1} we can choose $x_{R_1} = (t^5 - t^8, u^2 + u^6, v^3, w^2 + w^9)$.

We have:

$$R_2 = \mathbb{K}\left[\left[x_{R_1} = (t^5 - t^8, u^2 + u^6, v^3, w^2 + w^9), \frac{(t^6, u^2 + u^7 + u^{10}, v^7 - v^9, w^2 + w^7)}{x_{R_1}}\right]\right] =$$

$$= \mathbb{K}\left[\left[(t^5-t^8, u^2+u^6, v^3, w^2+w^9), \left(\frac{t}{1-t^3}, \frac{1+u^5+u^8}{1+u^4}, v^4-v^6, \frac{1+w^5}{1+w^7}\right)\right]\right].$$

Now we can verify that $\pi_{1,2}(R_2)$ is not local, $\pi_{1,3}(R_2)$ is local, $\pi_{1,4}(R_2)$ is not local and $\pi_{2,4}(R_2)$ is local, therefore $\mathfrak{P}(R_2) = \{P_{2,1} = \{1,3\}, P_{2,2} = \{2,4\}\}$. We have

$$R_2 \cong R_2^1 \times R_2^2,$$

where

$$R_2^1 = \mathbb{K}\left[\left[(t^5-t^8, v^3), \left(\frac{t}{1-t^3}, v^4-v^6\right)\right]\right],$$

$$R_2^2 = \mathbb{K}\left[\left[(u^2+u^6, w^2+w^9), \left(\frac{1+u^5+u^8}{1+u^4}, \frac{1+w^5}{1+w^7}\right)\right]\right] =$$

$$= \mathbb{K}\left[\left[(u^2+u^6, w^2+w^9), \left(\frac{-u^4+u^5+u^8}{1+u^4}, \frac{w^5-w^7}{1+w^7}\right)\right]\right],$$

and, following our conventions on the parametrization, we replace $\left(\frac{1+u^5+u^8}{1+u^4}, \frac{1+w^5}{1+w^7}\right)$ with $\left(\frac{1+u^5+u^8}{1+u^4}, \frac{1+w^5}{1+w^7}\right) - (1,1) = \left(\frac{-u^4+u^5+u^8}{1+u^4}, \frac{w^5-w^7}{1+w^7}\right)$.

We have $\mathrm{mult}(R_2^1) = (1,3)$ and we can choose as element of minimal value the sum $x_{R_2^1}$ of its two generators

$$x_{R_2^1} = \left(\frac{t+t^5(1-t^3)^2}{1-t^3}, v^3+v^4-v^6\right),$$

while $\mathrm{mult}(R_2^2) = (2,2)$ and we can choose as element of minimal value $x_{R_2^2} = (u^2+u^6, w^2+w^9)$. Then, we have $\mathrm{multset}(R_2) = \{(1,0,3,0), (0,2,0,2)\}$ and we can proceed with the computation of R_3. Thus

$$R_3 \cong \mathrm{Bl}(R_2^1) \times \mathrm{Bl}(R_2^2),$$

so we have to compute $\mathrm{Bl}(R_2^1)$ and $\mathrm{Bl}(R_2^2)$.

We have

$$\mathrm{Bl}(R_2^1) = \mathbb{K}\left[\left[\left(\phi_1^{(3)}(t), \psi_1^{(3)}(v)\right), \dots, \left(\phi_3^{(3)}(t), \psi_3^{(3)}(v)\right)\right]\right],$$

where

- $\left(\phi_1^{(3)}(t), \psi_1^{(3)}(v)\right) = \left(\dfrac{t+t^5(1-t^3)^2}{1-t^3}, v^3+v^4-v^6\right)$;
- $\left(\phi_2^{(3)}(t), \psi_2^{(3)}(v)\right) = \left(\dfrac{t^4(1-t^3)^2}{1+t^4(1-t^3)^2}, \dfrac{1}{1+v-v^3}\right)$;
- $\left(\phi_3^{(3)}(t), \psi_3^{(3)}(v)\right) = \left(\dfrac{1}{1+t^4(1-t^3)^2}, \dfrac{v-v^3}{1+v-v^3}\right)$.

We notice that the second generator has valuation $(4,0)$, then $\mathrm{Bl}(R_2^1)$ is not local in $\mathbb{K}[[t]] \times \mathbb{K}[[v]]$. Furthermore we have, with our notation, that multset($\mathrm{Bl}(R_2^1)$) $= \{(1,0),(0,1)\}$ in $\mathbb{K}[[t]] \times \mathbb{K}[[v]]$. Then we have

$$\mathrm{Bl}(R_2^1) = \mathbb{K}[[t]] \times \mathbb{K}[[v]].$$

Now we can compute $\mathrm{Bl}(R_2^2)$. We have

$$\mathrm{Bl}(R_2^2) = \mathbb{K}\left[\left[\left(u^2+u^6, w^2+w^9\right), \left(\frac{-u^2+u^3+u^6}{(1+u^4)^2}, \frac{w^3-w^5}{(1+w^7)^2}\right)\right]\right].$$

Thus $\mathrm{Bl}(R_2^2)$ is local in $\mathbb{K}[[u]] \times \mathbb{K}[w]]$ and mult($\mathrm{Bl}(R_2^2)$) $= (2,2)$ in this ring. Then $\mathfrak{P}(R_3) = \{P_{3,1} = \{1\}, P_{3,2} = \{3\}, P_{3,3} = \{2,4\}\}$ and

$$R_3 \cong R_3^1 \times R_3^2 \times R_3^3 = \mathbb{K}[[t]] \times \mathbb{K}[[v]] \times \mathrm{Bl}(R_2^2),$$

where multset(R_3) $= \{(1,0,0,0),(0,0,1,0),(0,2,0,2)\}$. As a minimal element of $\mathrm{Bl}(R_2^2) \cong R_3^3$ we can choose again $x_{R_3^3} = (u^2+u^6, w^2+w^9)$. Thus

$$R_4 \cong \mathrm{Bl}(R_3^1) \times \mathrm{Bl}(R_3^2) \times \mathrm{Bl}(R_3^3) = \mathbb{K}[[t]] \times \mathbb{K}[[v]] \times \mathrm{Bl}(R_3^3).$$

We have:

$$\mathrm{Bl}(R_3^3) = \mathbb{K}\left[\left[\left(u^2+u^6, w^2+w^9\right), \left(\frac{-1+u+u^4}{(1+u^4)^3}, \frac{w-w^3}{(1+w^7)^3}\right)\right]\right].$$

From this it is easy to show that $\mathrm{Bl}(R_3^3) \cong \mathbb{K}[[u]] \times \mathbb{K}[[w]]$.
Then $\mathfrak{P}(R_4) = \{P_{4,1} = \{1\}, P_{4,2} = \{2\}, P_{4,3} = \{3\}, P_{4,4} = \{4\}\}$ and

$$R_4 = \mathbb{K}[[t]] \times \mathbb{K}[[u]] \times \mathbb{K}[[v]] \times \mathbb{K}[[w]],$$

and we have reached the stop condition for our algorithm. We found that $N = 4$ and

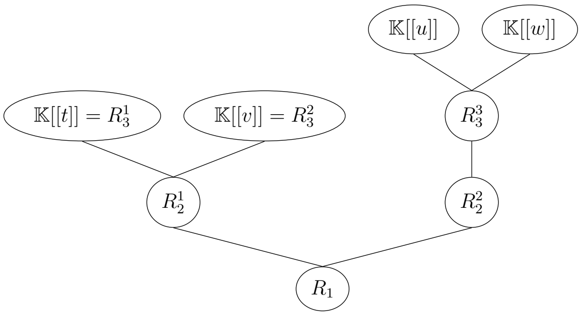
- multset(R_1) $= \{(5,2,3,2)\}$,
- multset(R_2) $= \{(1,0,3,0),(0,2,0,2)\}$,
- multset(R_3) $= \{(1,0,0,0),(0,0,1,0),(0,2,0,2)\}$,
- multset(R_4) $= \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

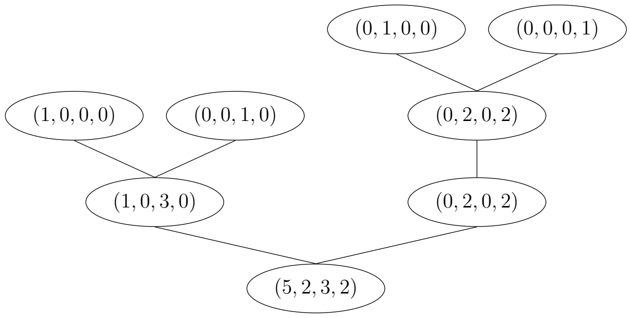
The corresponding minimal elements are:

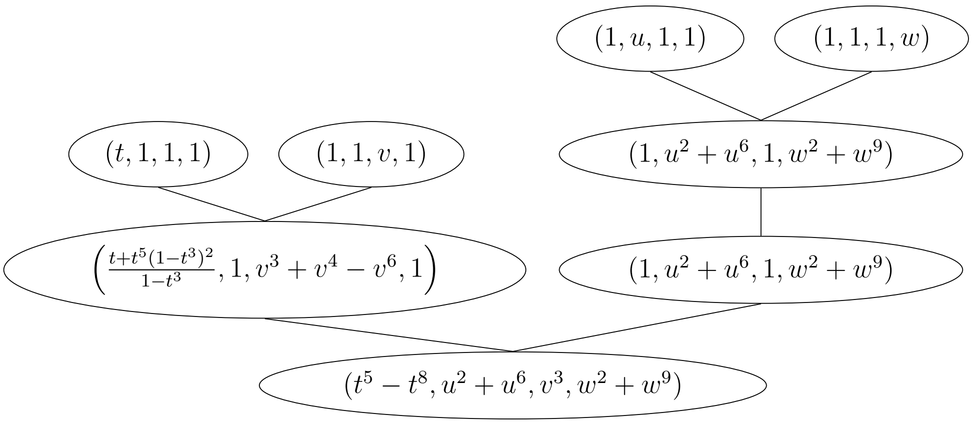
- $x_{R_1} = (t^5-t^8, u^2+u^6, v^3, w^2+w^9)$,
- $x_2^1 = \left(\frac{t+t^5(1-t^3)^2}{1-t^3}, 1, v^3+v^4-v^6, 1\right)$ and $x_2^2 = (1, u^2+u^6, 1, w^2+w^9)$;

- $x_3^1 = (t,1,1,1), x_3^2 = (1,1,v,1)$ and $x_3^3 = (1, u^2+u^6, 1, w^2+w^9)$;
- $x_4^1 = (t,1,1,1), x_4^2 = (1,u,1,1), x_4^3 = (1,1,v,1)$ and $x_4^4 = (1,1,1,w)$.

Then we have the following trees:







Then the multiplicity tree $T(R)$ of the Arf semigroup associated to ${}^*R$ is the tree described by the matrix

$$M(T(R))_E = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $E = \{[5],[2,2,2],[3,3],[2,2,2]\}$.

The conductor of $\nu({}^*R)$ is $c = (6,6,6,6)$, therefore

$$(t^6, u^6, v^6, w^6)\,(\mathbb{K}[[t]] \times \mathbb{K}[[u]] \times \mathbb{K}[[v]] \times \mathbb{K}[[w]]) \subseteq {}^*R.$$

We have that

$$\mathrm{Small}(\nu({}^*R)) = \{(5,2,3,2),(5,4,3,4),(5,6,3,6),(6,2,6,2),(6,4,6,4), c=(6,6,6,6)\}.$$

From the minimal tree we can recover the elements of ${}^*R$ with valuation belonging to Small$(\nu({}^*R))$. We can calculate the Arf closure truncating the terms with degree bigger than the conductor. So we obtain:

$$\left\{(t^5,u^2,v^3,w^2),(t^5,u^4,v^3,w^4),(t^5,0,v^3,0),(0,u^2,0,w^2),(0,u^4,0,w^4)\right\}.$$

Finally we have

$$\begin{gathered}{}^*R=\mathbb{K}(1,1,1,1)+\mathbb{K}(t^5,u^2,v^3,w^2)+\mathbb{K}(t^5,u^4,v^3,w^4)+\mathbb{K}(t^5,0,v^3,0)+\\ +\mathbb{K}(0,u^2,0,w^2)+\mathbb{K}(0,u^4,0,w^4)+(t^6,u^6,v^6,w^6)\left(\mathbb{K}[[t]]\times\mathbb{K}[[u]]\times\mathbb{K}[[v]]\times\mathbb{K}[[w]]\right).\end{gathered}$$

4.4 A bound for the series

In the previous sections, we have presented an algorithm that computes the Arf closure of an algebroid curve. The issues explained in Remark 1 for the algebroid branches occur also in this case, thus we would like to find a bound for the truncation of the series expansion in the parametrization, in order to improve the speed of the algorithm.

Remark 19. Using the bound of Arslan and Sahin (cf. Remark 1) on each branch would not solve our problem. In fact, although it guarantees the determination of the actual branches of the algebroid curve ${}^*R$, we can lose some important information on the splitting levels of its multiplicity tree.

Our strategy is based on the following theorem that generalizes the result of Arslan-Sahin to the case of two branches algebroid curves. Thus, in the following, we focus on the two branches case.

Let us fix some notation. Let R be a two-branches curve with parametrization

$$R=\mathbb{K}[[(\phi_1(t),\psi_1(u)),\dots,(\phi_n(t),\psi_n(u))]],$$

we call $c=(c[1],c[2])$ the conductor of $\nu({}^*R)$. Furthermore, we denote by $\overline{\phi_i(t)}$ and $\overline{\psi_i(u)}$ the formal power series obtained from $\phi_i(t)$ and $\psi_i(u)$ respectively by removing all monomials with degree greater than $c[1]+1$ and $c[2]+1$. Finally, we introduce:

$$\mathrm{Trunc}(R)=\mathbb{K}[[(\overline{\phi_1(t)},\overline{\psi_1(u)}),\dots,(\overline{\phi_n(t)},\overline{\psi_n(u)})]].$$

Theorem 4.4.1. *If we apply the algorithm to both R and Trunc(R) we obtain the same multiplicity tree.*

Proof Suppose that the multiplicity tree of ${}^*R$ is the tree T with $M(T)_E=\begin{pmatrix}0 & p_1\\ 0 & 0\end{pmatrix}$ where $E=\{m_1,m_2\}$. Consider an arbitrary element of the parametrization of R,

$$(\phi_i^{(1)}(t),\psi_i^{(1)}(u))=\left(\sum_{i\le c[1]+1}a_it^i+\sum_{i>c[1]+1}a_it^i,\ \sum_{i\le c[2]+1}b_iu^i+\sum_{i>c[2]+1}b_iu^i\right).$$

We denote by

$$(\chi_1^{(1)}(t), \chi_2^{(1)}(u)) = \left(\sum_{i>c[1]+1} a_i t^i, \sum_{i>c[2]+1} b_i u^i \right)$$

and

$$k = (k[1], k[2]) = (ord(\chi_1^{(1)}(t)), ord(\chi_2^{(1)}(u))) > (c[1]+1, c[2]+1).$$

Now, we want to follow the path of $\chi_1^{(1)}(t)$ and $\chi_2^{(1)}(u)$ in the algorithm in order to observe that by removing them from the parametrization, the result of the algorithm remains unchanged. We denote with $(\chi_1^{(i)}(t), \chi_2^{(i)}(u))$ the series obtained by $(\chi_1^{(1)}(t), \chi_2^{(1)}(u))$ at the i-th step of the algorithm.

To prove the thesis, it is necessary to prove that $(\chi_1^{(i)}(t), \chi_2^{(i)}(u))$ satisfies the following hypothesis at the i-th step:

i) $ord(\chi_1^{(i)}(t)) > m_{1,i}$ and $ord(\chi_2^{(i)}(u)) > m_{2,i}$;

ii) neither $ord(\chi_1^{(i)}(t))$ nor $ord(\chi_2^{(i)}(u))$ are 0.

If $i)$ is true we have that the monomials in $(\chi_1^{(i)}(t), \chi_2^{(i)}(u))$ are not involved in the choice of the minimal valuation elements at the i-th step. If $ii)$ is true they are not involved in the splits as consequence of Lemma 4.2.1.

So, if both hypothesis are true, the monomials in $(\chi_1^{(i)}(t), \chi_2^{(i)}(u))$ are not involved in the i-th step of the algorithm.

Recall that p_1 is the highest level were the branches in R are joined, then for all $i \leq p_1$, we have that:

$$\begin{aligned}
\nu(\chi_1^{(i)}(t), \chi_2^{(i)}(u)) &\geq (k[1] - m_{1,1} - \ldots - m_{1,i-1}, k[2] - m_{2,2}[1] - \ldots - m_{2,2}[i-1]) > \\
&> (c[1] + 1 - m_{1,1} - \ldots - m_{1,i-1}, c[2] + 1 - m_{2,1} - \ldots - m_{2,i-1}) = \\
&= \left(\sum_{j=1}^{\max(l_1,p_1)} m_{1,j} + 1 - \sum_{j=1}^{i-1} m_{1,j}, \sum_{j=1}^{\max(l_2,p_1)} m_{2,j} + 1 - \sum_{j=1}^{i-1} m_{2,j} \right) \geq \\
&\geq \left(\sum_{j=1}^{i} m_{1,j} + 1 - \sum_{j=1}^{i-1} m_{1,j}, \sum_{j=1}^{i} m_{2,j} + 1 - \sum_{j=1}^{i-1} m_{2,j} \right) = \\
&= (m_{1,i} + 1, m_{2,i} + 1) > (m_{1,i}, m_{2,i}) > (0,0).
\end{aligned}$$

So the hypothesis $i)$ and $ii)$ are satisfied for $\chi_1^{(i)}(t), \chi_2^{(i)}(u)$ with $i \leq p_1$. When $i > p_1$ the algorithm works individually on each branch, performing the computation of the Arf closure of an algebroid branch. Thus, because we have that $\chi_1^{(p_1+1)}(t)$ and $\chi_2^{(p_1+1)}(u)$ are elements with valuation strictly greater than the conductor of $\nu(R_1^{(p_1+1)})$ and $\nu(R_2^{(p_1+1)})$ respectively plus one, for the Arslan-Sahin theorem (cf.[2, Thm. 2.4]), $\chi_1^{(p_1+1)}(t), \chi_2^{(p_1+1)}(u)$ are not involved in the next steps of the algorithm and this concludes the proof. □

Remark 20. We want to point out that the previous theorem does not imply that the chains of blow-ups obtained applying the algorithm on R and $\text{Trunc}(R)$ are the same. In general, the parametrization of each blow-up and the minimal tree are different, but they are equal modulo $\langle t^{c[1]+2}, u^{c[2]+2}\rangle$ (when we truncate all the elements of degree greater than $c + \mathbf{1}$).

In the previous section, we have computed a presentation of the Arf closure starting by any minimal tree of the curve and it does not depends on the minimal tree chosen. For this reason we can enunciate the following obvious corollary.

Corollary 4.4.2. *R and Trunc(R) have the same Arf closure.*

From the previous corollary it follows that our new problem is to find a way to estimate the conductor of $\nu({}^*R)$ without actually knowing ${}^*R$.

Now we see how to do that by using the information given by the starting parametrization of R. Let us start by considering separately the two branches:

$$R^1 = \mathbb{K}[[\phi_1(t), \ldots, \phi_n(t)]] \qquad R^2 = \mathbb{K}[[\psi_1(u), \ldots, \psi_n(u)]].$$

We need to find the multiplicity sequences m_1 and m_2 of R^1 and R^2 respectively. Thus we compute the Arf closure of the branches (using the bound given by Arslan-Sahin we have an efficient way to do that).

Suppose that $m_1 \neq m_2$. In this case we have that the compatibility $\text{Comp}(m_1, m_2)$ is finite (recall that the compatibility is an upper bound for the splitting level p_1). If we set:

$$d_1 = \max\{l(m_1), \text{Comp}(m_1, m_2)\}, \qquad d_2 = \max\{l(m_2), \text{Comp}(m_1, m_2)\},$$

we have:

$$c[1] + 1 = \sum_{i=1}^{\max(l(m_1),p_1)} m_{1,i} + 1 \leq m_{1,1} + \ldots + m_{1,d_1} + 1,$$
$$c[2] + 1 = \sum_{i=1}^{\max(l(m_2),p_1)} m_{2,i} + 1 \leq m_{2,1} + \ldots + m_{1,d_2} + 1.$$

So, if we put:

$$b_1 = m_{1,1} + \ldots + m_{1,d_1} + 1,$$
$$b_2 = m_{2,1} + \ldots + m_{1,d_2} + 1,$$

as a consequence of Theorem 4.4.1, we can use the vector $b_O = (b_1, b_2)$ as a bound for the series expansions in the parametrization.

We have found a bound when $m_1 \neq m_2$ by only using the numeric properties of the multiplicity sequences. When $m_1 = m_2$ we cannot make assumptions on the split level by only using the m_i but we need to work directly on the parametrization in order to find a suitable bound.

Let us suppose that we have an algebroid curve with two branches such that $m_1 = m_2$. In this case we will do the following positions in order to simplify the notation. We denote with c_r the conductor of the branches R^1 and R^2 (in fact, in this case the two conductors are equal). We also set $l = l(m_1) = l(m_2)$. Now we define $Dis(1,2) = \{i \in \{1, \dots, n\} : \nu(\phi_i(t)) \neq \nu(\psi_i(u))\}$ and we call discrepancies the elements of this set. If $Dis(1,2) \neq \emptyset$, we define also

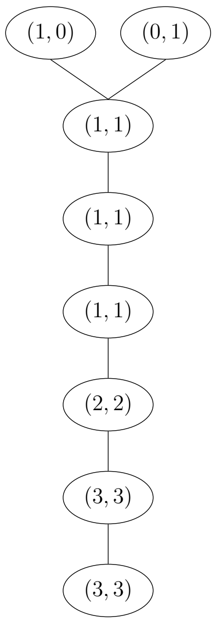
$$D = \min\{\min\{\nu(\phi_i(t)), \nu(\psi_i(u))\},\ i \in Dis(1,2)\}$$

which is the smallest order that causes a discrepancy.

Example 4.4.3. Let us consider the algebroid curve:

$$R = \mathbb{K}[[(t^3 + t^4, u^3 + u^7), (t^8 + t^9, u^8), (t^{12} + t^{15}, u^{13} + u^{14}), (t^{21}, u^{17} + u^{19})]].$$

The multiplicity tree associated to the ring is:



So we have: $m_1 = m_2$, $Dis(1,2) = \{3, 4\}$ and

$$D = \min\{\min\{12, 13\}, \min\{21, 17\}\} = \min\{12, 17\} = 12.$$

Lemma 4.4.4. *Let*

$$R = \mathbb{K}[[(\phi_1(t), \psi_1(u)), \dots, (\phi_n(t), \psi_n(u))]]$$

be an algebroid branch such that

i) $m_1 = m_2$ *;*

ii) $Dis(1,2) \neq \emptyset$.

Then we have $\max\{c_r, D\} \geq c[1] = c[2]$.

Proof From the definition of D, it follows that there exists an element of the type (D, x) in $\nu(R) \subseteq \nu(^*R)$ with $x > D$ (or equivalently of the type (y, D) with $y > D$). We know that there exists an integer k such that

$$D = \sum_{i=1}^{k} m_{1,i}.$$

Taking in account that the multiplicity tree $T(R)$ has two identical branches, it is easy to understand that $(D, x) \in \nu(^*R)$ with $x > D$ implies $p_1 \leq k$ (if we had $k < p_1$ the only possible element with valuation of the type (D, x) in $\nu(^*R)$ would be (D, D)). So we have

$$c[2] = c[1] = \sum_{i=1}^{\max(l(m_1),p_1)} m_{1,i} \leq \sum_{i=1}^{\max(l(m_1),k)} m_{1,i} = \max\{c_r, D\}.$$

$\square$

As a consequence of this theorem, we can take $b_D = (\max\{c_r, D\} + 1, \max\{c_r, D\} + 1)$ as a bound for an algebroid curve with $m_1 = m_2$ and $Dis(1, 2) \neq \emptyset$.

Now we only need to understand how to deal with the case of algebroid curves with $m_1 = m_2$ and $Dis(1, 2) = \emptyset$. In this case we have:

i) $m_1 = m_2$;

ii) $\nu(\phi_i(t)) = \nu(\psi_i(u))\ \forall i = 1, \ldots, n$.

Without loss of generality, we can rename the elements of the parametrization in order to have:

$$\nu(\phi_1(t), \psi_1(u)) \leq \nu(\phi_2(t), \psi_2(u)) \leq \ldots \leq \nu(\phi_n(t), \psi_n(u)).$$

Let $(\phi_i(t), \psi_i(u))$ be the first element with $i > 1$ such that at least one of the following holds

- $\phi_1(t) \neq \psi_1(t)$
- $\phi_i(t) \neq \psi_i(t)$,

(it must exist an element of this type because otherwise we would not have an algebroid curve with two branches). In this case we can always find $a, b, r, s \in \mathbb{N}$, such that

$$(\tilde{\phi}(t), \tilde{\psi}(u)) = a(\phi_1(t), \psi_1(u))^r + b(\phi_i(t), \psi_i(u))^s$$

with $ord(\tilde{\phi}(t)) > ord(\phi_1(t))$.

Now, let us consider

$$\tilde{R} = \mathbb{K}[[(\tilde{\phi}(t), \tilde{\psi}(u)), (\phi_2(t), \psi_2(u)), \ldots, (\phi_n(t), \psi_n(u))]],$$

and denote with $\tilde{c}$ the conductor of the Arf closure of $\tilde{R}$.
Since $\tilde{R} \subseteq R$, we have $c \leq \tilde{c}$. Now, if $\tilde{R}$ is an algebroid curve such that $\tilde{m_1} \neq \tilde{m_2}$ or $Dis(1,2) \neq \emptyset$, then we know how to find a bound for $\tilde{c}$.
Otherwise, we can apply the same idea starting by $\tilde{R}$ until we found an algebroid curve with a discrepancy for which we know to compute a bound; we will call this bound b_G. We note that this process necessarily produces a discrepancy since R is an algebroid curve that is not an algebroid branch.

Remark 21. We observe that it makes sense to compute b_G even when we have a discrepancy. A priori we do not know in this case which bound is better between b_D and b_G, so we will compute both of them and then we will choose the smaller one.

We will enunciate the following proposition that summarizes what we have seen above.

Proposition 4.4.5. *If R is an algebroid curve, c is the conductor of its Arf closure, m_1 and m_2 the multiplicity sequences of its branches, then the element*

$$b = \begin{cases} b_O & \textit{if} \quad m_1 \neq m_2; \\ \min\{b_D, b_G\} & \textit{if} \quad m_1 = m_2 \wedge Disc(1,2) \neq \emptyset; \\ b_G & \textit{if} \quad m_1 = m_2 \wedge Disc(1,2) = \emptyset, \end{cases}$$

is such that $b \geq (c[1]+1, c[2]+1)$.

As a consequence of the last proposition and Theorem 4.4.1, we have that b is a suitable bound for the algorithm.
Finally we show how the bound found in two-branches case can be used to determine a bound in the general case.

Remark 22. If R is an algebroid curve with r branches, parametrized by

$$R = \mathbb{K}[[(\phi_{11}(t_1), \ldots, \phi_{1r}(t_r)), \ldots, (\phi_{k1}(t_1), \ldots, \phi_{kr}(t_r))]].$$

We consider

$$\pi_{i,j}(R) = \mathbb{K}[[(\phi_{1i}(t_i), \phi_{1j}(t_j)), \ldots, (\phi_{ki}(t_i), \phi_{kj}(t_j))]],$$

the two-branch curve associated with the branches i and j for $i, j = 1, \ldots, r$, $i \neq j$.

We call $b_{\pi_{i,j}(R)} = (b_{\pi_{i,j}(R),i}, b_{\pi_{i,j}(R),j})$ the bound computed for the curve $\pi_{i,j}(R)$ where $b_{\pi_{i,j}(R),i}$ and $b_{\pi_{i,j}(R),j}$ are the components of the bound related to the branches i and j respectively. If we consider

$$b[i] = \max\{b_{\pi_{i,j}(R),i} \quad j = 1, \ldots, r, j \neq i\},$$

it is easy to observe that $b = (b[1], b[2], \ldots, b[r])$ is a suitable bound for the curve (because the general algorithm performs simultaneously the two-case one on each couple of branches).

***Remark* 23.** From Remark 22, it follows that a bound in the general case is still given by $c + \mathbf{1}$, where c is the conductor of the Arf closure of R. When the set of vectors

$$G = \{\nu((\phi_{11}(t_1), \dots, \phi_{1r}(t_r))), \dots, \nu((\phi_{k1}(t_1), \dots, \phi_{kr}(t_r)))\} \subseteq \mathbb{N}^r$$

satisfies the conditions of Theorem 2.2.1, we can speed up the computation of the bound. In fact we can compute the smallest Arf good semigroup S containing G that is in turn contained in $\nu(^*R)$. So $c(S) \geq c$ and we have a good bound for our procedure. However, as we show in the following example, the valuations of the elements in the parametrization are not forced to fulfil the aforementioned conditions. Furthermore, the bound obtained with this shortcut can be less accurate than the one obtained from the general strategy.

Example 4.4.6. We want to compute, using the truncation explained in the previous section, the Arf closure of the ring

$$R = R_1 = \mathbb{K}[[(t^5 - t^8, u^2 + u^6, v^3, w^2 + w^9), (t^6, u^2 + u^7 + u^{10}, v^7 - v^9, w^2 + w^7)]],$$

that appeared in Example 4.3.2.

If we use the algorithm of Arslan and Sahin to compute the Arf closure of the rings

$$R^1 = \mathbb{K}[[t^5 - t^8, t^6]], R^2 = \mathbb{K}[[u^2 + u^6, u^2 + u^7 + u^{10}]],$$
$$R^3 = \mathbb{K}[[v^3, v^7 - v^9]], R^4 = \mathbb{K}[[w^2 + w^9, w^2 + w^7]],$$

we find that the multiplicity tree T of $^*R$ belongs to $\tau(E)$, where

$$E = \{m_1 = [5], m_2 = [2, 2, 2], m_3 = [3, 3], m_4 = [2, 2, 2]\}.$$

We want to compute the bounds $b_{\pi_{ij}(R),i}$ with $i, j = 1, 2, 3, 4$, $i \neq j$. Since $b_{\pi_{ij}(R),i} = b_{\pi_{ji}(R),i}$ for all $i, j = 1, 2, 3, 4$, $i \neq j$, we can reduce to compute only $b_{\pi_{ij}(R),i}$ where $j > i$.

If $m_i \neq m_j$ we have seen that:

$$b_{\pi_{ij}(R),i} = \left(\sum_{k=1}^{\max(l(m_i),\mathrm{Comp}(m_i,m_j))} m_{i,k} \right) + 1 \text{ and } b_{\pi_{ij}(R),j} = \left(\sum_{k=1}^{\max(l(m_j),\mathrm{Comp}(m_i,m_j))} m_{j,k} \right) + 1.$$

We have:

- $\mathrm{Comp}(m_1, m_2) = 2 \Rightarrow$

$$b_{\pi_{1,2}(R),1} = \left(\sum_{k=1}^{\max(1,2)=2} m_{1,k} \right) + 1 = 5 + 1 + 1 = 7;$$
$$b_{\pi_{1,2}(R),2} = \left(\sum_{k=1}^{\max(3,2)=3} m_{2,k} \right) + 1 = 2 + 2 + 2 + 1 = 7.$$

- $\mathrm{Comp}(m_1, m_3) = 2 \Rightarrow$

$$\begin{aligned} b_{\pi_{1,3}(R),1} &= \left(\sum_{k=1}^{\max(1,2)=2} m_{1,k}\right) + 1 = 5 + 1 + 1 = 7; \\ b_{\pi_{1,3}(R),3} &= \left(\sum_{k=1}^{\max(2,2)=2} m_{3,k}\right) + 1 = 3 + 3 + 1 = 7. \end{aligned}$$

- $\mathrm{Comp}(m_1, m_4) = 2 \Rightarrow$

$$\begin{aligned} b_{\pi_{1,4}(R),1} &= \left(\sum_{k=1}^{\max(1,2)=2} m_{1,k}\right) + 1 = 5 + 1 + 1 = 7; \\ b_{\pi_{1,4}(R),4} &= \left(\sum_{k=1}^{\max(3,2)=3} m_{4,k}\right) + 1 = 2 + 2 + 2 + 1 = 7. \end{aligned}$$

- $\mathrm{Comp}(m_2, m_3) = 3 \Rightarrow$

$$\begin{aligned} b_{\pi_{2,3}(R),2} &= \left(\sum_{k=1}^{\max(3,3)=3} m_{2,k}\right) + 1 = 2 + 2 + 2 + 1 = 7; \\ b_{\pi_{2,3}(R),3} &= \left(\sum_{k=1}^{\max(2,3)=3} m_{3,k}\right) + 1 = 3 + 3 + 1 + 1 = 8. \end{aligned}$$

- $\mathrm{Comp}(m_3, m_4) = 3 \Rightarrow$

$$\begin{aligned} b_{\pi_{3,4}(R),3} &= \left(\sum_{k=1}^{\max(2,3)=3} m_{3,k}\right) + 1 = 3 + 3 + 1 + 1 = 8; \\ b_{\pi_{3,4}(R),4} &= \left(\sum_{k=1}^{\max(3,3)=3} m_{4,k}\right) + 1 = 2 + 2 + 2 + 1 = 7. \end{aligned}$$

We have $\mathrm{Comp}(m_2, m_4) = \infty$ because $m_2 = m_4 = [2, 2, 2]$, then to compute $b_{\pi_{2,4}(R)}$ we need to work on the parametrization of $\pi_{2,4}(R)$. We have:

$$\pi_{2,4}(R) = \mathbb{K}[[(u^2 + u^6, w^2 + w^9), (u^2 + u^7 + u^{10}, w^2 + w^7)]].$$

Both the generators of $\pi_{2,4}(R)$ have valuation $(2,2)$, therefore we have not discrepancies between the orders in the initial parametrization. So we have to produce an element of $\pi_{2,4}(R)$ with discrepancies by manipulating its generators. It suffices to take the difference between them, in fact we find:

$$\pi_{2,4}(R) \ni (u^2+u^6, w^2+w^9) - (u^2+u^7+u^{10}, w^2+w^7) = (u^6-u^7, -w^7+w^9),$$

with $\nu((u^6-u^7,-w^7+w^9)) = (6,7)$. Because $6 = \min(6,7)$ is less or equal than the conductor of $m_2 = [2,2,2]$ we can choose $b_{\pi_{2,4}(R)} = (6+1, 6+1) = (7,7)$.

Finally, denoting with $b[i]$ the bound on the i-th branch, we have:

- $b[1] = \max\left\{b_{\pi_{1,2}(R),1}, b_{\pi_{1,3}(R),1}, b_{\pi_{1,4}(R),1}\right\} = \max\{7,7,7\} = 7;$
- $b[2] = \max\left\{b_{\pi_{1,2}(R),2}, b_{\pi_{2,3}(R),2}, b_{\pi_{2,4}(R),2}\right\} = \max\{7,7,7\} = 7;$
- $b[3] = \max\left\{b_{\pi_{1,3}(R),3}, b_{\pi_{2,3}(R),3}, b_{\pi_{3,4}(R),3}\right\} = \max\{7,8,8\} = 8;$
- $b[4] = \max\left\{b_{\pi_{1,4}(R),4}, b_{\pi_{2,4}(R),4}, b_{\pi_{3,4}(R),4}\right\} = \max\{7,7,7\} = 7.$

Then on the i-th branch we can truncate all the terms with degree greater than $b[i]$ obtaining the new ring:

$$S = S_1 = \mathbb{K}[[(t^5, u^2+u^6, v^3, w^2), (t^6, u^2+u^7, v^7, w^2+w^7)]].$$

Let us show that ${}^*S = {}^*R$. We will use the same notations of Example 4.3.2.

It is easy to verify that $\pi_{1,2}(S)$, $\pi_{1,3}(S)$ and $\pi_{1,4}(S)$ are all local. Then for Lemma 4.3.1, it follows that $\mathfrak{P}(S) = \{\{1,2,3,4\}\}$, in other words S is local.

We have that $\mathrm{mult}(S_1) = (5,2,3,2)$. As the minimal value x_{S_1} we can choose $x_1 = (t^5, u^2+u^6, v^3, w^2)$.

We have:

$$\begin{aligned} S_2 &= \mathbb{K}\left[\left[(t^5, u^2+u^6, v^3, w^2), \frac{(t^6, u^2+u^7, v^7, w^2+w^7)}{x_{S_1}}\right]\right] = \\ &= \mathbb{K}\left[\left[(t^5, u^2+u^6, v^3, w^2), \left(t, \frac{1+u^5}{1+u^4}, v^4, 1+w^5\right)\right]\right]. \end{aligned}$$

Now we can verify that $\pi_{1,2}(S_2)$ is not local, $\pi_{1,3}(S_2)$ is local, $\pi_{1,4}(S_2)$ is not local and $\pi_{2,4}(S_2)$ is local, therefore $\mathfrak{P}(S_2) = \{P_{2,1} = \{1,3\}, P_{2,2} = \{2,4\}\}$. We have

$$S_2 \cong S_2^1 \times S_2^2,$$

where

$$S_2^1 = \mathbb{K}\left[\left[(t^5, v^3), (t, v^4)\right]\right],$$

$$S_2^2 = \mathbb{K}\left[\left[(u^2+u^6,w^2),\left(\frac{1+u^5}{1+u^4},1+w^5\right)\right]\right] =$$
$$= \mathbb{K}\left[\left[(u^2+u^6,w^2),\left(\frac{-u^4+u^5}{1+u^4},w^5\right)\right]\right].$$

where, following our conventions on the parametrization, we replace $\left(\frac{1+u^5}{1+u^4},1+w^5\right)$ with $\left(\frac{1+u^5}{1+u^4},1+w^5\right)-(1,1)=\left(\frac{-u^4+u^5}{1+u^4},w^5\right)$.

We have mult$(S_2^1)=(1,3)$ and we can choose as element of minimal value the sum $x_{S_2^1}$ of its two generators

$$x_{S_2^1} = \left(t+t^5, v^3+v^4\right),$$

while mult$(S_2^2)=(2,2)$ and we can choose as its minimal element $x_{S_2^2}=\left(u^2+u^6,w^2\right)$. Then we have multset$(S_2)=\{(1,0,3,0),(0,2,0,2)\}$ and we can proceed with the computation of S_3. Thus

$$S_2 \cong \mathrm{Bl}(S_2^1)\times \mathrm{Bl}(S_2^2),$$

so we have to compute Bl(S_2^1) and Bl(S_2^2).

We have

$$\mathrm{Bl}(S_2^1) = \mathbb{K}\left[\left[\left(\phi_1^{(3)}(t),\psi_1^{(3)}(v)\right),\dots,\left(\phi_3^{(2)}(t),\psi_3^{(2)}(v)\right)\right]\right],$$

where

- $\left(\phi_1^{(3)}(t),\psi_1^{(3)}(v)\right)=\left(t+t^5,v^3+v^4\right)$;
- $\left(\phi_2^{(3)}(t),\psi_2^{(3)}(v)\right)=\left(\dfrac{t^4}{1+t^4},\dfrac{1}{1+v}\right)$;
- $\left(\phi_3^{(3)}(t),\psi_3^{(3)}(v)\right)=\left(\dfrac{1}{1+t^4},\dfrac{v}{1+v}\right)$.

We notice that the second generator has valuation $(4,0)$, then Bl(S_2^1) is not local in $\mathbb{K}[[t]]\times\mathbb{K}[[v]]$. Furthermore we have multset(Bl(R_2^1)) $=\{(1,0),(0,1)\}$ in $\mathbb{K}[[t]]\times\mathbb{K}[[v]]$. Then we have

$$\mathrm{Bl}(S_2^1)=\mathbb{K}[[t]]\times\mathbb{K}[[v]].$$

Now we can compute Bl(S_2^2). We have

$$\mathrm{Bl}(S_2^2)=\mathbb{K}\left[\left[\left(u^2+u^6,w^2\right),\left(\frac{-u^2+u^3}{(1+u^4)^2},w^3\right)\right]\right].$$

Thus Bl(S_2^2) is local in $\mathbb{K}[[u]]\times\mathbb{K}[w]]$, and mult(Bl$(S_2^2)$) $=(2,2)$.

Then $\mathfrak{P}(S_3)=\{P_{3,1}=\{1\},P_{3,2}=\{3\},P_{3,3}=\{2,4\}\}$ and

$$S_3 \cong \mathbb{K}[[t]] \times \mathbb{K}[[v]] \times S_3^3,$$

with $\mathrm{multset}(S_3) = \{(1,0,0,0),(0,0,1,0),(0,2,0,2)\}$. As a minimal element of S_3^3 we can choose again $x_{S_3^3} = (u^2+u^6, w^2)$.

Thus

$$S_4 = \mathrm{Bl}(S_3^1) \times \mathrm{Bl}(S_3^2) \times \mathrm{Bl}(S_3^3) \cong \mathbb{K}[[t]] \times \mathbb{K}[[v]] \times \mathrm{Bl}(S_3^3).$$

We have:

$$\mathrm{Bl}(S_3^3) = \mathbb{K}\left[\left[\left(u^2+u^6, w^2\right), \left(\frac{-1+u}{(1+u^4)^3}, w\right)\right]\right].$$

From this it is easy to show that $\mathrm{Bl}(S_3^3) = \mathbb{K}[[u]] \times \mathbb{K}[[w]]$, hence

$$S_4 = \mathbb{K}[[t]] \times \mathbb{K}[[u]] \times \mathbb{K}[[v]] \times \mathbb{K}[[w]],$$

and we have reached the stop condition for our algorithm.

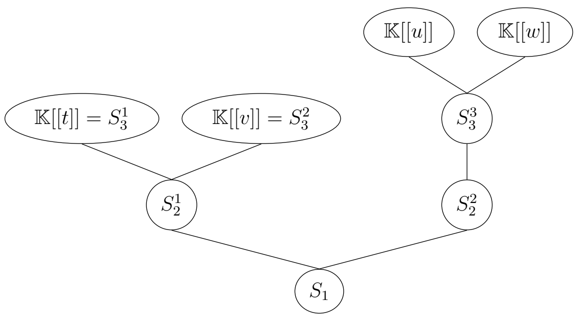
We found that $N = 4$ and

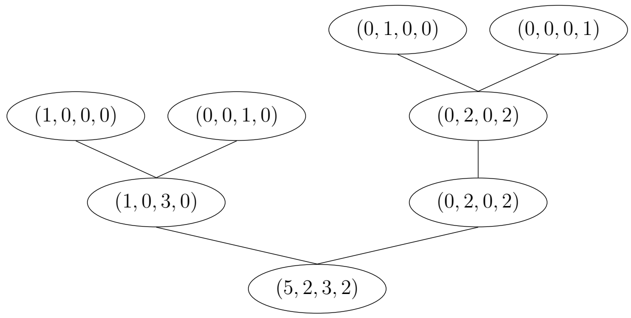
- $\mathrm{multset}(S_1) = \{(5,2,3,2)\}$,
- $\mathrm{multset}(S_2) = \{(1,0,3,0),(0,2,0,2)\}$,
- $\mathrm{multset}(S_3) = \{(1,0,0,0),(0,0,1,0),(0,2,0,2)\}$,
- $\mathrm{multset}(S_4) = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

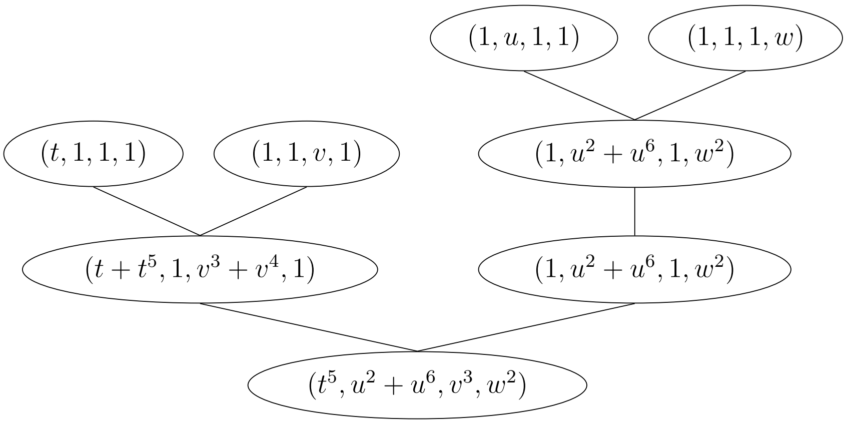
The corresponding minimal elements are:

- $x_{S_1} = (t^5, u^2+u^6, v^3, w^2)$,
- $x_2^1 = (t+t^5, 1, v^3+v^4, 1)$ and $x_2^2 = (1, u^2+u^6, 1, w^2)$;
- $x_3^1 = (t,1,1,1), x_3^2 = (1,1,v,1)$ and $x_3^3 = (1, u^2+u^6, 1, w^2)$;
- $x_4^1 = (t,1,1,1), x_4^2 = (1,u,1,1), x_4^3 = (1,1,v,1)$ and $x_4^4 = (1,1,1,w)$.

Then we have the following trees:







The conductor of $\nu({}^*S)$ is $c = (6,6,6,6)$ If we compare these tree with the tree computed starting by R in the Example 4.3.2, we can observe that the tree associated to the ring and the multiplicity tree are the same, while the minimal trees are equal module $c+1 = (7,7,7,7)$. Then we have $M(T(S))_E = M(T(R))_E$.
We have that

$$\begin{aligned}&\text{Small}(\nu({}^*S)) = \text{Small}(\nu({}^*R)) = \\ &= \{(5,2,3,2),(5,4,3,4),(5,6,3,6),(6,2,6,2),(6,4,6,4),c=(6,6,6,6)\}\,.\end{aligned}$$

From the minimal tree we can recover the elements of ${}^*S$ with valuation belonging to $\text{Small}(\mathcal{S}(T))$. We can calculate the Arf closure by truncating the terms with degree bigger than the conductor. They are:

$$\left\{(t^5,u^2,v^3,w^2),(t^5,u^4,v^3,w^4),(t^5,0,v^3,0),(0,u^2,0,w^2),(0,u^4,0,w^4)\right\}.$$

Finally we have

$${}^*S = {}^*R = \mathbb{K}(1,1,1,1) + \mathbb{K}(t^5,u^2,v^3,w^2) + \mathbb{K}(t^5,u^4,v^3,w^4) + \mathbb{K}(t^5,0,v^3,0) +$$

$$+\mathbb{K}(0,u^2,0,w^2) + \mathbb{K}(0,u^4,0,w^4) + (t^6,u^6,v^6,w^6)\left(\mathbb{K}[[t]]\times\mathbb{K}[[u]]\times\mathbb{K}[[v]]\times\mathbb{K}[[w]]\right).$$

Then, as expected, truncating the terms with valuation bigger than our bound did not change the output of our algorithm. Notice that the truncation can have a relevant impact on the speed of the computation, avoiding to take in account irrelevant terms when manipulating the parametrizations, as it is shown in the following output of the procedure implemented in GAP.

```
gap> R:=[[x_1^2+x_1^23,x_2^3+x_2^14,x_3^5+x_3^17],[x_1^4+x_1^21,x_2^6+x_2^30,x_3^6+x_3^22]];

gap> ArfClosurePresentation(R);
[ [ x_1^2, x_2^14+x_2^3, x_3^5 ], [ x_1^4, x_2^6, 0 ], [ x_1^6, x_2^9, 0 ], [ x_1^8, x_2^12, 0 ],
[ x_1^10, x_2^15, 0 ], [ x_1^12, 0, 0 ], [ x_1^14, 0, 0 ], [ x_1^16, 0, 0 ], [ x_1^18, 0, 0 ],
[ x_1^20, x_2^17, x_3^6 ] ]
gap> time;
249984
```

```
gap> ArfClosurePresentationWithTruncation(R);
[ [ x_1^2, x_2^14+x_2^3, x_3^5 ], [ x_1^4, x_2^6, 0 ], [ x_1^6, x_2^9, 0 ], [ x_1^8, x_2^12, 0 ],
[ x_1^10, x_2^15, 0 ], [ x_1^12, 0, 0 ], [ x_1^14, 0, 0 ], [ x_1^16, 0, 0 ], [ x_1^18, 0, 0 ],
[ x_1^20, x_2^17, x_3^6 ] ]
gap> time;
47
```

The function `ArfClosurePresentation` takes in input a list of polynomials Ψ and computes the Arf closure of the algebroid curve R parametrized by Ψ. It gives as an output a list of vectors $\{v_1, \ldots, v_t\}$ such that

$$^{*}R = \mathbb{K}(1,1,1,1) + \mathbb{K}v_1 + \mathbb{K}v_2 + \cdots + v_t\left(\mathbb{K}[[x_1]] \times \mathbb{K}[[x_2]] \times \cdots \times \mathbb{K}[[x_r]]\right).$$

The function `ArfClosurePresentationWithTruncation` works in the same way as `ArfClosurePresentation` but it applies the truncation explained in the previous section and it is considerably faster than the latter.

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