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From $(P.3), (P.4)$ and (9.3.29) we get

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dx \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n) \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^2 + \int_{-b}^{0} |u|^2_{m,B(3\sigma)} dt \right\}.$$

The last inequality and (9.3.25) allows us to conclude the proof. $\square$

applying Theorem 7.2.1, it follows

$$u \in L^2(-a, 0, H^{m+1}(B(\sigma), \mathbb{R}^N))$$

and

$$\int_{-a}^{0} |u|^2_{m+1,B(\sigma)} dt \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n) \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|^2_{m,B(3\sigma)}\, dt \right\}. \quad (9.3.25)$$

Finally we have to prove that $u \in H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N))$ and inequality (9.2.5). From inequality (9.2.3) we have

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|D''u\|^4 dx \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n) \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|^2_{m,B(3\sigma)} dt \right\} \quad (9.3.26)$$

then we have

$$D''u \in L^4(B(\sigma) \times (-a, 0), \mathcal{R}''). \quad (9.3.27)$$

Moreover, bearing in mind that, for $|\alpha| < m$, $a^\alpha(X, p)$ satisfies $(P.3)$, and for $|\alpha| = m$, $a^\alpha(X, p)$ satisfies $(P.4)$, we have

$$D^\alpha a^\alpha(X, p) \in L^2 \left(B(\sigma) \times (-a, 0), R^N \right) \qquad \forall \alpha : |\alpha| \leq m \quad (9.3.28)$$

Recalling the definition of weak solution, for every $\varphi \in C_0^\infty(Q, \mathbb{R}^N)$, proceeding as in [24], we have

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left(u \middle| \frac{\partial \varphi}{\partial t} \right) dx = \sum_{|\alpha| \leq m} \int_{-a}^{0} dt \int_{B(\sigma)} (D^\alpha a^\alpha(X, Du) | \, \varphi \,)\, dx, \quad (9.3.29)$$

and, bearing in mind (9.3.28), we obtain that

$$\exists \, \frac{\partial u}{\partial t} \in L^2 \left(B(\sigma) \times (-a, 0), \mathbb{R}^N \right). \quad (9.3.30)$$

every $|h| < h_0,$ it follows

$$\int_{B(2\sigma)} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 dx \leq \left(\int_{B(2\sigma)} \|\tau_{i,h} D'u\|^4 dx\right)^{\frac{1}{2}} \left(\int_{B(2\sigma)} \|D''u\|^4 dx\right)^{\frac{1}{2}} \leq$$

$$\leq |h|^2 \| D''u \|^2_{0,4,B(\frac{5}{2}\sigma)} \| D''u \|^2_{0,4,B(2\sigma)} \leq |h|^2 \mid u \mid^4_{m,4,B(\frac{5}{2}\sigma)} .$$

Integrating in $(-b^*, 0)$, from (9.3.23) it follows

$$\int_{-a}^{0} dt \int_{B(\sigma)} \| \tau_{i,h} D''u \|^2 dx \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n)|h|^2 \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|^2_{m,B(3\sigma)} dt \right\}. \tag{9.3.24}$$

If $h_0 \leq |h| < \frac{\sigma}{2}$, for every $i = 1, 2, \dots, n$ we easily obtain

$$\int_{-a}^{0} dt \int_{B\sigma)} \|\tau_{i,h} D''u\|^2 dx \leq 4 \int_{-a}^{0} dt \int_{B(3\sigma)} \|D''u\|^2 dx \leq 4 \frac{h^2}{h_0^2} \int_{-a}^{0} dt \int_{B(3\sigma)} \|D''u\|^2 dx \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n) h^2 \int_{-b}^{0} |u|^2_{m,B(3\sigma)} dt \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n)|h|^2 \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|^2_{m,B(3\sigma)} dt \right\}.$$

It is then proved, for every $|h| < \frac{\sigma}{2}$ and every $i \in \{1, 2, \dots, n\}$, that

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|\tau_{i,h} D''u\|^2 dx \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, m, n) |h|^2 \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|^2_{m,B(3\sigma)} dt \right\},$$

Multiplying each term for ρ_μ^2 and integrating respect to $(-b^*, -\frac{1}{\mu})$ and applying (9.3.5), we achieve

$$\int_{-b^*}^{-\frac{1}{\mu}} \rho_\mu^2\, dt \int_{B(\frac{5}{2}\sigma)} \left(|f^\alpha| + \|D''u\|^2\right) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|\, dx \le$$

$$\le \frac{\nu}{4\, c(K,m,n)} \int_{-b^*}^{-\frac{1}{\mu}} \rho_\mu^2\ dt \int_{B(2\sigma)} \psi^{2m} \,\|\, \tau_{i,h}\, D''u \,\|^2\ dx +$$
$$+ c(\nu, K, U, \lambda, \sigma, a, b, m, n) h^2 \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b^*}^{-\frac{1}{\mu}} |u|^2_{m,B(3\sigma)}\, dt \right\}.$$

Taking into consideration the last inequality and the properties of the function ψ, from (9.3.21) we deduce

$$\int_{-a}^{-\frac{2}{\mu}} dt \int_{B(\sigma)} \|\tau_{i,h}\, D''u\|^2\ dx \le$$
$$\le c(\nu, K, U, \lambda, \sigma, a, b, m, n)\, h^2 \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b^*}^{-\frac{1}{\mu}} |u|^2_{m,B(3\sigma)}\, dt \right\} +$$
$$+ c(\nu, K, \sigma, m, n) \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \,\|\, \tau_{i,h} D'\, u \,\|^2 \,\|\, D''u \,\|^2\ dx.$$

From which, passing the limit $\mu \to \infty$, we get

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|\tau_{i,h}\, D''u\|^2\ dx \le$$
$$\le c(\nu, K, \lambda, \sigma, m, n) h^2 \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b^*}^{0} |u|^2_{m,B(3\sigma)}\, dt \right\} +$$
$$+\ c(\nu, K, \sigma, m, n) \int_{-b^*}^{0} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \,\|\tau_{i,h} D'\, u\|^2 \,\|D''u\|^2\ dx. \quad (9.3.23)$$

Let us now estimate the last term in (9.3.23). Using Hölder inequality, applying Theorem 7.2.2 (for $p = 4$, $B(\frac{5}{2}\sigma)$ instead of $B(\sigma)$ and $t = \frac{4}{5}$) and formula (9.3.5), for

Let us focus our attention on the last term, taking into account that from (9.3.4), for
a. e. $t \in (-b^*, 0)$, we have

$$u(\cdot, t) \in H^{m,4}\left(B\left(\frac{5}{2}\sigma\right), \mathbb{R}^N\right)$$

then using Hölder and Young inequalities, for every α such that $|\alpha| < m$, for every $\varepsilon > 0$, it follows

$$\begin{aligned}
&\int_{B(\frac{5}{2}\sigma)} (|f^\alpha| + \|D''u\|^2) \, \|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\| \, dx \leq \\
&\leq \left(\int_{B(3\sigma)} |h|^{-2} \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 dx\right)^{\frac{1}{2}} \left(\int_{B(\frac{5}{2}\sigma)} h^2 (|f^\alpha| + \|D''u\|^2)^2 dx\right)^{\frac{1}{2}} \leq \\
&\leq \frac{\varepsilon}{2} |h|^{-2} \int_{B(3\sigma)} \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 dx + c(\varepsilon)\, h^2 \int_{B(\frac{5}{2}\sigma)} (|f^\alpha|^2 + \|D''u\|^4) \, dx.
\end{aligned}$$

Furthermore, for every α such that $|\alpha| < m$, from Theorem 7.2.2 for every $h \in \mathbb{R}$ with $|h| < h_0$ and for every $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{\varepsilon}{2} |h|^{-2} \int_{B(3\sigma)} \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 dx &\leq \frac{\varepsilon}{2} \int_{B(\frac{7}{2}\sigma)} \left\|D'' \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 dx \leq \\
&\leq \varepsilon \int_{B(2\sigma)} \psi^{2m} \, \|\, \tau_{i,h}\, D''u\,\|^2 \, dx + c(\sigma, \varepsilon) \int_{B(2\sigma)} \|\, \tau_{i,h} D'\, u\,\|^2 \, dx \leq \\
&\leq \varepsilon \int_{B(2\sigma)} \psi^{2m} \, \|\, \tau_{i,h}\, D''u\,\|^2 \, dx + c(\sigma, \varepsilon)\, h^2 \int_{B(3\sigma)} \|D''u\|^2 \, dx
\end{aligned}$$

the last inequality follows, as before, applying Theorem 7.2.2 for $p = 2$. Let us now choose $\varepsilon = \frac{\nu}{4\, c(K,m,n)}$, it ensures

$$\begin{aligned}
&\int_{B(\frac{5}{2}\sigma)} \left(|f^\alpha| + \|D''u\|^2\right) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\| \, dx \leq \\
&\qquad\leq \frac{\nu}{4\, c(K, m, n)} \int_{B(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 dx \\
&\qquad+ c(\nu, K, \sigma, m, n) h^2 \left\{\int_{B(3\sigma)} |f^\alpha|^2 dx + |u|^2_{m,B(3\sigma)} + |u|^4_{m,4,B(\frac{5}{2}\sigma)}\right\}.
\end{aligned} \tag{9.3.22}$$

Then, from (9.3.9) estimating the terms A, B, C, D and E, for every $\varepsilon > 0$, we have

$$\nu \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \, \|\tau_{i,h} D'' u\|^2 \, dx \leq$$
$$\leq \Big\{ 3\varepsilon + c(K, U, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda} \right) \Big\} \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \, \|\tau_{i,h} D'' u\|^2 dx +$$
$$+ c(K, \sigma, a, b, m, n, \varepsilon) \, h^2 \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(3\,\sigma)} \left(1 + \|D'' u\|^2 \right) \, dx + c(\sigma, a, b, n,) K h^2 +$$
$$+ \; c(K, \sigma, m, n, \varepsilon) \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \, \|\tau_{i,h} D' \, u\|^2 \, \|D'' u\|^2 \, dx +$$
$$+ \, c(K, m, n) \sum_{|\alpha| < m} \int_{-b^*}^{-\frac{1}{\mu}} \rho_\mu^2 dt \int_{B(\frac{5}{2}\sigma)} (|f^\alpha| + \|D'' u\|^2) \|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u \right) \| \, dx. \quad (9.3.20)$$

We observe that the function

$$h \longrightarrow c(K, U, \sigma, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda} \right)$$

is continuous in the origin, then $\exists \, h_0(\nu, K, U, \lambda, \sigma, m, n)$, $0 < h_0 < \min\{1, \frac{\sigma}{2}\}$, such that for every $|h| < h_0$, we have

$$c(K, U, \sigma, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda} \right) < \frac{\nu}{4}.$$

For each integer $i = 1, \ldots, n$, for $\varepsilon = \frac{\nu}{12}$ and every h such that $|h| < h_0 (< 1)$, it follows

$$\frac{\nu}{2} \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \, \|\tau_{i,h} D'' u\|^2 \, dx \leq$$
$$\leq c(\nu, K, \sigma, a, b, m, n) \, |h|^2 \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(3\,\sigma)} \left(1 + \|D'' u\|^2 \right) \, dx \; +$$
$$+ \; c(\nu, K, \sigma, m, n) \int_{-b^*}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m} \rho_\mu^2 \, \|\tau_{i,h} D' \, u\|^2 \, \|D'' u\|^2 \, dx \; +$$
$$+ \; c(K, m, n) \sum_{|\alpha| < m} \int_{-b^*}^{-\frac{1}{\mu}} \rho_\mu^2 dt \int_{B(\frac{5}{2}\sigma)} (|f^\alpha| \; + \|D'' u\|^2) \, \|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u \right) \| dx.$$
$$(9.3.21)$$

The term B can be estimated, for every $\varepsilon > 0$, as follows

$$|B| \leq \left\{\varepsilon + c(K,U,m,n)\left(|h|^{\lambda} + |h|^{2\lambda}\right)\right\} \int_{-b}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m}\rho_\mu^2 \left\|\tau_{i,h}D''u\right\|^2 dx +$$
$$+ c(K,\sigma,m,n,\varepsilon)h^2 \int_{-b}^{-\frac{1}{\mu}} dt \int_{B(3\sigma)} \left\|D''u\right\|^2 dx +$$
$$+ c(K,m,n,\varepsilon) \int_{-b}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m}\rho_\mu^2 \left\|\tau_{i,h}D'u\right\|^2 \left\|D''u\right\|^2 dx. \quad (9.3.16)$$

Let us consider the term C, for every $\varepsilon > 0$, we have

$$|C| \leq \left\{\varepsilon + c(K,m,n)\left(h^2 + |h|\right)\right\} \int_{-b}^{-\frac{1}{\mu}} dt \int_{B(2\sigma)} \psi^{2m}\rho_\mu^2 \left\|\tau_{i,h}D''u\right\|^2 dx +$$
$$+ c(K,\sigma,m,n,\varepsilon)\, h^2 \int_{-b}^{-\frac{1}{\mu}} dt \int_{B(3\sigma)} \left(1 + \left\|D''u\right\|^2\right) dx.$$

To estimate the term D, we firstly observe that

$$(\rho'_\mu \rho_\mu)(t) \begin{cases} = 0 & \text{if } t \leq -b,\ -a \leq t \leq -\frac{2}{\mu},\ t \geq -\frac{1}{\mu} \\ \leq \frac{1}{b-a} & \text{if } -b \leq t \leq -a \\ \leq 0 & \text{if } -\frac{2}{\mu} \leq t \leq -\frac{1}{\mu} \end{cases} \quad (9.3.17)$$

then, using Theorem 7.2.2, we obtain

$$D \;=\; \int_Q \psi^{2m}\, \rho'_\mu\, \rho_\mu \,\|\tau_{i,h}\, u\|^2 \, dX \;=$$
$$= \int_{-b}^{-a} dt \int_{B_{(2\sigma)}} \psi^{2m}\rho'_\mu\rho_\mu\|\tau_{i,h}u\|^2 dx + \int_{-\frac{2}{\mu}}^{-\frac{1}{\mu}} dt \int_{B_{(2\sigma)}} \psi^{2m}\rho'_\mu\rho_\mu\|\tau_{i,h}u\|^2 dx \leq$$
$$\leq \frac{1}{b\,-\,a} \int_{-b}^{-a} dt \int_{B_{(2\sigma)}} \|\tau_{i,h}u\|^2 \, dx \;\leq\; \frac{h^2}{b-a} \int_{-b}^{-a} dt \int_{B_{(3\sigma)}} \|D_i u\|^2 dx. \quad (9.3.18)$$

Finally, using $(P.3)$ condition, the term E can be expressed as follows

$$|E| \leq c(K,m,n) \sum_{|\alpha|<m} \int_{-b}^{-\frac{1}{\mu}} \rho_\mu^2 \, dt \int_{B(\frac{5}{2}\sigma)} (|f^\alpha| + \|D''u\|^2) \|\tau_{i,-h} D^\alpha \left(\psi^{2m}\tau_{i,h}u\right)\| \, dx. \quad (9.3.19)$$

we have

$$\nu\int_{-b}^{-\frac{1}{\mu}} dt\ \int_{B(2\sigma)}\psi^{2m}\rho_\mu^2\ \|\tau_{i,h}\,D''u\|^2\ dx = \nu\int_{-b}^{-\frac{1}{\mu}} dt\ \int_{B(2\sigma)}\psi^{2m}\rho_\mu^2\sum_{|\alpha|=m}\|\tau_{i,h}D^\alpha u\|^2\,dx \le$$

$$\le\ \int_Q \psi^{2m}\rho_\mu \sum_{|\alpha|=|\beta|=m}\ \sum_{k=1}^{N}\left(\left(\tau_{i,h}\,D^\beta u_k(X)\right)\ \widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}}\right|\ \rho_\mu\,\tau_{i,h}\,D^\alpha u\Big)\,dX\ \le$$

$$\le A+B+C+D+E, \tag{9.3.9}$$

where

$$A = -\sum_{|\alpha|=|\beta|=m}\ \sum_{\gamma<\alpha}\sum_{k=1}^{N}\int_Q c_{\alpha\gamma}(\psi)\psi^m\rho_\mu^2\left(\left(\tau_{i,h}\,D^\beta u_k(X)\right)\widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}}\right|(\tau_{i,h}D^\gamma u)(X)\Big)dX, \tag{9.3.10}$$

$$B = -\sum_{|\alpha|=m}\ \sum_{|\beta|<m}\ \sum_{k=1}^{N}\int_Q\left(\left(\tau_{i,h}\,D^\beta u_k(X)\right)\widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}}\ \right|\ D^\alpha\left(\psi^{2m}\rho_\mu^2\tau_{i,h}u(X)\right)\Big)\,dX, \tag{9.3.11}$$

$$C = -h\sum_{|\alpha|=m}\ \int_Q\left(\ \widetilde{\frac{\partial a^\alpha}{\partial x_i}}\ \right|\ D^\alpha\ \left(\psi^{2m}\,\rho_\mu^2\,\tau_{i,h}u)(X)\right)\Big)\,dX, \tag{9.3.12}$$

$$D\ =\ \int_Q \psi^{2m}\ \rho_\mu'\,\rho_\mu\,\|\,\tau_{i,h}\,u\,\|^2\,dX, \tag{9.3.13}$$

$$E = -\sum_{|\alpha|<m}\int_Q\left(a^\alpha\left(X,Du\right)|\tau_{i,-h}D^\alpha\left(\psi^{2m}\rho_\mu^2\tau_{i,h}u\right)\right)dX. \tag{9.3.14}$$

We observe that, for every $\varepsilon>0$, we have

$$|A|\le\varepsilon\int_{-b}^{-\frac{1}{\mu}}dt\int_{B(2\sigma)}\psi^{2m}\rho_\mu^2\,\|\tau_{i,h}D''u\|^2dx + c(K,\sigma,m,n,\varepsilon)\,h^2\int_{-b}^{-\frac{1}{\mu}}dt\int_{B(3\sigma)}\left(1+\|D''u\|^2\right)dx. \tag{9.3.15}$$

Then, equality (9.3.8) becomes

$$\int_Q \sum_{|\alpha|=m} \left(h\, \widetilde{\frac{\partial a^\alpha}{\partial x_i}} + \sum_{|\beta|\le m} \sum_{k=1}^{N} \left(\tau_{i,h}\, D^\beta u_k(X)\right) \widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}} \middle| D^\alpha \left(\psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g_s \right] \right) \right) dX =$$

$$= \int_Q \psi^{2m} \rho'_\mu \left(\tau_{i,h} u | (\rho_\mu \tau_{i,h} u) * g_s\right) dX + \int_Q \left(\tau_{i,h} u |\, \psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g_s \right]' \right) dX -$$

$$\sum_{|\alpha|<m} \int_Q \left(a^\alpha (X, Du) \,|\tau_{i,-h} D^\alpha \{ \psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g_s \right] \} \right) dX.$$

Taking into account, for $\alpha : |\alpha| = m$, that

$$D^\alpha \left(\psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g_s \right] \right) = \psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} D^\alpha u) * g_s \right] + \psi^m \rho_\mu \sum_{\gamma<\alpha} c_{\alpha\gamma}(\psi) \left[(\rho_\mu \tau_{i,h} D^\gamma u) * g_s \right]$$

where

$$|c_{\alpha\gamma}(\psi)| \le \frac{c(m,n)}{\sigma^{m-|\gamma|}},$$

we obtain

$$\int_Q \psi^{2m} \rho_\mu \sum_{|\alpha|=|\beta|=m} \sum_{k=1}^{N} \left(\left(\tau_{i,h}\, D^\beta u_k(X)\right) \widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}} \middle| (\rho_\mu\, \tau_{i,h}\, D^\alpha u) * g_s \right) dX =$$

$$= - \sum_{|\alpha|=|\beta|=m} \sum_{\gamma<\alpha} \sum_{k=1}^{N} \int_Q \left(\left(\tau_{i,h}\, D^\beta u_k(X)\right) \widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}} \middle| \psi^m \rho_\mu c_{\alpha\gamma}(\psi) \left[(\rho_\mu \tau_{i,h} D^\gamma u) * g_s \right] \right) dX -$$

$$- \sum_{|\alpha|=m} \sum_{|\beta|<m} \sum_{k=1}^{N} \int_Q \left(\left(\tau_{i,h}\, D^\beta\, u_k(X)\right) \widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}} \middle| D^\alpha \left(\psi^{2m} \rho_\mu \left[(\rho_\mu\, \tau_{i,h} u) * g_s \right] \right) \right) dX -$$

$$- h \sum_{|\alpha|=m} \int_Q \left(\widetilde{\frac{\partial a^\alpha}{\partial x_i}} \middle| D^\alpha \left(\psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g_s \right] \right) \right) dX + \int_Q \psi^{2m} \rho'_\mu \left(\tau_{i,h} u | (\rho_\mu \tau_{i,h} u) * g_s\right) dX +$$

$$+ \int_Q \left(\tau_{i,h} u |\, \psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g'_s \right] \right) dX - \sum_{|\alpha|<m} \int_Q \left(a^\alpha (X, Du) \,|\tau_{i,-h} D^\alpha \left\{ \psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g_s \right] \right\} \right) dX.$$

For $s \to +\infty$, using ellipticity condition (3.6), symmetry hypothesis, convolution property of g_s and that

$$\lim_{s\to+\infty} \int_Q \left(\tau_{i,h} u |\, \psi^{2m} \rho_\mu \left[(\rho_\mu \tau_{i,h} u) * g'_s \right] \right) dX = 0,$$

Let i be a positive integer, $i \le n$, and h a real number such that $|h| < \frac{\sigma}{2}$. For every $\mu > \frac{2}{a}$ and for every $s > max\{\mu, \frac{1}{T-b}\}$ let us define the following "test function"

$$\varphi(X) = \tau_{i,-h}\left\{\psi^{2m}\rho_\mu\left[(\rho_\mu\tau_{i,h}u) * g_s\right]\right\}, \quad \forall X = (x,t) \in Q. \tag{9.3.7}$$

Substituting in (9.2.1) the above defined function φ, we have

$$\begin{aligned}
&\int_Q \sum_{|\alpha|=m}\left(\tau_{i,h}a^\alpha(X,Du)\,|D^\alpha\left(\psi^{2m}\rho_\mu\left[(\rho_\mu\tau_{i,h}u) * g_s\right]\right)\right)dX = \\
&\quad = \int_Q \left(\tau_{i,h}u|\,\psi^{2m}\{\rho_\mu\left[(\rho_\mu\tau_{i,h}u) * g_s\right]\}'\right)dX - \\
&\qquad - \sum_{|\alpha|<m}\int_Q \left(a^\alpha(X,Du)\,|\tau_{i,-h}D^\alpha\left\{\psi^{2m}\rho_\mu\left[(\rho_\mu\tau_{i,h}u) * g_s\right]\right\}\right)dX.
\end{aligned} \tag{9.3.8}$$

For every $\alpha : |\alpha| = m$ and a. e. $X = (x,t) \in Q$, we have

$$\begin{aligned}
&\tau_{i,h}\,a^\alpha(X, Du(X)) = \; a^\alpha\left(x + he^i, t, Du(x+he^i,t)\right) - a^\alpha(X, Du(X)) = \\
&= \int_0^1 \frac{d}{d\eta}\,a^\alpha\left(x + \eta he^i, t, Du(X) + \eta\,\tau_{i,h}Du(X)\right)\,d\eta = \\
&= \; h\int_0^1 \frac{\partial}{\partial x_i}\,a^\alpha\left(x + \eta he^i, t, Du(X) + \eta\,\tau_{i,h}Du(X)\right)d\eta + \\
&+ \sum_{|\beta|\le m}\sum_{k=1}^N\left(\tau_{i,h}D^\beta u_k(X)\right)\int_0^1\frac{\partial}{\partial p_k^\beta}a^\alpha\left(x+\eta he^i, t, Du(X)+\eta\tau_{i,h}Du(X)\right)d\eta = \\
&= \; h\,\widetilde{\frac{\partial a^\alpha}{\partial x_i}} + \sum_{|\beta|\le m}\sum_{k=1}^N\left(\tau_{i,h}\,D^\beta u_k(X)\right)\widetilde{\frac{\partial a^\alpha}{\partial p_k^\beta}},
\end{aligned}$$

where, if $b = b(X,p)$, for simplicity of notation, we set

$$\tilde{b}(X) = \int_0^1 b\left(x + \eta he^i, t, Du(X) + \eta\,\tau_{i,h}Du(X)\right)\,d\eta.$$

then, using (9.3.1), written with $\theta = 1 - \frac{\lambda}{2}$, and (9.3.2)–(9.3.4), we have

$$\int_{-b^*}^{0} \|u\|^4_{m,4,B\left(\frac{5}{2}\sigma\right)}\, dt \le c(\sigma) \int_{-b^*}^{0} \|u\|^4_{m,4+\frac{4\lambda}{n-\lambda},B\left(\frac{5}{2}\sigma\right)}\, dt \le$$

$$\le c(\theta,\lambda,\sigma,m,n) \int_{-b^*}^{0} \|D''u\|^2_{1-\frac{\lambda}{2},B\left(\frac{5}{2}\sigma\right)} \|u\|^2_{C^{m-1,\lambda}(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N)}\, dt \le$$
$$\le c(\nu,K,U,\lambda,\sigma,m,n) \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)}\, dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|^2_{m,B(3\sigma)}\, dt \right\}, \tag{9.3.5}$$

then it follows the requested inequality (9.2.3). □

Proof. Let us fix $B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$, $a, b \in (0,T)$ with $a < b$ and $h \in \mathbb{R}$ such that $|h| < \frac{\sigma}{2}$, set $b^* = \frac{a+b}{2}$ and let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ a real function satisfying the following properties $0 \le \psi \le 1$ in $\mathbb{R}^n$, $\psi = 1$ in $B(\sigma), \psi = 0$ in $\mathbb{R}^n \setminus B(2\sigma)$, $\|D\psi\| \le \frac{c}{\sigma}$ in $\mathbb{R}^n$.

Let us also define the function $\rho_\mu(t)$, for $\mu > \frac{2}{a}, \mu$ integer, the following real function

$$\rho_\mu(t) = \begin{cases} 1 & \text{if } -a \le t \le -\frac{2}{\mu} \\ 0 & \text{if } t \le -b \text{ and } t \ge -\frac{1}{\mu} \\ \frac{t+b}{b-a} & \text{if } -b < t < -a \\ -(\mu t + 1) & \text{if } -\frac{2}{\mu} < t < -\frac{1}{\mu}. \end{cases} \tag{9.3.6}$$

Moreover set $\{g_s(t)\}$ the sequence of symmetric regularizing functions such that

$$g_s(t) \in C_0^\infty(\mathbb{R}), \qquad g_s(t) \ge 0, \qquad g_s(t) = g_s(-t),$$
$$\operatorname{supp} g_s \subset \left[-\frac{1}{s}, \frac{1}{s}\right], \qquad \int_{\mathbb{R}} g_s(t)\, dt = 1.$$

and

$$\int_{-b^*}^{0} |D''u|^2_{\vartheta,B(\frac{5}{2}\sigma)}\, dt \le$$
$$\le c(\nu,K,U,\vartheta,\lambda,\sigma,a,b,m,n)\left\{1+\sum_{|\alpha|<m}\left(\int_{-b}^{0}\|f^\alpha\|_{0,B(3\sigma)}dt\right)^{\frac{1+\vartheta}{2}}+\int_{-b^*}^{0}|u|^2_{m,B(3\sigma)}\,dt\right\}. \tag{9.3.1}$$

Hence, we remark that $u \in C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)$, then, it results, for $a.\,e.\, t \in (-b^*,0)$,

$$u(x,t) \in H^{m+\vartheta}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right) \cap C^{m-1,\lambda}\left(\overline{B\left(\frac{5}{2}\sigma\right)},\mathbb{R}^N\right), \quad \forall\, 0<\vartheta<1,\ \forall\, B(3\sigma)\subset\subset\Omega.$$

Then, from Theorem 7.2.4 with $\Omega = B\left(\frac{5}{2}\sigma\right)$, $1-\lambda<\theta<1$, for $\delta=\frac{1}{2}$, and for $a.e.\, t\in(-b^*,0)$:

$$u(x,t) \in H^{m,p}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right),$$

and there exists a constant $c = c(\theta,\lambda,\sigma,m,n)$ such that

$$\|u\|_{m,p,B\left(\frac{5}{2}\sigma\right)} \le c\|u\|^{\frac{1}{2}}_{m+\theta,B\left(\frac{5}{2}\sigma\right)}\|u\|^{\frac{1}{2}}_{C^{m-1,\lambda}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right)},$$

where $p = 4+\frac{8(\theta+\lambda-1)}{n-2(\theta+\lambda-1)} > 4$.

The choice $\theta = 1-\frac{\lambda}{2}(>1-\lambda)$ ensures that for a. e. $t\in(-b^*,0)$ we have

$$u(x,t) \in H^{m,p}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right), \qquad \text{with } p = 4+\frac{4\lambda}{n-\lambda}, \quad \forall B(3\sigma)\subset\subset\Omega. \tag{9.3.2}$$

and

$$\|u\|_{m,p,B\left(\frac{5}{2}\sigma\right)} \le c(\theta,\lambda,\sigma,m,n)\|u\|^{\frac{1}{2}}_{m+1-\frac{\lambda}{2},B\left(\frac{5}{2}\sigma\right)}\|u\|^{\frac{1}{2}}_{C^{m-1,\lambda}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right)}, \tag{9.3.3}$$

where $p = 4+\frac{4\lambda}{n-\lambda} > 4$.

Then we have, for a. e. $t\in(-b^*,0)$, the following inclusion between Sobolev spaces

$$u(x,t) \in H^{m,p}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right) \subset\subset H^{m,4}\left(B\left(\frac{5}{2}\sigma\right),\mathbb{R}^N\right) \tag{9.3.4}$$

and the following estimate holds

$$\int_{-a}^{0} \|u\|_{m,4,B(\sigma)}^4 dt \le c(\nu, K, U, \lambda, \sigma, a, b, m, n) \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)}\, dt \right)^{\frac{1+\vartheta}{2}} + \int_{-b}^{0} |u|_{m,B(3\sigma)}^2 dt \right\} \tag{9.2.3}$$

where $K = \sup_Q \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{Q},\mathbb{R}^N)}$.

Theorem 9.2.2. ***(main result).*** *If* $u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap C^{m-1,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (6) *and if the assumptions (P.1) – (3.6) hold, then* $\forall B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$, $\forall a, b \in (0, T)$, $a < b$ *it results*

$$u \in L^2(-a, 0, H^{m+1}(B(\sigma), \mathbb{R}^N)) \cap H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N)) \tag{9.2.4}$$

and the following estimate holds

$$\int_{-a}^{0} \left\{ |u|_{m+1,B(\sigma)}^2 + \left| \frac{\partial u}{\partial t} \right|_{0,B(\sigma)}^2 \right\} dt \le$$
$$\le c(\nu, K, U, \lambda, \sigma, a, b, m, n) \left\{ 1 + \sum_{|\alpha|<m} \left(\int_{-b}^{0} \|f^\alpha\|_{0,B(3\sigma)} dt \right)^2 + \int_{-b}^{0} |u|_{m,B(3\sigma)}^2 dt \right\} \tag{9.2.5}$$

where $K = \sup_Q \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{Q},\mathbb{R}^N)}$.

9.3 Proofs of the main results

Proof. Let us observe that, using Theorem 2.III in [11], for every $0 < \vartheta < 1$ and $b^* = \frac{a+b}{2}$, we have

$$u \in L^2 \left(-b^*, 0, H^{m+\vartheta} \left(B \left(\frac{5}{2} \sigma \right), \mathbb{R}^N \right) \right),$$

we have

$$\|a^\alpha\| + \sum_{r=1}^{n} \left\| \frac{\partial\, a^\alpha}{\partial\, x_r} \right\| + \sum_{k=1}^{N} \sum_{|\beta|<m} \left\| \frac{\partial\, a^\alpha}{\partial\, p_k^\beta} \right\| \le M(K)\,(1 + \|p''\|)\,,$$

$$\sum_{k=1}^{N} \sum_{|\beta|=m} \left\| \frac{\partial\, a^\alpha}{\partial\, p_k^\beta} \right\| \le M(K);$$

(3.6) $\exists\, \nu = \nu(K) > 0$ *such that:*

$$\sum_{h,k=1}^{N} \sum_{|\alpha|=|\beta|=m} \frac{\partial\, a_h^\alpha(X,p)}{\partial\, p_k^\beta}\, \xi_h^\alpha\, \xi_k^\beta \ge \nu(K) \sum_{|\beta|=m} \left\|\xi^\beta\right\|_N^2 = \nu\|\xi\|^2\,,$$

for every $\xi = (\,\xi^\alpha\,) \in \mathcal{R}''$ *and for every* $(X,p) \in Q \times \mathcal{R}$*, with* $\|p'\| \le K$ *.*

If the coefficients a^α satisfy condition (3.6) we say that the system (6) is *strictly elliptic* in Ω.

9.2 Main results

We say a function $u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(Q, \mathbb{R}^N)$, N positive integer and $0 < \lambda < 1$, *weak solution* in Q to the nonlinear parabolic system of order $2m$

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha\, a^\alpha\,(X, Du)\; + \;\frac{\partial\, u}{\partial\, t} = 0$$

if

$$\int_Q \left\{ \sum_{|\alpha|\le m} (a^\alpha(X,Du)|\, D^\alpha\varphi) \;-\; \left(u|\frac{\partial\,\varphi}{\partial\,t}\right)\right\} dX \;=\; 0, \quad \forall\varphi \;\in\; C_0^\infty(Q, \mathbb{R}^N). \quad (9.2.1)$$

Theorem 9.2.1.. *If* $u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap C^{m-1,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (6) *and if the assumptions (P.1) – (3.6) hold, then* $\forall B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$, $\forall a, b \in (0, T)$, $a < b$, *it results*

$$u \in L^4(-a, 0, H^{m,4}(B(\sigma), \mathbb{R}^N)) \quad (9.2.2)$$

and $p = \{p^\alpha\}_{|\alpha|\le m}$, $p^\alpha \in \mathbb{R}^N$, the generic point of $\mathcal{R}$. If $p \in \mathcal{R}$, we set $p = (p', p'')$ where $p' = \{p^\alpha\}_{|\alpha|<m} \in \mathcal{R}' = \prod_{|\alpha|<m} \mathbb{R}^N_\alpha$, $p'' = \{p^\alpha\}_{|\alpha|=m} \in \mathcal{R}'' = \prod_{|\alpha|=m} \mathbb{R}^N_\alpha$, and

$$\|p\|^2 = \sum_{|\alpha|\le m} \|p^\alpha\|^2_N, \quad \|p'\|^2 = \sum_{|\alpha|<m} \|p^\alpha\|^2_N, \qquad \|p''\|^2 = \sum_{|\alpha|=m} \|p^\alpha\|^2_N.$$

We consider, as usual,

$$D_i = \frac{\partial}{\partial x_i},\ i = 1, \ldots, n; \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n};$$

$$Du = \{D^\alpha u\}_{|\alpha|\le m}, \quad D'u = \{D^\alpha u\}_{|\alpha|<m}, \quad D''u = \{D^\alpha u\}_{|\alpha|=m}.$$

Let us consider the following differential nonlinear variational parabolic system of order $2m$:

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha a^\alpha (X, Du) + \frac{\partial u}{\partial t} = 0 \tag{9.1.1}$$

where $a^\alpha(X, p) = a^\alpha(X, p', p'')$ are functions of $\Lambda = Q \times \mathcal{R}$ in $\mathbb{R}^N$, satisfying the following conditions:

(P.1) *for every $\alpha : |\alpha| < m$ and every $p \in \mathcal{R}$, the function $X \longrightarrow a^\alpha(X, p)$, defined in Q having values in $\mathbb{R}^N$, is measurable in X;*

(P.2) *for every $\alpha : |\alpha| < m$ and every $X \in Q$, the function $p \longrightarrow a^\alpha(X, p)$, defined in $\mathcal{R}$ having values in $\mathbb{R}^N$, is continuous in p;*

(P.3) *for every $\alpha : |\alpha| < m$ and every $(X, p) \in \Lambda$, such that $\|p'\| \le K$, we have*

$$\|a^\alpha(X, p)\| \le M(K) \left(|f^\alpha(X)| + \|p''\|^2\right),$$

where $f^\alpha \in L^2(Q)$;

(P.4) *for every $\alpha : |\alpha| = m$, the function $a^\alpha(X, p', p'')$, defined in $Q \times \mathcal{R}$ having values in $\mathbb{R}^N$, are of class C^1 in $Q \times \mathcal{R}$ and, for every $(X, p', p'') \in Q \times \mathcal{R}$ with $\|p'\| \le K$,*

Chapter 9

Nonlinear parabolic systems

In this chapter, we investigate differentiability of the solutions of nonlinear parabolic systems of order $2\,m$ in divergence form of the following type

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha}\, a^{\alpha}\,(X, Du)\; +\; \frac{\partial\, u}{\partial\, t} = 0.$$

The results are achieved inspired by the papers [23] and [25]. This chapter can be viewed as a continuation of the study of regularity properties for solutions of elliptic systems started in [15] and continued in [16] and [18], and also as a generalization of the paper [7] where regularity properties of the solutions of nonlinear elliptic systems of order $2m$ with quadratic growth are reached.

9.1 Problem formulation

Let us set m, N positive integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_n$ the order of α. We denote by $\mathcal{R}$ the Cartesian product

$$\mathcal{R} = \prod_{|\alpha|\leq m} \mathbb{R}^N_{\alpha}$$

where $K = \sup_{\overline{\Omega}} \|D'u\|$.
Therefore, because we are exactly in the same situation studied in n. 3 Chapt. IV of [4], we get the conclusion.

$\square$

and, $\forall \alpha : |\alpha| = m$,

$$G^{\alpha s}(x, Du) = -\frac{\partial a^{\alpha}(x, Du)}{\partial x_s} - \sum_{|\beta|<m} \sum_{k=1}^{N} (D_s\, D^{\beta} u_k) \frac{\partial a^{\alpha}(x, Du)}{\partial p_k^{\beta}}. \tag{8.4.4}$$

Let us also assume in (8.4.2) $\theta = D_s \varphi$ with $\varphi \in C_0^{\infty}(\Omega_0, \mathbb{R}^N)$, summing from 1 to n respect to s, we gain that the function $u \in H^{m+1}(\Omega_0, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega_0}, \mathbb{R}^N)$ is solution of the following quasilinear system of order $2\,(m+1)$

$$\int_{\Omega_0} \sum_{|\alpha|=|\beta|=m} \sum_{r,s=1}^{n} \left(B_{\alpha r, \beta s}(x, Du) D_s\, D^{\beta}\, u | D_r\, D^{\alpha}\, \varphi\right)\, dx =$$
$$= \int_{\Omega_0} \sum_{|\alpha|=m} \sum_{s=1}^{n} (G^{\alpha s}(x, Du) + \delta_{\alpha s} \sum_{|\beta|<m} a^{\beta}(x, Du) | D^{\beta}\, D_s \varphi)\, dx\,, \quad \forall \varphi \in C_0^{\infty}(\Omega_0, \mathbb{R}^N) \tag{8.4.5}$$

where

$$B_{\alpha r \beta s} = \delta_{rs}\, A_{\alpha\beta}\,. \tag{8.4.6}$$

We point out that system (8.1.1) is strictly monotone but, because of $a^{\alpha} \in C^1(\Omega \times \mathcal{R}, \mathbb{R}^N)$, for $|\alpha| = m$, this condition is equivalent to that of strict ellipticity. Let us prove that the same is also true of system (8.4.5) with the same ellipticity constant ν. Indeed, thanks to (8.4.6) and (8.4.3), for every system $\{\eta^{\alpha s}\}_{\alpha, s=1,2,\ldots,n}$ of vectors of $\mathbb{R}^N$, we have

$$\sum_{|\alpha|=|\beta|=m} \sum_{r,s=1}^{n} \left(B_{\alpha r, \beta s}\, \eta^{\beta s} |\, \eta^{\alpha r}\right) = \sum_{s=1}^{n} \sum_{|\alpha|=|\beta|=m} \left(A_{\alpha\beta}\, \eta^{\beta s} |\, \eta^{\alpha s}\right) =$$
$$= \sum_{s=1}^{n} \sum_{|\alpha|=|\beta|=m} \sum_{h,k=1}^{N} A_{\alpha\beta}^{hk}\, \eta_h^{\beta s}\, \eta_k^{\alpha s} = \sum_{s=1}^{n} \sum_{|\alpha|=|\beta|=m} \sum_{h,k=1}^{N} \frac{\partial a_h^{\alpha}}{\partial p_k^{\beta}} \eta_h^{\beta s}\, \eta_k^{\alpha s} \geq \nu \sum_{s=1}^{n} \sum_{|\alpha|=m} \|\eta^{\alpha s}\|_N^2\,.$$

Moreover from the hypotheses $(E.3)$ and $(E.4)$ it follows

$$\|G^{\alpha s} + \delta_{\alpha s} \sum_{|\beta|<m} a^{\beta}(x, Du)\| \leq c(K)\{1 + \sum_{|\alpha|<m} |f^{\alpha}| + \|D'' u\|^2\},$$

m.

Theorem 8.4.1. *Let $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$, a weak solution of the system* (8.1.1)*, are true the hypotheses* (*E*.1)*,* (*E*.2)*,* (*E*.4)*,* (*E*.5)*,* (*E*.3) *for $f^\alpha \in L^{\frac{2n}{n-2\lambda}}(\Omega)$, $|\alpha| < m$, and $a^\alpha(x, Du) \in C^1(\Omega \times \mathcal{R}, \mathbb{R}^N)$ for $|\alpha| = m$,. Then, there exists a closed set $\Omega_0 \subset \Omega$, such that*

$$H_{n-q}(\Omega_0) = 0 \text{ for a number } q > 2,\ u \in C^{m,\gamma}(\Omega \setminus \Omega_0, \mathbb{R}^N) \text{ for a suitable } \gamma \in (0,1)\,,$$

where $H_{n-q}(\Omega_0)$ is the $(n-q)$-dimensional Hausdorff measure of Ω_0.

Proof. ***of Theorem 4.1.*** Let us fix a positive number s, $s \leq n$, and assume in the definition of weak solution (9.2.1) $\varphi = D_s\,\theta$, for $\theta \in C_0^\infty(\Omega_0, \mathbb{R}^N)$, $\Omega_0 \subset\subset \Omega$, we have

$$\int_{\Omega_0} \sum_{|\alpha| \leq m} (D_s\, a^\alpha(x, Du) | D^\alpha \theta)\; dx = 0, \quad \forall\, \theta \in C_0^\infty(\Omega_0, \mathbb{R}^N)\,. \tag{8.4.1}$$

we can write the derivatives:

$$D_s\, a^\alpha(x, Du) = \frac{\partial a^\alpha}{\partial x_s} + \sum_{|\beta| < m} \sum_{k=1}^{N} (D_s D^\beta\, u_k) \frac{\partial a^\alpha}{\partial p_k^\beta} + \sum_{|\beta| = m} \sum_{k=1}^{N} (D_s\, D^\beta\, u_k) \frac{\partial a^\alpha}{\partial p_k^\beta}.$$

Applying the previous theorem we have that $u \in H^{m+1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, thus we are able to write (8.4.1) as follows

$$\begin{aligned} &\int_{\Omega_0} \sum_{|\alpha|=|\beta|=m} \big(A_{\alpha\beta}(x, Du) D_s\, D^\beta\, u | D^\alpha\, \theta\big)\; dx = \\ &= \int_{\Omega_0} \{ \sum_{|\alpha|=m} (G^{\alpha,s}(x, Du)|D^\alpha \theta) - \sum_{|\alpha|<m} (a^\alpha(x, Du)|D^\alpha D_s \theta)\} dx, \quad \forall \theta \in C_0^\infty(\Omega_0, \mathbb{R}^N) \end{aligned} \tag{8.4.2}$$

where $\forall \alpha, \beta : |\alpha| = |\beta| = m$,

$$A_{\alpha\beta} = \{A^{hk}_{\alpha\beta}\}\,,\ A^{hk}_{\alpha\beta} = \frac{\partial a^\alpha_h(x, Du)}{\partial p^\beta_k}, \qquad h, k\ = 1, \ldots, N \tag{8.4.3}$$

Let us now estimate the last term using the Hölder inequality

$$\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx \le \left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^4 \, dx\right)^{\frac{1}{2}} \left(\int_{Q(2\sigma)} \|D''u\|^4 \, dx\right)^{\frac{1}{2}},$$

Then, applying Theorem 7.2.2 (for $p = 4$, $Q(\frac{5}{2}\sigma)$ instead of $Q(\sigma)$ and $t = \frac{4}{5}$), for every $|h| < h_0$, it follows

$$\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx \le h^2 \|D''u\|^2_{0,4,Q(\frac{5}{2}\sigma)} \|D''u\|^2_{0,4,Q(2\sigma)} \le h^2 |u|^4_{m,4,Q(3\sigma)} \,. \tag{8.3.52}$$

From (8.3.51) and (8.3.52), for every i $(1 \le i \le n)$ and every $|h| < h_0$, we gain the following estimate

$$\int_{Q(\sigma)} \|\tau_{i,h} \, D''u\|^2 \, dx \le c(\nu, K, \sigma, m, n) \, h^2 \{1+(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)})^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)}\}.$$

If $h_0 \le |h| < \frac{\sigma}{2}$, as in (8.3.19), we have that

$$\int_{Q(\sigma)} \|\tau_{i,h} D''u\|^2 \, dx \le 4\int_{Q(3\sigma)} \|D''u\|^2 dx \le 4\,\frac{h^2}{h_0^2}\int_{Q(3\sigma)} \|D''u\|^2 dx \le c(\nu, K, U, \lambda, \sigma, m, n) h^2 |u|^2_{m,Q(3\sigma)} \le$$

$$\le c(\nu, K, U, \lambda, \sigma, n) \, h^2 \, \{ 1+(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)})^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)} \}, \quad \forall i = 1, 2, \ldots, n \,.$$

It is then proved, for every $|h| < \frac{\sigma}{2}$ and every $i \in \{1, 2, \ldots, n\}$, that

$$\int_{Q(\sigma)} \|\tau_{i,h} \, D''u\|^2 \, dx \le c(\nu, K, U, \lambda, \sigma, m, n) \, h^2 \{1+(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)})^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)}\},$$

applying Theorem 7.2.1, it follows (8.2.9) and (8.2.10). □

8.4 . Partial Hölder continuity of higher order derivatives

As application of the previous differentiability properties for solutions of system (8.1.1) we have the following result of partial Hölder continuity of derivatives of order

Exploiting Theorem 8.2.3 we can achieve that $u \in H^{m,4}_{\text{loc}}(\Omega, \mathbb{R}^N)$, then we can estimate the last term as follows

$$\sum_{|\alpha|<m} \int_{Q(3\sigma)} (|f^\alpha| + \|D''u\|^2) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\| \, dx \le$$

$$\le \sum_{|\alpha|<m} \Big(\int_{Q(3\sigma)} |h|^{-2} \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_{Q(3\sigma)} h^2 (|f^\alpha| + \|D''u\|^2)^2 \, dx\Big)^{\frac{1}{2}} \le$$

$$\le \frac{\varepsilon}{2} |h|^{-2} \sum_{|\alpha|<m} \int_{Q(3\sigma)} \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 \, dx \, + c(\varepsilon)\, h^2 \sum_{|\alpha|<m} \int_{Q(3\sigma)} (|f^\alpha|^2 + \|D''u\|^4) \, dx \, .$$

Furthermore, from Theorem 7.2.2 (for $p = 2$, $Q(\frac{7}{2}\sigma)$ instead of $Q(\sigma)$ and $t = \frac{6}{7}$), for every $h \in \mathbb{R}$ con $|h| < h_0$ and every $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{\varepsilon}{2} |h|^{-2} \int_{Q(3\sigma)} \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 \, dx &\le \frac{\varepsilon}{2} \int_{Q(\frac{7}{2}\sigma)} \left\|D'' \left(\psi^{2m} \tau_{i,h}\, u\right)\right\|^2 \, dx \ \le \\
&\le \varepsilon \int_{Q(2\sigma)} \psi^{2m} \, \|\, \tau_{i,h}\, D''u \,\|^2 \, dx \, + c(\sigma, \varepsilon) \int_{Q(2\sigma)} \|\, \tau_{i,h} D'u \,\|^2 \, dx \le \\
&\le \varepsilon \int_{Q(2\sigma)} \psi^{2m} \, \|\, \tau_{i,h}\, D''u \,\|^2 \, dx \, + c(\sigma, \varepsilon)\, h^2 \int_{Q(3\sigma)} \|D''u\|^2 \, dx.
\end{aligned}$$

the last inequality follows, as before, applying Theorem 7.2.2 (for $p = 2$, $Q(3\sigma)$ instead of $Q(\sigma)$ and $t = \frac{2}{3}$). Let us now choose $\varepsilon = \frac{\nu}{4\, c(K)}$, it ensure

$$\sum_{|\alpha|<m} \int_{Q(3\sigma)} \left(|f^\alpha| + \|D''u\|^2\right) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\| \, dx \le$$

$$\le \frac{\nu}{4c(K)} \int_{Q(2\sigma)} \psi^{2m} \, \|\tau_{i,h} D''u\|^2 \, dx + c(\nu, K, \sigma, m) h^2 \left\{ \Big(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)}\Big)^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)} \right\} .$$

Taking into consideration the last inequality and the properties of the function ψ, from (8.3.50) we deduce

$$\begin{aligned}
\int_{Q(\sigma)} \|\tau_{i,h} D''u\|^2 \, dx \le c(\nu, K, \sigma, m, n) h^2 \{1 + \Big(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)}\Big)^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)}\} + \\
+ \ c(\nu, K, \sigma, n) \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, \|D''u\|^2 \, dx \, . \quad (8.3.51)
\end{aligned}$$

we assure that $\vartheta^* \in (0,1)$ exists and is such that $\frac{2(1+\vartheta^*)n}{n-2\vartheta^*\lambda} > 4$. Let us set p^* in $\left(4, \frac{2(1+\vartheta^*)n}{n-2\vartheta^*\lambda}\right)$, from (9.3.2), we have

$$u \in H^{m,p^*}(Q(\sigma), \mathbb{R}^N)\,, \forall Q(\sigma) \subset\subset \Omega$$

from which, because of $p^* > 4$, it follows

$$u \in H^{m,4}(Q(\sigma), \mathbb{R}^N)\,. \tag{8.3.49}$$

We end the conclusion remarking that (8.3.49) is true for every $Q(\sigma) \subset\subset \Omega$.

□

8.3.2 Proof of local differentiability result in H^{m+1} space

Proof. ***of Theorem 3.4.*** Let us consider $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ the *cut-off* function above defined in (8.3.1), $Q(4\sigma) \subset\subset \Omega$ a generic cube, $i \leq n$ a positive integer and h a real number such that $|h| < \frac{\sigma}{2}$. Carrying on as in the proof of Theorem 8.2.1, we obtain

$$\begin{aligned}\frac{\nu}{2}\int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D'' u\|^2 \, dx \leq c(\nu, K, \sigma, m, n)\, h^2 \int_{Q(3\sigma)} \left(1 + \|D''u\|^2\right) dx + \\ + c(\nu, K, \sigma, m, n) \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, \|D''u\|^2 \, dx + \\ + c(K) \sum_{|\alpha|<m} \int_{Q(3\sigma)} (|f^\alpha| + \|D''u\|^2) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\right\| dx.\end{aligned} \tag{8.3.50}$$

and we reach the inequality

$$|D''u|^2_{\vartheta_i,Q(4^{-i}\rho)} \le c(\nu,K,U,\vartheta,\lambda,\rho,m,n)(1+(\sum_{|\alpha|<m}\|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(4\rho)})^{1+\vartheta_{i-1}}+|u|^2_{m,Q(4\rho)}). \tag{8.3.47}$$

Let us fix arbitrarily $x_0\in\Omega$, $Q(\sigma)=Q(x^0,\sigma)\subset\subset Q(\sigma_0)=Q(x^0,\sigma_0)\subset\subset\Omega$ and assume $\rho=\frac{\sigma_0-\sigma}{8}$. The set of cubes

$$\mathcal{F}=\left\{Q(y^0,4^{-i-1}\rho),\ y^0\in Q(\sigma)\right\}$$

is an open cover of $\overline{Q(\sigma)}$, let us then extract the finite cover

$$Q(y^{(1)},4^{-i-1}\rho)\,,\ Q(y^{(2)},4^{-i-1}\rho)\,,\dots,Q(y^{(t)},4^{-i-1}\rho).$$

After that, set $\Omega_k=Q(y^{(k)},4^{-i-1}\rho)\cap Q(\sigma),\ k=1,2,\dots,t,$

$$\bigcup_{k=1}^t\Omega_k=Q(\sigma),\ Q(y^{(k)},4\rho)\subset\subset Q(\sigma_0)\subset\subset\Omega\,,\,\forall k=1,2,\dots,t\,,$$

from (8.3.47) and Theorem 7.1.6 (if $\Omega=Q(\sigma)$, $\vartheta=\vartheta_i$ and $\sigma=\frac{3\rho}{4^{i+1}}$), we have

$$|D''u|^2_{\vartheta_i,Q(\sigma)}\le c(\nu,K,U,\vartheta,\lambda,\sigma,\sigma_0,m,n)\{1+(\sum_{|\alpha|<m}\|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(\sigma_0)})^{1+\vartheta_{i-1}}+|u|^2_{m,Q(\sigma_0)}\}$$

we gain (8.2.6) and (9.3.1) bearing in mind that $\vartheta_{i-1}<\vartheta\le\vartheta_i$.

To prove (8.2.8) we remark that $u\in C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)$, then

$$u\in H^{m+\vartheta}(Q(\sigma),\mathbb{R}^N)\cap C^{m-1,\lambda}(\overline{Q(\sigma)},\mathbb{R}^N),\qquad\forall\,0<\vartheta<1,\qquad\forall\,Q(\sigma)\subset\subset\Omega.$$

In addition, Theorem 7.1.7 ensures that

$$u\in H^{m,p}(Q(\sigma),\mathbb{R}^N),\quad\forall\,1\le p<\frac{2(1+\vartheta)n}{n-2\vartheta\lambda},\quad\forall\,0<\vartheta<1,\quad\forall Q(\sigma)\subset\subset\Omega. \tag{8.3.48}$$

and observing that

$$\lim_{\vartheta\to1^-}\frac{2(1+\vartheta)n}{n-2\vartheta\lambda}=\frac{4n}{n-2\lambda}>4\,,$$

i) $\vartheta_s = 1 - (1-\vartheta_0)^{s+1}$;

ii) $0 < \vartheta_s < \vartheta_{s+1} < 1$;

iii) $\vartheta_{s+1} - \vartheta_s = \vartheta_0(1-\vartheta_0)^{s+1}$;

iv) $\vartheta_{s+1} < \vartheta_s + \frac{\lambda}{2}(1-\vartheta_s)$;

v) $q_s = \frac{2(1+\vartheta_s)n}{n-2\vartheta_s\lambda} < \frac{4n}{n-2\lambda}$.

It ensure that $f^\alpha \in L^{\frac{q_s}{2}}(\Omega)$, for every $s = 0, 1, 2, \dots$ and every α such that $|\alpha| < m$. Due to $\lim\limits_{s\to+\infty} \vartheta_s = 1$, fixing arbitrarily $\vartheta \in (\vartheta_0, 1)$ exists a positive integer $i = i(\vartheta, \lambda)$ such that $\vartheta_{i-1} < \vartheta \le \vartheta_i < 1$.
Additionally, from Theorem 8.2.1 we deduce

$$u \in H^{m+\vartheta_0}(Q(4\rho), \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{Q(4\rho)}, \mathbb{R}^N), \qquad \forall Q(4\rho) \subset\subset \Omega$$

and

$$|D''u|^2_{\vartheta_0, Q(\rho)} \le c(\nu, K, U, \lambda, \rho, m, n)(1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0, \frac{2n}{n-2\lambda}, Q(4\rho)} + |u|^2_{m, Q(4\rho)}). \quad (8.3.45)$$

Exploiting Theorem 8.2.2 for $\vartheta = \vartheta_0$ $q = q_0$, $\vartheta' = \vartheta_1$ and $\Omega = Q(4\rho)$, as well as iv) and v) for $s = 0$, we have

$$u \in H^{m+\vartheta_1}(Q(\rho), \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{Q(\rho)}, \mathbb{R}^N)$$

and

$$\begin{aligned} |D''u|^2_{\vartheta_1, Q(4^{-1}\rho)} \le & c(\nu, K, U, \lambda, \rho, m, n)(1 + (\sum_{|\alpha|<m} \|f^\alpha\|_{0, \frac{q_0}{2}, Q(\rho)})^{1+\vartheta_0} + |u|^2_{m, Q(\rho)} + |D''u|^2_{\vartheta_0, Q(\rho)}) \le \\ \le & c(\nu, K, U, \lambda, \rho, m, n) \ (1 + (\sum_{|\alpha|<m} \|f^\alpha\|_{0, \frac{2n}{n-2\lambda}, Q(4\rho)})^{1+\vartheta_0} + |u|^2_{m, Q(4\rho)}). \end{aligned}$$

making use i times of Theorem 8.2.2 we establish, $\forall Q(4\rho) \subset\subset \Omega$, so that

$$u \in H^{m+\vartheta_i}(Q(4^{-i+1}\rho), \mathbb{R}^N) \quad (8.3.46)$$

then, for every $h : 0 < |h| < 2\sigma$ is integrable in $[-2\sigma, 2\sigma]$ the second member of

$$\sum_{i=1}^{n} \frac{1}{|h|^{1+2\vartheta'}} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2\, dx \;\le\; c(\nu, K, U, \vartheta, \lambda, m, n, \sigma)\cdot$$
$$\cdot\, \{1+(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(3\sigma)})^{1+\vartheta} + |u|^2_{m,Q(3\sigma)} + |D''u|^2_{\vartheta,Q(3\sigma)}\} \frac{1}{|h|^{1+2\vartheta'-2\vartheta-\lambda(1-\vartheta)}} \qquad (8.3.43)$$

and thus also the first one is integrable.

It is then proved that

$$\sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta'}} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2\, dx \;\le\; c(\nu, K, U, \vartheta, \vartheta', \lambda, m, n, \sigma)\cdot$$
$$\cdot\, \{1+(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(3\sigma)})^{1+\vartheta} + |u|^2_{m,Q(3\sigma)} + |D''u|^2_{\vartheta,Q(3\sigma)}\}, \quad \forall\, 0 < \vartheta' < \vartheta + \frac{\lambda}{2}(1-\vartheta).$$
$$(8.3.44)$$

Because of $u \in H^m(\Omega, \mathbb{R}^N)$ from Theorem 7.1.3, we have

$$D''u \in H^{\vartheta'}(Q(\sigma), \mathcal{R}")$$

and

$$|D''u|^2_{\vartheta',Q(\sigma)} \le c(n) \sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta'}} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2\, dx \;\le$$
$$\le c(\nu, K, U, \vartheta, \vartheta', \lambda, m, n, \sigma) \{1+(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(3\sigma)})^{1+\vartheta} + |u|^2_{m,Q(3\sigma)} + |D''u|^2_{\vartheta,Q(3\sigma)}\},$$

we achieve our goal. □

Proof. ***of Theorem 3.3.*** Let us fix $\vartheta_0 = \frac{\lambda}{4}$ and make a point of the geometric series

$$1 + (1-\vartheta_0) + (1-\vartheta_0)^2 + \cdots + (1-\vartheta_0)^r + \cdots\cdots .$$

For $s = 0, 1, \ldots$, let us set $\vartheta_s = \vartheta_0 \sum_{r=0}^{s} (1-\vartheta_0)^r$. We achieve, for every $s = 0, 1, \ldots$, that

Therefore for every $\varepsilon > 0$ and every $2 < p < \min(4, q)$ we reach

$$\frac{\nu}{2}\int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \le c(\nu, K, \sigma, m, n) h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \|D''u\|^2\right) dx +$$
$$+ c(\nu, K, U, \vartheta, \lambda, m, n, \sigma) \mid h \mid^{2\vartheta + 2\lambda(1-\vartheta)} \left\{ \mid D''u \mid^2_{\vartheta, Q(\frac{5}{2}\sigma)} + 1 \right\} +$$
$$+ 2\varepsilon \int_{Q(2\sigma)} \psi^{2m} \| \tau_{i,h} D''u \|^2 \, dx + c(\sigma, \varepsilon) \, h^2 \int_{Q(\frac{5}{2}\sigma)} \| D''u \|^2 \, dx +$$
$$+ c(K, U, p, \varepsilon) \, |h|^{p-2+\lambda(2-\frac{p}{2})} \int_{Q(\frac{5}{2}\sigma)} ((\sum_{|\alpha|<m} |f^\alpha|)^{\frac{p}{2}} + \| D''u \|^p) \, dx.$$

Let us now set in the last inequality $\varepsilon = \frac{\nu}{8}$ and $p = 2(1+\vartheta) \in \big(2, \min(4, q)\big)$. We have, for every $h : |h| < h_0 \, (< 1)$, that

$$\frac{\nu}{4}\int_{Q(2\sigma)} \psi^{2m} \| \tau_{i,h} D''u \|^2 \, dx \le c(\nu, K, \sigma, m, n) h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \|D''u\|^2\right) dx +$$
$$+ c(\nu, K, U, \vartheta, \lambda, m, n, \sigma) \mid h \mid^{2\vartheta+2\lambda(1-\vartheta)} \left\{ \mid D''u \mid^2_{\vartheta, Q(\frac{5}{2}\sigma)} + 1 \right\} +$$
$$+ c(\nu, K, U, \vartheta) \, |h|^{2\vartheta+\lambda(1-\vartheta)} \int_{Q(\frac{5}{2}\sigma)} ((\sum_{|\alpha|<m} |f^\alpha|)^{1+\vartheta} + \|D''u\|^{2(1+\vartheta)}) \, dx \le$$
$$\le c(\nu, K, U, \vartheta, \lambda, m, n, \sigma) \mid h \mid^{2\vartheta+\lambda(1-\vartheta)} \cdot$$
$$\{1 + |u|^2_{m, Q(\frac{5}{2}\sigma)} + |D''u|^2_{\vartheta, Q(\frac{5}{2}\sigma)} + |u|^{2(1+\vartheta)}_{m, 2(1+\vartheta), Q(\frac{5}{2}\sigma)} + (\sum_{|\alpha|<m} \|f^\alpha\|_{0, \frac{q}{2}, Q(\frac{5}{2}\sigma)})^{1+\vartheta}\}.$$

From (8.3.34), for $|h| < h_0$, we gain

$$\sum_{i=1}^{n} \int_{Q(\sigma)} \|\tau_{i,h} D''u\|^2 \, dx \le$$
$$\le c(\nu, K, U, \vartheta, \lambda, m, n, \sigma) \, |h|^{2\vartheta+\lambda(1-\vartheta)} \{1 + (\sum_{|\alpha|<m} \|f^\alpha\|_{0, \frac{q}{2}, Q(3\sigma)})^{1+\vartheta} + |u|^2_{m, Q(3\sigma)} + |D''u|^2_{\vartheta, Q(3\sigma)}\}. \tag{8.3.42}$$

The procedure if $h_0 \le |h| < 2\sigma$ is similar to the one used in the proof of Theorem 8.2.1 .

Combining both results we obtain that (8.3.42) is true for $|h| < 2\sigma$.

Let us now choose $0 < \vartheta' < \vartheta + \frac{\lambda}{2}(1-\vartheta)$, it implies that $1 + 2\vartheta' - 2\vartheta - \lambda(1-\vartheta) < 1$

inequality, for $2 < p < \min(4,q)$, we carry out

$$|D|\leq \sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}[M(K)(|f^\alpha|+\|D''u\|^2)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^{\frac{4}{p}-1}][\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right\|^{2-\frac{4}{p}}]dx\leq$$

$$\leq c(K,p)\sum_{|\alpha|<m}\left(\int_{Q(\frac{5}{2}\sigma)}\left(|f^\alpha|^{\frac{p}{2}}+\|D''u\|^p\right)\left\|\tau_{i,-h}D^\alpha(\psi^{2m}\tau_{i,h}u)\right\|^{\frac{4-p}{p}\frac{p}{2}}\,dx\right)^{\frac{2}{p}}\cdot$$

$$\cdot\left(\int_{Q(\frac{5}{2}\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\right)^{\frac{p-2}{p}}=$$

$$=c(K,p)\sum_{|\alpha|<m}\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{p-2}\left(|f^\alpha|^{\frac{p}{2}}+\|D''u\|^p\right)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^{\frac{4-p}{2}}\,dx\right)^{\frac{2}{p}}\cdot$$

$$\cdot\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{-2}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2dx\right)^{\frac{p-2}{p}}. \tag{8.3.40}$$

The use of the suitable consequence of Young inequality $ab\leq \varepsilon a^{1+s}+\varepsilon^{-\frac{1}{s}}b^{1+\frac{1}{s}}$, denoting with

$$s=\frac{2}{p-2}\,,\qquad a=\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{-2}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\right)^{\frac{p-2}{p}},$$

$$b=c(K,p)\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{p-2}\left(|f^\alpha|^{\frac{p}{2}}+\|D''u\|^p\right)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right\|^{\frac{4-p}{2}}\,dx\right)^{\frac{2}{p}}$$

and the hypothesis $u\in C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)$ allows us to have

$$|D|\leq\varepsilon\,|h|^{-2}\sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\;+c(K,U,p,\varepsilon)\,|h|^{p-2+\lambda(2-\frac{p}{2})}\cdot$$

$$\cdot\sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}\left(|f^\alpha|^{\frac{p}{2}}+\|D''u\|^p\right)\,dx\,,\quad\forall\varepsilon>0,\,\forall\;2<p<\min(4,q). \tag{8.3.41}$$

Thus we also need Theorem 7.2.2 to obtain

$$\varepsilon\,|h|^{-2}\sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\;\leq\varepsilon\int_{Q(3\sigma)}\left\|D''\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\;\leq$$

$$\leq 2\varepsilon\int_{Q(2\sigma)}\psi^{4m}\left\|\tau_{i,h}\,D''u\right\|^2\,dx\;+\;c(\sigma,\varepsilon)\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}D'u\right\|^2\,dx\;\leq$$

$$\leq 2\varepsilon\int_{Q(2\sigma)}\psi^{2m}\,\left\|\tau_{i,h}\,D''u\right\|^2\,dx\;+\;c(\sigma,\,\varepsilon)\,h^2\int_{Q(\frac{5}{2}\sigma)}\left\|D''u\right\|^2\,dx\,.$$

the last inequality is obtained considering that $\|\tau_{i,h}D'u\|^{\frac{2\vartheta p}{p-2}} \in L^{\frac{p-2}{2\vartheta}}(Q(2\sigma))$, $\|\tau_{i,h}D'u\|^{\frac{2(1-\vartheta)p}{p-2}} \in L^{\frac{p-2}{p-2(1+\vartheta)}}(Q(2\sigma))$, $\frac{2\vartheta}{p-2}+\frac{p-2(1+\vartheta)}{p-2} = 1$. From Theorem 7.2.2 for $t=\frac{4}{5}$ and $Q(\frac{5}{2}\sigma)$ in place of $Q(\sigma)$, $\forall h \in \mathbb{R},\ |h| < \frac{\sigma}{2}$, we attain the inequality

$$\left(\int_{Q(2\sigma)} \|\tau_{i,h}D'u\|^p \, dx\right)^{\frac{2\vartheta}{p}} \le |h|^{2\vartheta} \left(\int_{Q(\frac{5}{2}\sigma)} \|D''u\|^p \, dx\right)^{\frac{2\vartheta}{p}}, \qquad \forall\, p,\, q \,:\, 2(1+\vartheta) < p < q\,. \tag{8.3.36}$$

Using the hypothesis $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ we deduce, for every $2(1+\vartheta) < p < q$, that

$$\left(\int_{Q(2\sigma)} \|\tau_{i,h}D'u\|^{\frac{2(1-\vartheta)p}{p-2(1+\vartheta)}} \, dx\right)^{\frac{p-2(1+\vartheta)}{p}} \le U^{2(1-\vartheta)}\, |h|^{2\lambda(1-\vartheta)} \,[\operatorname{mis} Q(2\sigma)]^{\frac{p-2(1+\vartheta)}{p}}\,. \tag{8.3.37}$$

From (8.3.35)–(8.3.37) we reach

$$\int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h}D'u\|^2 \|D''u\|^2 \, dx \le c(U,\vartheta,n,p,\sigma)\, |h|^{2\vartheta+2\lambda(1-\vartheta)} |u|^{2(1+\vartheta)}_{m,p,Q(\frac{5}{2}\sigma)}\,,\ \forall\, 2(1+\vartheta) < p < q,$$

that, for $p = 1+\vartheta+\frac{q}{2}$ and combined with (8.3.34) for $\rho = \frac{5}{2}\sigma$ and for $p = 1+\vartheta+\frac{q}{2}$, gives

$$\begin{aligned} &\int_{Q(2\sigma)} \psi^{2m}\, \|\tau_{i,h}D'u\|^2\, \|D''u\|^2 \, dx \le \\ &\qquad\qquad \le\; c(K,U,\vartheta,\lambda,n,\sigma)\, |h|^{2\vartheta+2\lambda(1-\vartheta)} \left\{ |D''u|^2_{\vartheta,Q(\frac{5}{2}\sigma)} + 1 \right\}, \end{aligned} \tag{8.3.38}$$

$$\begin{aligned} \frac{\nu}{2} \int_{Q(2\sigma)} \psi^{2m}\, \|\tau_{i,h}\, D''u\|^2 \, dx \le\; & c(\nu,K,\sigma,m,n) h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \|D''u\|^2\right) dx \,+ \\ & +\; c(\nu,K,U,\vartheta,\lambda,n,m,\sigma)\, |h|^{2\vartheta+2\lambda(1-\vartheta)} \left\{ |D''u|^2_{\vartheta,Q(\frac{5}{2}\sigma)} + 1 \right\} \,+\, |\,D\,|\,. \end{aligned} \tag{8.3.39}$$

Let us focus our attention on the term D. Combining (E.3), (8.3.31) and Hölder

and

$$\int_{Q(\rho)} \left\| D''u - (D''u)_{Q(\rho)} \right\|^p dx \leq c(\vartheta, \lambda, m, n, p)\, (\operatorname{mis} Q(\rho))^{1-\frac{p}{q}} \, [D'u]_{\lambda,\overline{\Omega}}^{\frac{p\vartheta}{1+\vartheta}} \, |D''u|_{\vartheta,Q(\rho)}^{\frac{p}{1+\vartheta}} \, . \tag{8.3.32}$$

Thus we deduce that

$$\begin{aligned} |\, u\,|_{m,p,Q(\rho)}^{2(1+\vartheta)} &= \left(\int_{Q(\rho)} \|\, D''u \,\|^p \, dx \right)^{\frac{2(1+\vartheta)}{p}} \leq \\ &\leq 2^{(p-1)\frac{2(1+\vartheta)}{p}} \left\{ \int_{Q(\rho)} \left\| D''u - (D''u)_{Q(\rho)} \right\|^p dx + \int_{Q(\rho)} \left\| (D''u)_{Q(\rho)} \right\|^p dx \right\}^{\frac{2(1+\vartheta)}{p}} \leq \\ &\leq c(U, \vartheta, \lambda, m, n, p) \left\{ |D''u|_{\vartheta,Q(\rho)}^2 + \left\| (D''u)_{Q(\rho)} \right\|^{2(1+\vartheta)} \right\} \leq \\ &\leq c(U, \vartheta, \lambda, m, n, p) \left\{ |\, D''u\,|_{\vartheta,Q(\rho)}^2 + |\, u\,|_{m,Q(\rho)}^{2(1+\vartheta)} \right\}. \end{aligned} \tag{8.3.33}$$

Using interpolation inequality contained in Theorem 7.1.4 we derive

$$\begin{aligned} |u|_{m,Q(\rho)}^{2(1+\vartheta)} &\leq c(\vartheta, n) \left\{ |D''u|_{\vartheta,Q(\rho)}^2 \, \|u\|_{m-1,Q(\rho)}^{2\vartheta} + \rho^{-2(1+\vartheta)} \, \|u\|_{m-1,Q(\rho)}^{2(1+\vartheta)} \right\} \leq \\ &\leq c(K, \vartheta, m, n, \rho) \, \left\{ |D''u|_{\vartheta,Q(\rho)}^2 + 1 \right\} \end{aligned}$$

then, $\forall\, Q(\rho) \subset\subset \Omega$ and $\forall 2 \leq p < q = \frac{2(1+\vartheta)n}{n-2\vartheta\lambda}$, we have

$$|u|_{m,p,Q(\rho)}^{2(1+\vartheta)} \leq c(K, U, \vartheta, \lambda, m, n, p, \rho) \left\{ |D''u|_{\vartheta,Q(\rho)}^2 + 1 \right\}. \tag{8.3.34}$$

By (8.3.31), the hypothesis $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ and Hölder inequality, for every $2(1+\vartheta) < p < q$, it follows

$$\int_{Q(2\sigma)} \psi^{2m} \, \|\tau_{i,h} \, D'u\|^2 \, \|D''u\|^2 \, dx \leq \left(\int_{Q(2\sigma)} \|D''u\|^p \, dx \right)^{\frac{2}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} \, D'u\|^{\frac{2p}{p-2}} \, dx \right)^{\frac{p-2}{p}} = \tag{8.3.35}$$

$$= \left(\int_{Q(2\sigma)} \|\, D''u\,\|^p \, dx \right)^{\frac{2}{p}} \left(\int_{Q(2\sigma)} \|\, \tau_{i,h} D'u \,\|^{\frac{2\vartheta p}{p-2}} \, \|\, \tau_{i,h} D'u\,\|^{\frac{2(1-\vartheta)p}{p-2}} \, dx \right)^{\frac{p-2}{p}} \leq$$

$$\leq \left(\int_{Q(2\sigma)} \|D''u\|^p dx \right)^{\frac{2}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} \, D'u\,\|^p \, dx \right)^{\frac{2\vartheta}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^{\frac{2(1-\vartheta)p}{p-2(1+\vartheta)}} dx \right)^{\frac{p-2(1+\vartheta)}{p}}$$

Recalling that $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, we have

$$\|\tau_{i,h} D' u(x)\| \leq U \; |h|^{\lambda}, \quad \forall x \in Q(2\sigma). \tag{8.3.27}$$

Moreover applying Theorem 7.2.2, for $p = 2$, $t = \frac{4}{5}$ and $Q(\frac{5}{2}\sigma)$ in replacement of $Q(\sigma)$, for every $h \in \mathbb{R}$ such that $|h| < \frac{\sigma}{2}$, we achieve

$$\int_{Q(2\sigma)} \|\tau_{i,h} D' u\|^2 \, dx \leq \; h^2 \int_{Q(\frac{5}{2}\sigma)} \|D'' u\|^2 \, dx, \quad i = 1, 2, \dots, n. \tag{8.3.28}$$

From (8.3.24) – (8.3.28) it follows

$$\begin{aligned}
\nu \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} \, D'' u\|^2 \, dx \leq \{3\varepsilon + c(K, U, \sigma, m, n)(|h| + h^2 + |h|^{\lambda} + |h|^{2\lambda})\} \cdot \\
\cdot \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} \, D'' u\|^2 \, dx \; + c(K, \sigma, m, n, \varepsilon) \, h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \|D'' u\|^2\right) dx + \\
+ \; c(K, \sigma, m, n, \varepsilon) \int_{Q(2\sigma)} \psi^{2m} \, \| \, \tau_{i,h} D' u \, \|^2 \, \| \, D'' u \, \|^2 \; dx \; + \; | \, D \, | \; , \qquad \forall \varepsilon > 0.
\end{aligned} \tag{8.3.29}$$

As in the proof of Theorem 8.2.1 there exists $h_0(\nu, K, U, \lambda, \sigma, m, n), 0 \; < h_0 < \min\{1, \frac{\sigma}{2}\}$, such that

$$c(K, U, \sigma, m, n) \left(|h| + h^2 + |h|^{\lambda} + |h|^{2\lambda}\right) < \frac{\nu}{4}, \quad |h| < h_0$$

then, for $\varepsilon = \frac{\nu}{12}$ we have

$$\begin{aligned}
\frac{\nu}{2} \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} \, D'' u\|^2 \, dx \leq \; c(\nu, K, \sigma, m, n) h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \|D'' u\|^2\right) dx + \\
+ \; c(\nu, K, \sigma, m, n) \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D' u\|^2 \, \|D'' u\|^2 \, dx + |D| \, .
\end{aligned} \tag{8.3.30}$$

Let us now estimate the last two terms in (8.3.30).
Applying Theorem 7.1.7 we have, $\forall \, Q(\rho) = Q(x^0, \rho) \subset\subset \Omega$ and $2 \leq p < q = \frac{2(1+\vartheta)n}{n - 2\vartheta\lambda}$, that

$$u \in H^{m,p}(Q(\rho), \mathbb{R}^N) \tag{8.3.31}$$

and

$$|D''u|^2_{\vartheta,Q(\sigma)} \le c(n) \sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h} D''u\|^2 \, dx, \qquad \forall\, 0 < \vartheta < \frac{\lambda}{2}. \qquad (8.3.23)$$

From (8.3.22), (8.3.23) and (8.3.21) we reach the conclusion. □

Proof. ***of Theorem 3.2.*** Let us fix $x_0 \in \Omega$ and the cube $Q(4\sigma) = Q(x^0, 4\sigma) \subset\subset \Omega$. Let us also consider a positive integer $i \le n$, and a real number h such that $|h| < \frac{\sigma}{2}$. As in the proof of Theorem 8.2.1, for every $\varepsilon > 0$, we have

$$\nu \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h}\, D''u\|^2 \, dx \le \varepsilon \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h}\, D''u\|^2 \, dx +$$
$$+ c(K, \sigma, m, n, \varepsilon) \int_{Q(2\sigma)} \|\tau_{i,h} D'\, u\|^2 \, dx \; + B + C + D \;, \quad (8.3.24)$$

where ψ is the above defined *cut-off* function (see (8.3.1)) and the terms B, C and D are considered in (8.3.7) – (8.3.9).

The terms $|B|$ and $|C|$ can be estimated ,$\forall \varepsilon > 0$, as follows

$$|B| \le \left\{\varepsilon + c(K, \sigma, m, n) \left(|h| + h^2\right)\right\} \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h}\, D''u\|^2 \, dx +$$
$$+ c(K, \sigma, m, n, \varepsilon) h^2 \int_{Q(2\sigma)} \psi^{2m} \left(1 + \|D''u\|^2\right) dx + \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx, \quad (8.3.25)$$

$$|C| \le \int_{Q(2\sigma)} \left\{\varepsilon + c(K, \sigma, m, n) \left(\|\tau_{i,h} D'u\| + \|\tau_{i,h} D'u\|^2\right)\right\} \psi^{2m} \|\tau_{i,h}\, D''u\|^2 \, dx +$$
$$+ c(K, \sigma, m, n, \varepsilon) \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx + c(K, \sigma, m, n, \varepsilon) \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx, \quad (8.3.26)$$

similarly to the proof of Theorem 8.2.1.

Otherwise if $h_0 \leq |h| < 2\sigma$, we get for $i = 1, 2, \ldots, n$,

$$\int_{Q(\sigma)} \|\tau_{i,h} D'' u\|^2 \, dx \leq 2 \int_{Q(\sigma)} \left\|D'' u\left(x + he^i\right)\right\|^2 dx + 2 \int_{Q(\sigma)} \|D'' u(x)\|^2 \, dx \leq$$
$$\leq 2 \int_{Q(3\sigma)} \| D'' u(x) \|^2 \, dx + 2 \int_{Q(\sigma)} \| D'' u(x) \|^2 \, dx \leq$$
$$\leq 4 \int_{Q(3\sigma)} \| D'' u \|^2 \, dx \leq 4 \frac{|h|^\lambda}{h_0^\lambda} \int_{Q(3\sigma)} \| D'' u \|^2 \, dx \leq$$
$$\leq c(\nu, K, U, \lambda, \sigma, m, n) |h|^\lambda \{1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)}\} \quad (8.3.19)$$

From (8.3.18) and (8.3.19), for every $0 < |h| < 2\sigma$, it follows that

$$\sum_{i=1}^n \frac{1}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h} \, D'' u\|^2 \, dx \leq$$
$$\leq c(\nu, K, U, \lambda, \sigma, m, n) \left\{ 1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)} \right\} \frac{1}{|h|^{1+2\vartheta-\lambda}} . \quad (8.3.20)$$

The hypothesis $0 < \vartheta < \frac{\lambda}{2}$ assures that $1 + 2\vartheta - \lambda < 1$, then the function of the variable h that appears in the second member of (8.3.20) is integrable in $[-2\sigma, 2\sigma]$, it implies the integrability in $[-2\sigma, 2\sigma]$ of the left term of inequality (8.3.20) and it follows

$$\sum_{i=1}^n \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h} \, D'' u\|^2 \, dx \leq$$
$$\leq c(\nu, K, U, \vartheta, \lambda, \sigma, m, n) \left\{ 1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)} \right\} . \quad (8.3.21)$$

Finally, recalling that $u \in H^m(\Omega, \mathbb{R}^N)$, from (8.3.21) it follows that $D'' u$ satisfy the hypotheses of Theorem 7.1.3, we can conclude that

$$D'' u \in H^\vartheta(Q(\sigma), \mathcal{R}") \quad (8.3.22)$$

$$|D| \leq \sum_{|\alpha|<m} \int_{Q(3\sigma)} \|a^\alpha(x, D'u, D''u)\| \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)\right\| \, dx \leq$$

$$\leq \; c(K) \sum_{|\alpha|<m} \int_{Q(3\sigma)} \left(|f^\alpha| + \|D''u\|^2\right) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\right\| dx. \tag{8.3.14}$$

On the other hand, using the hypothesis that $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, we easily obtain

$$\left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right)(x)\right\| \leq 2\, U\, |h|^\lambda, \qquad \forall x \in Q(3\sigma). \tag{8.3.15}$$

From (8.3.14) and (8.3.15) we have

$$|D| \leq \; c(K, U, m) \; |h|^\lambda \left(\sum_{|\alpha|<m} \int_{Q(3\sigma)} |f^\alpha| + \|D''u\|^2 \; dx \right). \tag{8.3.16}$$

From (8.3.5), (9.3.15), (9.3.16), (9.3.19), choose $\varepsilon = \frac{\nu}{12}$ we deduce that

$$\nu \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \; dx \; \leq \left\{ \frac{\nu}{4} + c(\nu, K, U, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda}\right) \right\} \cdot$$

$$\cdot \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx + c(\nu, K, U, \sigma, m, n)(h^2 + |h|^\lambda + |h|^{2\lambda}) \int_{Q(3\sigma)} (1 + \sum_{|\alpha|<m} |f^\alpha| + \|D''u\|^2) dx. \tag{8.3.17}$$

Because of the continuity of the function $h \longrightarrow c(K, U, \sigma, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda}\right)$ in the origin, $\exists\, h_0(\nu, K, U, \lambda, \sigma, n)$, $0 < h_0 < \min\{1, \sigma\}$, such that for every $|h| < h_0$, we have

$$c(K, U, \sigma, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda}\right) < \frac{\nu}{4}.$$

Let us consider, at first, that $|h| < h_0 < 1$.

Recalling that $0 < \lambda < 1$ we have $h^2 + |h|^\lambda + |h|^{2\lambda} \leq 3\,|h|^\lambda$ and taking into consideration that $\psi(x) = 1$ in $Q(\sigma)$, from (8.3.17), it follows, for $i = 1, 2, \dots, n$,

$$\frac{\nu}{2} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2 \; dx \leq c(\nu, K, U, \sigma, m, n)\, |h|^\lambda \, \{1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)}\}. \tag{8.3.18}$$

Similarly, using (E.4), for the term C we have

$$|C| \leq \int_{Q(2\sigma)} \sum_{|\alpha|=m} \|a^\alpha(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) -$$

$$- \ a^\alpha(x, D'u(x), D''u(x) + \tau_{i,h} D''u(x))\| \left\| D^\alpha \left(\psi^{2m} \tau_{i,h} u\right) \right\| \, dx \ \leq$$

$$\leq c(K,n) \int_{Q(2\sigma)} \|\tau_{i,h} \, D'u\| \, (1 + \|D''u\| + \|\tau_{i,h} D''u\|) \left(c(\sigma,m) \psi^{2m-1} \|\tau_{i,h} \, D'u\| + \psi^{2m} \|\tau_{i,h} D''u\|\right) dx =$$

$$= \int_{Q(2\sigma)} [\psi^m \, \|\tau_{i,h} D''u\|] \, [c(K,n) \psi^m \, \|\tau_{i,h} D'u\| \, (1 + \|D''u\|)] \ dx +$$

$$+ \ c(\, K, n\,) \int_{Q(2\sigma)} \psi^{2m} \ \| \, \tau_{i,h} \, D'u \, \| \ \| \, \tau_{i,h} \ D''u \, \|^2 \ dx +$$

$$+ \int_{Q(2\sigma)} \left[c(\sigma,m) \|\tau_{i,h} \, D'u\|\right] \left[c(K,n) \psi^{2m-1} \|\tau_{i,h} \, D'u\| (1 + \|D''u\| + \|\tau_{i,h} D''u\|)\right] dx.$$

Because of $u \in C^{m-1,\lambda}(\Omega, \mathbb{R}^N)$, for every $\varepsilon > 0$, it follows

$$|C| \leq \varepsilon \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 dx + c(K,n,\varepsilon) \int_{Q(2\sigma)} \psi^{2m} \, \|\tau_{i,h} D'u\|^2 \left(1 + \|D''u\|^2\right) dx +$$

$$+ \, c(K,U,n) \, |h|^\lambda \int_{Q(2\sigma)} \psi^{2m} \, \|\tau_{i,h} D''u\|^2 dx + c(\sigma,m) \int_{Q(2\sigma)} \|\tau_{i,h} \, D'u\|^2 dx +$$

$$+ \, c(K,n) \int_{Q(2\sigma)} \psi^{2m} \, \| \, \tau_{i,h} D'u \, \|^2 \left(1 + \| \, D''u \, \|^2 + \| \, \tau_{i,h} \, D''u \, \|^2\right) \ dx \ \leq$$

$$\leq \left\{ \varepsilon + c \, (K,U,n) \left(| \ h \ |^\lambda + | \ h \ |^{2\lambda}\right) \right\} \int_{Q(2\sigma)} \psi^{2m} \, \| \, \tau_{i,h} \ D''u \, \|^2 \ dx \ +$$

$$+ \, c(K,\sigma,m,n,\varepsilon) \int_{Q(2\sigma)} \|\tau_{i,h} \, D'u\|^2 \, dx + c(K,U,n,\varepsilon) |h|^{2\lambda} \int_{Q(2\sigma)} \|D''u\|^2 dx.$$

Using again (8.3.11), we get

$$|C| \leq \left\{ \varepsilon + c(K,U,n) \left(|h|^\lambda + |h|^{2\lambda}\right) \right\} \int_{Q(2\sigma)} \psi^{2m} \, \|\tau_{i,h} D''u\|^2 \ dx +$$

$$+ \, c(K,U,\sigma,m,n,\varepsilon) \left(h^2 + |h|^{2\lambda}\right) \int_{Q(3\sigma)} \|D''u\|^2 \ dx \, , \quad \forall \varepsilon > 0. \tag{8.3.13}$$

Finally, let us estimate the terms D. For the hypothesis (E.3), we have

From (8.3.10) and (8.3.11), we have

$$|A| \le \varepsilon \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 \, dx + c(K,\sigma,m,n,\varepsilon)\, h^2 \int_{Q(3\sigma)} \left\|D''u\right\|^2 \, dx\,, \quad \forall \varepsilon > 0\,. \tag{8.3.12}$$

From (E.4) we can majorize the term B as follows

$$\begin{aligned}
|B| \le & \int_{Q(2\sigma)} \sum_{|\alpha|=m} \big\| a^{\alpha}(x + he^i, D'u(x) + \tau_{i,h}\, D'u(x), D''u(x) + \tau_{i,h}\, D''u(x)) - \\
& - a^{\alpha}(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x))\| \left\| D^{\alpha}\left(\psi^{2m} \tau_{i,h}\, u\right)\right\| \, dx \le \\
& \le c(K,n)\, |h| \int_{Q(2\sigma)} (1 + \|D''u\| + \|\tau_{i,h} D''u\|) \left(\frac{2mk}{\sigma} \psi^{2m-1} \left\|\tau_{i,h}\, D'u\right\| + \psi^{2m} \left\|\tau_{i,h} D''u\right\| \right) dx \le \\
& \le c(K,m,n)\, |h| \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 \, dx + \int_{Q(2\sigma)} \big[c(K,m,n)\, |h|\, \psi^m \left(1 + \|D''u\|\right)\big] \\
& \big[\psi^m \left\|\tau_{i,h} D''u\right\|\big] dx + \int_{Q(2\sigma)} [c(\sigma,m) \left\|\tau_{i,h} D'u\right\|] \, [c(K,n)\, |h|\, \psi^m \left(1 + \|D''u\| + \|\tau_{i,h} D''u\|\right)] \, dx.
\end{aligned}$$

Then, for every $\varepsilon > 0$, we have

$$\begin{aligned}
|B| \le & \, c(K,m,n)\, |h| \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 \, dx + \varepsilon \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 \, dx + \\
& + c(K,m,n,\varepsilon) h^2 \int_{Q(2\sigma)} \psi^{2m} \left(1 + \left\|D''u\right\|^2\right) dx + c(\sigma,m) \int_{Q(2\sigma)} \left\|\tau_{i,h} D'u\right\|^2 dx + \\
& + c(K,m,n)\; h^2 \int_{Q(2\sigma)} \psi^{2m} \left(1 + \left\|D''u\right\|^2 + \left\|\tau_{i,h} D''u\right\|^2 \right) \, dx \; = \\
& = \left\{ \varepsilon + c(K,m,n)\, \left(h^2 + |h|\right) \right\} \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}\, D''u\right\|^2 \, dx \; + \\
& + c(K,m,n,\varepsilon) h^2 \int_{Q(2\sigma)} \psi^{2m} (1 + \left\|D''u\right\|^2) dx + c(\sigma,m) \int_{Q(2\sigma)} \left\|\tau_{i,h}\, D'u\right\|^2 dx.
\end{aligned}$$

Using (8.3.11), we can estimate $|B|$ as follows

$$\begin{aligned}
|B| \le & \left\{\varepsilon + c(K,m,n) \left(h^2 + |h|\right)\right\} \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 \, dx + \\
& + c(K,\sigma,m,n,\varepsilon)\, h^2 \int_{Q(3\sigma)} \left(1 + \left\|D''u\right\|^2\right) \, dx, \quad \forall \varepsilon > 0.
\end{aligned}$$

$$B = -\int_\Omega \sum_{|\alpha|=m} \Big(a^\alpha(x+he^i, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) -$$

$$- a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) \,|\, D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\Big)\,dx, \tag{8.3.7}$$

$$C = -\int_\Omega \sum_{|\alpha|=m} (a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) -$$

$$- a^\alpha(x, D'u(x), D''u(x) + \tau_{i,h}\,D''u(x)) \,|\, D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\Big)\,dx, \tag{8.3.8}$$

$$D = -\sum_{|\alpha|<m} \int_\Omega \Big(a^\alpha\,(x, D'u(x), D''u) \,|\, \tau_{i,-h}\,D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\Big)\,dx. \tag{8.3.9}$$

Let us estimate the terms A, B, C and D.

Applying hypothesis (E.5) and the properties of the function ψ, we have

$$\begin{aligned} |A| \le 2m\int_{Q(2\sigma)} \psi^{2m-1} \sum_{|\alpha|=m} \|a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x)) - \\ - a^\alpha\,(x, D'u(x), D''u(x))\| \,\Big\|\,(\,D^\alpha\,\psi\,)\,\tau_{i,h}\,u\,\Big\|\,dx \le \\ \le c(\,K,\,m\,)\int_{Q(2\sigma)} \psi^{2m-1} \sum_{|\alpha|=m} |D^\alpha\psi|\,\|\tau_{i,h}D''u\|\,\|\tau_{i,h}D'u\|\,dx \le \\ \le c\,(K,\sigma,m,n)\int_{Q(2\sigma)} \psi^{2m-1}\,\|\tau_{i,h}D''u\|\|\tau_{i,h}D'u\|\,dx \le \\ \le c\,(K,\sigma,m,n)\int_{Q(2\sigma)} \psi^{m}\,\|\tau_{i,h}D''u\|\,\|\tau_{i,h}D'u\|\,dx. \end{aligned}$$

Then, for every $\varepsilon > 0$, we have

$$|A| \le \varepsilon\int_{Q(2\sigma)} \psi^{2m}\,\|\tau_{i,h}D''u\|^2\,dx + c(K,\sigma,m,n,\varepsilon)\int_{Q(2\sigma)} \|\tau_{i,h}D'u\|^2\,dx\,, \tag{8.3.10}$$

On the other hand, using Theorem 7.2.2 for $p = 2$, $t = \frac{2}{3}$ and $Q(3\sigma)$ instead of $Q(\sigma)$, for every $h \in \mathbb{R}$ such that $|h| < \left(1 - \frac{2}{3}\right)3\sigma = \sigma$, we have

$$\int_{Q(2\sigma)} \|\tau_{i,h}u\|_N^2\,dx \le h^2\int_{Q(3\sigma)} \|D_iu\|_N^2\,dx\,, \qquad i = 1, 2, \ldots, n\,. \tag{8.3.11}$$

formula (9.3.8) becomes:

$$\int_{\Omega}\psi^{2m}\sum_{|\alpha|=m}(a^{\alpha}(x,D'u(x),D''u(x)+\tau_{i,h}D''u(x))-a^{\alpha}(x,D'u(x),D''u(x))|\tau_{i,h}D^{\alpha}u)\ dx= \tag{8.3.4}$$

$$=-2m\int_{\Omega}\psi^{2m-1}\sum_{|\alpha|=m}(a^{\alpha}(x,D'u(x),D''u(x)+\tau_{i,h}D''u(x))-a^{\alpha}(x,D'u(x),D''u(x))|\,(D^{\alpha}\psi)\,\tau_{i,h}u)dx-$$

$$-\int_{\Omega}\sum_{|\alpha|=m}(a^{\alpha}(x,D'u(x)+\tau_{i,h}D'u(x),D''u(x)+\tau_{i,h}D''u(x))-a^{\alpha}(x,D'u(x),D''u(x)+\tau_{i,h}D''u(x))|$$

$$|\ D^{\alpha}\left(\psi^{2m}\tau_{i,h}\,u\right)\ dx-\int_{\Omega}\sum_{|\alpha|=m}\left(a^{\alpha}(x+he^{i},D'u(x)\ +\ \tau_{i,h}\ D'u(x),\ D''u(x)\ +\ \tau_{i,h}\ D''u(x))-\right.$$

$$\left.-\ a^{\alpha}\,(\,x,\ D'u(x)\ +\ \tau_{i,h}\,D'u(x),\ D''u(x)\ +\ \tau_{i,h}\,D''u(x)\,)\ |\ D^{\alpha}\left(\psi^{2m}\ \tau_{i,h}\,u\right)\right)dx\ -$$

$$-\sum_{|\alpha|<m}\int_{\Omega}\left(a^{\alpha}\,(\,x,D'u,D''u\,)\ |\ \tau_{i,-h}\,D^{\alpha}\,\left(\psi^{2m}\,\tau_{i,h}\,u\right)\right)\ dx.$$

Using hypotheses (E.5) we can minimize the first member of (8.3.4), as follows

$$\int_{\Omega}\psi^{2m}\sum_{|\alpha|=m}(a^{\alpha}(x,D'u(x),\ D''u(x)+\tau_{i,h}D''u(x))-a^{\alpha}(x,D'u(x),D''u(x))|\tau_{i,h}D^{\alpha}u)\ dx=$$

$$=\int_{Q(2\sigma)}\psi^{2m}\ (a(x,D'u(x),D''u(x)\ +\ \tau_{i,h}\,D''u(x))\ -\ a(x,D'u(x),D''u(x))\,|\,\tau_{i,h}D''u)\ \ dx\geq$$

$$\geq\ \nu\int_{Q(2\sigma)}\psi^{2m}\ \left\|\tau_{i,h}D''u\right\|^{2}\ dx,$$

then we obtain

$$\nu\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}D''u\right\|^{2}\,dx\ \leq A+B+C+D\,, \tag{8.3.5}$$

where

$$A=-2m\int_{\Omega}\psi^{2m-1}\sum_{|\alpha|=m}(a^{\alpha}(x,D'u(x),D''u(x)+\tau_{i,h}D''u(x))\ -$$

$$-\,a^{\alpha}(x,D'u(x),D''u(x))\,|\ (D^{\alpha}\psi)\ \tau_{i,h}u)\,dx, \tag{8.3.6}$$

Let us also consider $i \leq n$ a positive integer, h a real number, $|h| < \sigma$, and let us also set

$$\varphi = \tau_{i,-h}\left(\psi^{2m}\tau_{i,h}\, u\right), \tag{8.3.2}$$

it follows that $\varphi \in H_0^m(\Omega, \mathbb{R}^N) \cap H^{m-1,\infty}(\Omega, \mathbb{R}^N)$. From (9.2.1), written for this "test function" φ, it follows

$$\int_\Omega \sum_{|\alpha|=m} \left(\tau_{i,h} a^\alpha\left(x, Du\right) | D^\alpha\left(\psi^{2m}\tau_{i,h}\, u\right)\right) dx$$
$$= -\sum_{|\alpha|<m} \int_\Omega \left(a^\alpha\left(x, Du\right) | \tau_{i,-h} D^\alpha\left(\psi^{2m}\tau_{i,h}\, u\right)\right) dx. \tag{8.3.3}$$

On the other hand, for every α such that $|\alpha| = m$ and for a. e. $x \in Q(2\sigma)$, it follows:

$$\tau_{i,h} a^\alpha(x, Du(x)) \;=\; \tau_{i,h} a^\alpha(x, D'u(x), D''u(x))$$
$$=\; a^\alpha(x + he^i, D'u(x + he^i), D''u(x + he^i)) - a^\alpha(x, D'u(x), D''u(x))$$
$$= a^\alpha(x + he^i, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) - a^\alpha(x, D'u(x), D''u(x))$$
$$= \big[a^\alpha(x + he^i, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x))$$
$$-\; a^\alpha(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x))]$$
$$+\; [a^\alpha(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x))$$
$$- a^\alpha(x, D'u(x), D''u(x) + \tau_{i,h} D''u(x))]$$
$$+[\, a^\alpha(x, D'u(x), D''u(x) + \tau_{i,h} D''u(x)) - a^\alpha(x, D'u(x), D''u(x))]\ .$$

Regarding in mind that

$$D^\alpha(\psi^{2m}\tau_{i,h}\, u) = \psi^{2m}\tau_{i,h} D^\alpha u + 2m\psi^{2m-1}\left(D^\alpha\psi\right)\tau_{i,h} u\,,$$

8.2.2 Local differentiability result in H^{m+1} space

Let us now apply the previous local differentiability properties in $H^{m+\vartheta}(\Omega, \mathbb{R}^N)$, $0 < \vartheta < 1$ to reach the main objective of this chapter (see also [18]).

Theorem 8.2.4. *(Main result) If $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$, is a weak solution of the system* (8.1.1) *satisfying the hypotheses* $(E.1)$*,* $(E.2)$*,* $(E.4)$*,* $(E.5)$ *and, for $f^\alpha \in L^{\frac{2n}{n-2\lambda}}(\Omega)$ assumption* $(E.3)$*, then*

$$u \in H^{m+1}_{\text{loc}}(\Omega, \mathbb{R}^N) \tag{8.2.9}$$

and, for every cube $Q(4\sigma) \subset\subset \Omega$, the following inequality is true

$$\begin{aligned} &|u|^2_{m+1,Q(\sigma)} \\ &\leq c(\nu, K, U, \lambda, \sigma, m, n) \left(1 + \Big(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(4\sigma)} \Big)^2 + |u|^2_{m,Q(4\sigma)} + |u|^4_{m,4,Q(4\sigma)} \right), \end{aligned} \tag{8.2.10}$$

where $K = \sup_{\overline{\Omega}} \|u\|$ and $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)}$.

8.3 Proofs of main goals

In this section we give the proof of the main results of this chapter.

8.3.1 Proofs of local differentiability results in $H^{m+\vartheta}$ spaces

We start with the proof of local differentiability results in $H^{m+\vartheta}(\Omega, \mathbb{R}^N)$, $0 < \vartheta < \frac{\lambda}{2}$.

Proof. ***of Theorem 8.2.1*** Let us choose $x_0 \in \Omega$ and a generic cube $Q(4\sigma) = Q(x^0, 4\sigma) \subset\subset \Omega$, let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ a *cut-off* function having the following properties:

$$0 \leq \psi \leq 1 \text{ in } \mathbb{R}^n, \ \psi = 1 \text{ in } Q(\sigma), \ \psi = 0 \text{ in } \mathbb{R}^n \setminus Q(2\sigma), \ \|D\psi\| \leq \frac{k}{\sigma} \text{ in } \mathbb{R}^n. \tag{8.3.1}$$

where $|D''u|^2_{\vartheta,Q(\sigma)} = \sum_{|\alpha|=m} |D^\alpha u|^2_{\vartheta,Q(\sigma)}$, $K = \sup_{\overline{\Omega}} \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)}$.

Theorem 8.2.2. *If* $u \in H^{m+\vartheta}(\Omega,\mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N), 0 < \vartheta, \lambda < 1$, *is a weak solution of the system* (8.1.1), *the assumptions* $(E.1)$, $(E.2)$, $(E.4)$, $(E.5)$ *and the condition* $(E.3)$ *with* $f^\alpha \in L^{\frac{q}{2}}(\Omega), q = \frac{2(1+\vartheta)n}{n-2\vartheta\lambda}$, *are true, we have*

$$u \in H^{m+\vartheta'}_{\text{loc}}(\Omega,\mathbb{R}^N), \qquad \forall \vartheta' \in \left(0, \vartheta + \frac{\lambda}{2}(1-\vartheta)\right), \tag{8.2.4}$$

and, for every cube $Q(4\sigma) \subset\subset \Omega$, *we have the following inequality*

$$\begin{aligned} |D''u|^2_{\vartheta',Q(\sigma)} &\le c(\nu, K, U, \vartheta, \vartheta', \lambda, \sigma, m, n) \\ &\cdot \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(4\sigma)} \right)^{1+\vartheta} + |u|^2_{m,Q(4\sigma)} + |D''u|^2_{\vartheta,Q(4\sigma)} \right\}, \end{aligned} \tag{8.2.5}$$

where $|D''u|^2_{\vartheta,Q(\sigma)} = \sum_{|\alpha|=m} |D^\alpha u|^2_{\vartheta,Q(\sigma)}$, $K = \sup_{\overline{\Omega}} \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)}$.

Applying an iterative method, we have the following result.

Theorem 8.2.3. *If* $u \in H^m(\Omega,\mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (8.1.1), *the hypotheses* $(E.1)$, $(E.2)$, $(E.4)$, $(E.5)$ *and the condition* $(E.3)$ *with* $f^\alpha \in L^{\frac{2n}{n-2\lambda}}(\Omega)$ *are verified, then*

$$u \in H^{m+\vartheta}_{\text{loc}}(\Omega,\mathbb{R}^N), \qquad \forall\, \vartheta \,:\, 0 < \vartheta < 1. \tag{8.2.6}$$

Moreover, for every cube $Q(\sigma) \subset\subset Q(\sigma_0) \subset\subset \Omega$, *we have*

$$\begin{aligned} &|D''u|^2_{\vartheta,Q(\sigma)} \\ &\le c(\nu, K, U, \vartheta, \lambda, \sigma, \sigma_0, m, n) \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(\sigma_0)} \right)^{1+\vartheta} + |u|^2_{m,Q(\sigma_0)} \right\}, \end{aligned} \tag{8.2.7}$$

where $K = \sup_{\overline{\Omega}} \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)}$.
Moreover

$$u \in H^{m,4}_{\text{loc}}(\Omega,\mathbb{R}^N). \tag{8.2.8}$$

(E.5) *for every* $(x,p') \in \Omega \times \mathcal{R}'$, *the functions* $p'' \longrightarrow a^\alpha(x,p',p'')$, $|\alpha| = m$, *are strictly monotone with non-linearity* $q = 2$, *so that there exist two positive constants* $M(K)$ *and* $\nu(K)$ *such that* $\forall (x,p') \in \Omega \times \mathcal{R}'$, *with* $\|p'\| \le K$, *and* $\forall p'', q'' \in \mathcal{R}''$, *we obtain:*

$$\|a(x,p',p'') - a(x,p',q'')\| \le M(K)\, \|p'' - q''\| \ ,$$
$$(a(x,p',p'') - a(x,p',q'')|p'' - q'') \ge \nu(K)\, \|p'' - q''\|^2 \ .$$

Remark 8.1.1. We point out that the assumptions $(E.1)$ — $(E.5)$ are more general than the one used by Campanato and Cannarsa in [7].

8.2 Main results

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \ge 2$.

We say *weak solution* of the system (9.2) a function $u \in H^m(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega \sum_{|\alpha|\le m} (a^\alpha(x,Du)|\, D^\alpha \varphi)\; dx = 0, \ \ \forall\, \varphi \in H^m_0(\Omega,\mathbb{R}^N) \,\cap\, H^{m-1,\infty}(\Omega,\mathbb{R}^N). \quad (8.2.1)$$

8.2.1 Local fractional differentiability results

Let us now state the local fractional differentiability results (see also [18]).

Theorem 8.2.1. *If* $u \in H^m(\Omega,\mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (8.1.1) *and the assumptions* $(E.1)$ — $(E.5)$ *are satisfied, then*

$$u \in H^{m+\vartheta}_{\mathrm{loc}}(\Omega,\mathbb{R}^N), \quad \forall \vartheta \in \left(0, \frac{\lambda}{2}\right), \quad (8.2.2)$$

moreover, for every cube $Q(4\sigma) \subset\subset \Omega$, *we have the following inequality*

$$|D''u|^2_{\vartheta,Q(\sigma)} \le c(\nu,K,U,\vartheta,\lambda,\sigma,m,n) \left(1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(4\sigma)} + |u|^2_{m,Q(4\sigma)} \right), \quad (8.2.3)$$

and $p = \{p^\alpha\}_{|\alpha|\le m}$, $p^\alpha \in \mathbb{R}^N$, the generic point of $\mathcal{R}$. If $p \in \mathcal{R}$, we set $p = (p', p'')$ where $p' = \{p^\alpha\}_{|\alpha|<m} \in \mathcal{R}' = \prod_{|\alpha|<m} \mathbb{R}^N_\alpha$, $p'' = \{p^\alpha\}_{|\alpha|=m} \in \mathcal{R}'' = \prod_{|\alpha|=m} \mathbb{R}^N_\alpha$, and

$$\|p\|^2 = \sum_{|\alpha|\le m} \|p^\alpha\|^2_N, \quad \|p'\|^2 = \sum_{|\alpha|<m} \|p^\alpha\|^2_N, \qquad \|p''\|^2 = \sum_{|\alpha|=m} \|p^\alpha\|^2_N.$$

We consider, as usual,

$$D_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n; \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}.$$

Let us consider the following differential nonlinear variational system of order $2m$:

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha a^\alpha(x, Du) = 0 \tag{8.1.1}$$

where $a^\alpha(x,p) = a^\alpha(x,p',p'')$ are functions of $\Lambda = \Omega \times \mathcal{R}$ in $\mathbb{R}^N$, satisfying the following conditions:

(E.1) *for every α and for every $p \in \mathcal{R}$, the function $x \longrightarrow a^\alpha(x,p)$, defined in Ω having values in $\mathbb{R}^N$, is measurable in x;*

(E.2) *for every α and for every $x \in \Omega$, the function $p \longrightarrow a^\alpha(x,p)$, defined in $\mathcal{R}$ having values in $\mathbb{R}^N$, is continuous in p;*

(E.3) *for every α, such that $|\alpha| < m$, for every $(x,p',p'') \in \Omega \times \mathcal{R}$, with $\|p'\|_N \le K$, we have:*

$$\|a^\alpha(x,p',p'')\| \le M(K)\left(|f^\alpha(x)| + \sum_{|\alpha|=m} \|p^\alpha\|^2_N\right) = M(K)\left(|f^\alpha(x)| + \|p''\|^2\right),$$

where $f^\alpha \in L^1(\Omega)$;

(E.4) *for every $x \in \Omega, \forall y \in Q\left(x, \frac{1}{\sqrt{n}} d_x\right)$ $\forall p', q' \in \mathcal{R}'$, where $\|p'\|, \|q'\| \le K$ and for every $p'' \in \mathcal{R}''$, we have:*

$$\|a(x,p',p'')\| \le M(K)\,(1 + \|p''\|)$$

$$\|a(x,p',p'') - a(y,q',p'')\| \le M(K)\,(\|x-y\| + \|p'-q'\|)\,(1 + \|p''\|)\,;$$

where $a\,(x,p) \equiv (a^\alpha(x,p))_{|\alpha|=m}$ and $d_x = \operatorname{dist}(\{x\}, \partial\Omega) > 0$.

Chapter 8

Nonlinear elliptic systems

We continue the study of regularity properties for solutions of elliptic systems started in [15] and continued in [18] (see also [16]), proving, in a bounded open set Ω of $\mathbb{R}^n$, local differentiability and partial Hölder continuity of the weak solutions u of nonlinear elliptic systems of order $2m$ in divergence form

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha}\, a^{\alpha}\,(x, Du) = 0.$$

Specifically, are generalized the results obtained by Campanato and Cannarsa, contained in [7], under the hypothesis that the coefficient $a^{\alpha}\,(x, Du)$, are strictly monotone with nonlinearity q = 2.

8.1 Problem formulation

Let us set m, N positive integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_n$ the order of α. We denote by $\mathcal{R}$ the Cartesian product

$$\mathcal{R} = \prod_{|\alpha|\leq m} \mathbb{R}^N_{\alpha}$$

Theorem 7.2.2.. *Let $u \in H^{1,p}(B(\rho), \mathbb{R}^N)$ for $a, \rho > 0$, $1 \le p < +\infty$ and N be a positive integer. Then, for every $\tau \in (0,1)$ and every $h \in \mathbb{R}$, $|h| < (1-\tau)\rho$, we have*

$$\|\tau_{i,h} u\|_{0,p,B(\tau\rho)} \le |h| \, \|D_i u\|_{0,p,B(\rho)}\,, \quad i = 1, 2, \dots, n.$$

Theorem 7.2.3. *(see [30], [27]). Let N be a positive integer and Ω a cube of $\mathbb{R}^n$. If*

$$u \in W^{m,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N),$$

with $m \ge 2$, m integer, $1 < r < \infty$, $s \ge 0$, s integer, $0 < \lambda < 1$, $s < m-1$, then, for each integer j with $s + \lambda < j < m$, there exists two constants c_1 and c_2 (depending on $\Omega, m, r, s, \lambda, j$) such that

$$\max_{|\alpha|=j} |D^\alpha u|_{0,p,\Omega} \le c_1 \left(\max_{|\alpha|=m} |D^\alpha u|_{0,r,\Omega} \right)^{\delta} \cdot \left(\max_{|\alpha|=s} [D^\alpha u]_{\lambda,\Omega} \right)^{1-\delta} + \; c_2 \max_{|\alpha|=s} [D^\alpha u]_{\lambda,\Omega}$$

where $\frac{1}{p} = \frac{j}{n} + \delta \left(\frac{1}{r} - \frac{m}{n} \right) - (1-\delta) \frac{s+\lambda}{n}, \qquad \forall\, \delta \in \left[\frac{j-s-\lambda}{m-s-\lambda}, 1 \right[.$

Theorem 7.2.4. *(see [23]). Let N be a positive integer and Ω a cube of $\mathbb{R}^n$. If*

$$u \in W^{m+\theta,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N),$$

with $m \ge 1$, m integer, $0 < \theta < 1$, $1 < r < \infty$, $s \ge 0$, s integer, $0 < \lambda < 1$, $s < m$, then, for each integer j with $max(s + \lambda, m + \theta - \frac{n}{r}) < j < m + \theta$, it results

$$u \in W^{j,p}(\Omega, \mathbb{R}^N)$$

and there exists a constant c (depending on $\Omega, m, \theta, r, s, \lambda, j, n, \delta$) such that

$$\|u\|_{j,p,\Omega} \le c \|u\|^{\delta}_{m+\theta,r,\Omega} \, \|u\|^{1-\delta}_{C^{s,\lambda}(\Omega, \mathbb{R}^N)}\,,$$

where $\frac{1}{p} = \frac{j}{n} + \delta \left(\frac{1}{r} - \frac{m+\theta}{n} \right) - (1-\delta) \frac{s+\lambda}{n}, \qquad \forall \delta \in \left[\frac{j-s-\lambda}{m+\theta-s-\lambda}, 1 \right[$ *with* $(1-\delta)(s+\lambda) + \delta(m+\theta)$ *non integer.*

Let k a positive integer, $p \in [1,+\infty[$, $\vartheta \in (0,1)$, in the following we will consider the spaces

$$L^p(-T,0,H^{k,p}(\Omega,\mathbb{R}^N))$$
$$=\left\{u(x,t)|u(\cdot,t) \in H^{k,p}(\Omega,\mathbb{R}^N) \text{ for a.e. } t \in (-T,0) \text{ and } \int_{-T}^{0} \|u(\cdot,t)\|^p_{k,p,\Omega}\, dt < \infty\right\}$$

and

$$L^p(-T,0,H^{k+\theta,p}(\Omega,\mathbb{R}^N))$$
$$=\left\{u(x,t)|u(\cdot,t) \in H^{k+\theta,p}(\Omega,\mathbb{R}^N) \text{ for a.e. } t \in (-T,0) \text{ and } \int_{-T}^{0} \|u(\cdot,t)\|^p_{k+\theta,p,\Omega}\, dt < \infty\right\}.$$

Let us now state some properties useful in the sequel.

Let $\tau \in]0,1[$, ρ and a two positive numbers and $h \in \mathbb{R}\setminus\{0\}$, where $|h| < (1-\tau)\rho$. If u is a function from $B(\rho)\times(-a,0)$ in $\mathbb{R}^N$ and $X=(x,t) \in B(\tau\rho)\times(-a,0)$, we set

$$\tau_{i,h}u(X) = u(x+he^i,t) - u(X), \quad i=1,2,\ldots,n, \tag{7.2.1}$$

where $\{e^i\}_{i=1,2,\ldots,n}$ is the canonic basis of $\mathbb{R}^n$.

Let us now state the following results, proved in [20] and [30], useful to achieve the main result of the note.

Theorem 7.2.1.. *If* $u \in L^p(-a,0,L^p(B(2\rho),\mathbb{R}^N))$, $a,\rho>0$, $1<p<+\infty$, N *is a positive integer and exists* $M>0$ *such that*

$$\int_{-a}^{0} dt \int_{B(\rho)} \|\tau_{i,h}\, u\|^p\, dx \le |h|^p M, \quad \forall\ |h| < (1-\tau)\rho,\ \forall i=1,\ldots,n,$$

then $u \in L^p(-a,0,H^{1,p}(B(\rho),\mathbb{R}^N))$ *and*

$$\int_{-a}^{0} dt \int_{B(\rho)} \|D_i u\|^p\, dx \le M, \quad \forall\ i=1,\ldots,n.$$

where $q = \dfrac{2(1+\vartheta)n}{n-2\vartheta\lambda}$. *Specifically*

$$D_i\, u \in L^p(Q(\sigma), \mathbb{R}^N), \qquad \forall 1 \le p < q,$$

and is true the following inequality

$$\int_{Q(\sigma)} \left\| D_i\, u - (D_i\, u)_{Q(\sigma)} \right\|^p dx \le c(\vartheta, n, p, q)(\operatorname{mis} Q(\sigma))^{1-\frac{p}{q}} \, [\, u\,]^{\frac{p\vartheta}{1+\vartheta}}_{\lambda, Q(\sigma)} \sum_{j=1}^{n} |D_j\, u|^{\frac{p}{1+\vartheta}}_{\vartheta, Q(\sigma)} \, .$$

7.2 Parabolic systems: notations and preliminary results

Let Ω be an bounded open set in $\mathbb{R}^n$, $n > 2$, $x = (x_1, x_2, \dots, x_n)$ denotes a generic point therein, $0 < T < \infty$ and Q the cylinder $\Omega \times (-T, 0)$, let N be a positive integer. In Q we consider the following parabolic metric

$$d(X,Y) \,=\, max\{||x-y||_n,\, |t-\tau|^{\frac{1}{2}}\}, \quad X \,=(x,t),\, Y \,=\, (y,\tau).$$

Let us set k a positive integer greater than 1, $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ respectively the scalar product and the norm in $\mathbb{R}^k$. If there is no ambiguity we omit the index k.

Let k be a nonnegative integer and $\lambda \in]0,1]$. We denote by $C^{k,\lambda}(\overline{Q}, \mathbb{R}^N)$ the subspace of $C^k(\overline{Q}, \mathbb{R}^N)$ of functions $u : \overline{Q} \longrightarrow \mathbb{R}^N$ which satisfy a Hölder condition of exponent λ, together with all their derivatives $D^\alpha u, |\alpha| \le k$. If $u \in C^{k,\lambda}(\overline{Q}, \mathbb{R}^N)$, then we set

$$\|u\|_{C^{k,\lambda}(\overline{Q},\mathbb{R}^N)} = \sum_{|\alpha|\le k} \sup_{\overline{Q}} \|D^\alpha u\|_N + \sum_{|\alpha|=k} [D^\alpha u]_{\lambda,\overline{Q}}$$

where

$$[D^\alpha u]_{\lambda,\overline{Q}} = \sup_{\substack{X,Y\in\overline{Q}\\ X\neq Y}} \frac{\|D^\alpha u(X) - D^\alpha u(Y)\|_N}{d^\lambda(X,Y)} < +\infty, \; \forall \alpha \,:\, |\alpha| = k.$$

The space $C^{k,\lambda}(\overline{Q}, \mathbb{R}^N)$ is a Banach space, provided with the norm

$$\|u\|_{C^{k,\lambda}(\overline{Q},\mathbb{R}^N)} = \|u\|_{C^k(\overline{Q},\mathbb{R}^N)} + \sum_{|\alpha|=k} [D^\alpha u]_{\lambda,\overline{Q}} \, .$$

Theorem 7.1.3. *If $u \in L^2(Q(3\sigma),\mathbb{R}^N)$ and, for $\vartheta \in (0,1)$, is finite*

$$\sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h}\, u(x)\|^2 \, dx,$$

then $u \in H^{\vartheta}(Q(\sigma),\mathbb{R}^N)$ and

$$|u|^2_{\vartheta,Q(\sigma)} \leq c(n) \sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h}\, u(x)\|^2 \, dx\,.$$

We mention the following interpolation inequality, fundamental for the sequel of the work (see e.g. [7], Appendix, Lemma 1).

Theorem 7.1.4. *If $u \in H^{1+\vartheta}(Q(\sigma),\mathbb{R}^N)$, for $0 < \vartheta < 1$, then*

$$|u|_{1,Q(\sigma)} \leq c(n,\vartheta) \left\{ \left(\sum_{i=1}^{n} |D_i\, u|^2_{\vartheta,Q(\sigma)} \right)^{\frac{1}{2(1+\vartheta)}} \|u\|^{\frac{\vartheta}{1+\vartheta}}_{0,Q(\sigma)} + \sigma^{-1} \|u\|_{0,Q(\sigma)} \right\}.$$

Theorem 7.1.5. *([7], Appendix, Lemma 2). Let us consider $u \in H^{1+\vartheta}(Q(\sigma),\mathbb{R}^N)$, for $0 < \vartheta < 1$, then*

$$\sum_{i=1}^{n} \|D_i\, u - (D_i\, u)_{Q(\sigma)}\|^2_{0,Q(\sigma)} \leq c(n,\vartheta) \left(\sum_{i=1}^{n} |D_i\, u|^2_{\vartheta,Q(\sigma)} \right)^{\frac{1}{1+\vartheta}} \|\, u - u_{Q(\sigma)}\|^{\frac{2\vartheta}{1+\vartheta}}_{0,Q(\sigma)}.$$

Theorem 7.1.6. *([7], Lemma I.3). Let us set $\Omega, \Omega_1, \Omega_2, \dots, \Omega_m$ $m+1$ bounded open sets of $\mathbb{R}^n$ such that $\bigcup_{k=1}^{m} \Omega_k = \Omega$, σ and ϑ two positive real numbers, $\vartheta < 1$ and $u \in H^{\vartheta}(\Omega_k,\mathbb{R}^N)$, for every $k = 1,2,\dots,m$. Then, there exists a positive constant $c(\vartheta,\sigma)$ such that*

$$|u|^2_{\vartheta,\Omega} \leq c(\vartheta,\sigma) \Bigg\{ \|u\|^2_{0,\Omega} + \sum_{k=1}^{m} \int_{\Omega_{k,\sigma}\cap\Omega} dx \int_{\Omega_k} \frac{\|u(x)-u(y)\|^2}{\|x-y\|^{n+2\vartheta}}\, dy \Bigg\},$$

where $\Omega_{k,\sigma}$, $k = 1,2,\dots,m$, is the set of points of $\mathbb{R}^n$ away from $\overline{\Omega_k}$ less than σ.

Theorem 7.1.7. *(see [7], Teorema 2.I). If $u \in H^{1+\vartheta}(Q(\sigma),\mathbb{R}^N) \cap C^{0,\lambda}(\overline{Q(\sigma)},\mathbb{R}^N)$, $0 < \vartheta \leq 1$ and $0 < \lambda \leq 1$. Then, for every $t > 0$ and every $i = 1,2,\dots,n$, we have*

$$\operatorname{mis}\Big\{ x \in Q(\sigma) : \Big\| D_i u(x) - (D_i\, u)_{Q(\sigma)} \Big\| > t \Big\} \leq c^q(n,\vartheta) \frac{\sum_{j=1}^{n} |D_j\, u|^{\frac{q}{1+\vartheta}}_{\vartheta,Q(\sigma)} \cdot [u]^{\frac{q\vartheta}{1+\vartheta}}_{\lambda,\overline{Q(\sigma)}}}{t^q}$$

$1, 2, \ldots, n,$ *then* $u \in H^{1,p}(Q(t\sigma), \mathbb{R}^N)$ *and*

$$\|D_i u\|_{0,p,Q(t\sigma)} \le M, \quad \forall\, i = 1, 2, \ldots, n.$$

Theorem 7.1.2. *(see e.g. [4], [15]). Let* $u \in H^{1,p}(Q(\sigma), \mathbb{R}^N)$ *for* $1 \le p < +\infty$ *and* N *be a positive integer. Then, for every* $t \in (0, 1)$ *and every* $h \in \mathbb{R}$, $|h| < (1 - t)\sigma$, *we have*

$$\|\tau_{i,h} u\|_{0,p,Q(t\sigma)} \le |h| \, \|D_i u\|_{0,p,Q(\sigma)}\,, \quad i = 1, 2, \ldots, n. \tag{7.1.6}$$

7.1.3 Sobolev spaces with fractionary exponent $H^{k+\vartheta,p}$

Let Ω be an open bounded set in $\mathbb{R}^n$, $\vartheta \in (0, 1)$, $p \in [1, +\infty[$ and N a positive integer.

Definition 7.1.2. We say that a function u defined in Ω having values in $\mathbb{R}^N$ belongs to $H^{\vartheta,p}(\Omega, \mathbb{R}^N)$ if $u \in L^p(\Omega, \mathbb{R}^N)$ and is finite

$$|u|^p_{\vartheta,p,\Omega} = \int_\Omega dx \int_\Omega \frac{\|u(x) - u(y)\|^p_N}{\|x - y\|^{n+\vartheta p}_n} \, dy.$$

Definition 7.1.3. If k is a nonnegative integer, we mean for $H^{k+\vartheta,p}(\Omega, \mathbb{R}^N)$ the subspace of $H^{k,p}(\Omega, \mathbb{R}^N)$ of functions $u \in H^{k,p}(\Omega, \mathbb{R}^N)$ such that

$$D^\alpha u \in H^{\vartheta,p}(\Omega, \mathbb{R}^N), \quad \forall\, \alpha \,:\, |\alpha| = k.$$

We stress that $H^{k+\vartheta,p}(\Omega, \mathbb{R}^N)$ is a Banach space equipped with the following norm

$$\|u\|_{k+\vartheta,p,\Omega} = \Big(\|u\|^p_{k,p,\Omega} + \sum_{|\alpha|=k} |D^\alpha u|^p_{\vartheta,p,\Omega} \Big)^{\frac{1}{p}}.$$

If $p = 2$, then we shall simply write $H^{k+\vartheta}(\Omega, \mathbb{R}^N)$ and $\|u\|_{k+\vartheta,\Omega}$.

The result below is used recurrently throughout the paper (see the proof in [2], Lemma II.3).

7.1.2 Sobolev spaces

Definition 7.1.1 (Sobolev Spaces)**.** (see e.g. [1], [21]). Let k and j be two positive integers, $k \geq j$. If $p \in [1, +\infty[$ and $u \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$, so we set

$$|u|_{j,p,\Omega} = \left(\int_\Omega \sum_{|\alpha|=j} \|D^\alpha u\|_N^p \, dx \right)^{\frac{1}{p}}, \quad \|u\|_{k,p,\Omega} = \left(\sum_{j=0}^{k} |u|_{j,p,\Omega}^p \right)^{\frac{1}{p}} \tag{7.1.3}$$

and denote respectively by $H^{k,p}(\Omega, \mathbb{R}^N)$ and $H_0^{k,p}(\Omega, \mathbb{R}^N)$ the spaces obtained as closure of $C^\infty(\overline{\Omega}, \mathbb{R}^N)$ and $C_0^\infty(\Omega, \mathbb{R}^N)$ regarding the norm $\|u\|_{k,p,\Omega}$.
The spaces $H^{k,p}(\Omega, \mathbb{R}^N)$ and $H_0^{k,p}(\Omega, \mathbb{R}^N)$ are known in literature as Sobolev Spaces.

We remark that $H^{0,p}(\Omega, \mathbb{R}^N) = L^p(\Omega, \mathbb{R}^N)$, $1 \leq p < +\infty$. If $p = 2$, then we shall simply write $H^k(\Omega, \mathbb{R}^N)$, $H_0^k(\Omega, \mathbb{R}^N)$, $|u|_{j,\Omega}$, $\|u\|_{k,\Omega}$.

Let us now state some properties useful in the sequel.

We set, for $x^0 \in \mathbb{R}^n$ and $\sigma > 0$, $Q(\sigma) = Q(x^0, \sigma)$ the cube of $\mathbb{R}^n$ defined by

$$\left\{ x \in \mathbb{R}^n : \left| x_i - x_i^0 \right| < \sigma, \ i = 1, 2, \ldots, n \right\}, \tag{7.1.4}$$

we also consider $t \in (0,1), \sigma > 0$, $h \in \mathbb{R} \setminus \{0\}$, where $|h| < (1-t)\sigma$.
If there is no ambiguity we only write the radius and not also the center of the cube.

Let u be a function defined in $Q(\sigma)$ in $\mathbb{R}^N$ and $x \in Q(t\sigma)$, we set

$$\tau_{i,h} u(x) = u(x + he^i) - u(x), \quad i = 1, 2, \ldots, n, \tag{7.1.5}$$

where $\{e^i\}_{i=1,2,\ldots,n}$ is the canonic basis of $\mathbb{R}^N$.

Let us now state Nirenberg's Theorem (see [4], Chapt. I, Theorem 3.X.), useful to achieve the main result of the note.

Theorem 7.1.1.. *If* $u \in L^p(Q(\sigma), \mathbb{R}^N)$, $1 < p < +\infty$, N *is a positive integer and exists* $M > 0$ *such that* $\| \tau_{i,h} u \|_{0,p,Q(t\sigma)} \leq M |h|$, $\forall \ |h| < (1-t)\sigma$, $i =$

Chapter 7

Preliminaries

7.1 Some function spaces and preliminary results

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, having diameter d_Ω and boundary $\partial\,\Omega$, $x = (x_1, x_2, \ldots, x_n)$ denotes a generic point therein. Let us set k a positive integer greater than 1, $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ respectively the scalar product and the norm in $\mathbb{R}^k$. If there is no ambiguity we omit the index k.

7.1.1 Hölder continuous functions

Let k be a nonnegative integer and $\lambda \in]0,1]$. We denote by $C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ the subspace of $C^k(\overline{\Omega}, \mathbb{R}^N)$ of functions $u : \overline{\Omega} \longrightarrow \mathbb{R}^N$ which satisfy a Hölder condition of exponent λ, together with all their derivatives $D^\alpha u, |\alpha| \leq k$; if $u \in C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, then

$$\|u\|_{C^{k,\lambda}(\overline{\Omega},\mathbb{R}^N)} = \sup_{\Omega} \sum_{|\alpha|\leq k} \|D^\alpha u\| + \sum_{|\alpha|=k} [D^\alpha u]_{\lambda,\overline{\Omega}} \tag{7.1.1}$$

where

$$[D^\alpha u]_{\lambda,\overline{\Omega}} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{\|D^\alpha u(x) - D^\alpha u(y)\|_N}{\|x-y\|_n^\lambda} < +\infty, \ \forall \alpha \ : \ |\alpha| = k.$$

The space $C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ is a Banach space, provided with the norm

$$\|u\|_{C^{k,\lambda}(\overline{\Omega},\mathbb{R}^N)} = \|u\|_{C^k(\overline{\Omega},\mathbb{R}^N)} + \sum_{|\alpha|=k} [D^\alpha u]_{\lambda,\overline{\Omega}} \,. \tag{7.1.2}$$

Let us also mention the considerable note by [25] where the authors prove that a solution u of nonlinear parabolic systems of order 2 with natural growth and coefficients uniformly monotone in $D\,u$ belongs to

$$L^2(-a,0,H^2(B(\sigma),\mathbb{R}^N))\cap H^1(-a,0,L^2(B(\sigma),\mathbb{R}^N))$$

Results similar to those obtained by Marino and Maugeri in [23], with stronger assumptions, are obtained by Naumann in [28] and by Naumann and Wolf in [29]. Let us also bear in mind the study made by Sergio Campanato in [8] on parabolic systems in divergence form.

We want to finish this historical overview, concerning interior differentiability of weak solutions, recalling the recent note [18] where similar results are achieved for elliptic systems of order $2m$.

if, preliminarily, is not ensured the regularity

$$D_i u \in L^4(-a, 0, L^4(B(\sigma), \mathbb{R}^N)), \quad i = 1, \dots, n \tag{6.0.5}$$

for every $a \in (0, T)$, and for every $B(2\sigma) \subset\subset \Omega$,

The technique used in [12] allows the author to achieve, instead of (6.0.5), the condition

$$D_i u \in L^{2\,(1+\theta)}(-a, 0, L^4(B(\sigma), \mathbb{R}^N)), \quad i = 1, \dots, n,$$

for every $a \in (0, T)$, $\forall B(\sigma) \subset\subset \Omega$ and every $\theta \in \left(\frac{n}{n+4\lambda}, 1\right)$, which is *not* enough to ensure that is true (6.0.4)

In [23], under the same assumptions of the previous result [12], the differentiability result (6.0.4) is proved, for u satisfying (6.0.3).

Key of this note is the use of interpolation theorems of Gagliardo-Nirenberg type.

The use of interpolation theory, made in [23] and in [25] with monotonicity assumption and quadratic growth, has recently allowed Fattorusso and Marino to obtain differentiability also for weak solutions of nonlinear parabolic systems of second order having nonlinearity $1 < q < 2$ (see for details [14]).

Inspired by the note mentioned above by Marino and Maugeri, in the present note the authors extend differentiability properties to the case of parabolic systems of order $2m$. More precisely, let Ω be an open subset of $\mathbb{R}^n$, $n > 2$, and $0 < T < \infty$, aim of this note is to study, in the cylinder $Q = \Omega \times (-T, 0)$, the problem of interior local differentiability for solutions

$$u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap C^{m-1,\lambda}(Q, \mathbb{R}^N), \quad 0 < \lambda < 1$$

of the nonlinear parabolic systems of order $2m$ of variational type

$$\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha a^\alpha (X, Du) + \frac{\partial u}{\partial t} = 0.$$

Using the above explained idea is proved the following local differentiability with respect to the spatial derivatives

$$u \in L^2(-a, 0, H^{m+1}(B(\sigma), \mathbb{R}^N)) \cap H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N)), \ \forall a \in (0, T), \forall B(\sigma) \subset\subset \Omega.$$

and applying an iterative method we attain that

$$u \in H^{m+\vartheta}_{\text{loc}}(\Omega, \mathbb{R}^N)\,, \qquad \forall\, 0 < \vartheta < 1.$$

Therefore in paragraph 2.2 the main result (Theorem 8.2.4) allows us to reach the differentiability (6.0.2) and in paragraph 2.3 using it, is established partial Hölder regularity for the derivatives $D^{m+1}u$ of the system (9.2) (see Theorem 8.4.1).

PARABOLICOThe study of regularity for solutions of partial differential equations and systems has received considerable attention over the last thirty years. On the other hand little is known concerning parabolic systems in divergence form of order $2\,m$ with quadratic growth and the corresponding analytic properties of solutions. To such classes of systems our attention is devoted.

This note is a natural continuation of the study, carried out in the last decade and a half, of embedding results of Gagliardo-Nirenberg type from which we deduce local differentiability theorems, making use of interpolation theory in Besov spaces (see e.g. [31] and [32]).

In this respect we mention at first the note [12] where the author proves that, let $\Omega \subset \mathbb{R}^n$ an open set, $0 < T < \infty$ and $Q = \Omega \times (-T, 0)$, $x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \in \Omega$, $\rho > 0$ and $B(\rho) = B(x^0, \rho) = \{x = (x_1, x_2, \ldots, x_n) : |x_i - x^0_i| < \rho,\ i = 1, 2, \ldots, n\}$, if

$$u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(Q, \mathbb{R}^N)), \quad \forall 0 < \lambda < 1 \tag{6.0.3}$$

is a solution of a second order nonlinear parabolic system of variational type and under the assumptions that the coefficients $a^\alpha(x, Du)$ have quadratic growth is obtained that

$$u \in L^2(-a, 0, H^{1+\theta}(B(\sigma), \mathbb{R}^N)),$$

for every $a \in (0, \frac{T}{2}), \forall\theta \in (0, 1)$ and for each cube $B(2\,\sigma) \subset\subset \Omega$.

In the same paper Fattorusso stressed that it is *not* possible to improve this result in such a way to achieve, for each solution u to the above system, the differentiability

$$u \in L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N)), \tag{6.0.4}$$

precisely exploiting natural growth and coefficients uniformly monotone in Du, at first in [13], later the complete extension of the results contained in [7] is achieved in [25]. The crucial step in the two mentioned papers by Fattorusso, Marino and Maugeri is the use of interpolation estimates of Gagliardo-Nirenberg's type in generalized Sobolev spaces. Recently, as announced in [26], the use of interpolation inequalities allows, in [14], the authors to establish differentiability results for weak solutions of nonlinear parabolic systems of second order endowed with nonlinearity $q \in (1,2)$. The present note can be view as an extension from second order nonlinear elliptic systems to order $2m$ of the results established by one of the authors in [15].

Thus we can see that nonlinear systems of second order in divergence form have been extensively studied, much less depth if we talk about order $2m$.

The aim of this note is to give an answer to the starting problem using as assumptions that the vectors $a^\alpha(x, Du)$, $|\alpha| = m$, are strictly monotone and endowed with nonlinearity 2.

The technique used in this note to obtain Hölder regularity is not the classic one, founded on representation formulas of solutions and their derivatives, it is based on Campanato spaces $\mathcal{L}^{p,\lambda}$. They allows us to characterize Hölder functions using integral inequality and then it is very useful to study the regularity of weak solutions of elliptic and parabolic equations and systems (see e.g. [20], [5], [8]).

We wish to recall the study made by Giusti in [19] where this technique is used and appreciated.

This paper is organized as follows. In Section 1 we set the definitions of Sobolev spaces and fractionary Sobolev spaces, as well as useful preliminary Gagliardo-Nirenberg estimates. In Section 2 are established local differentiability results for weak solutions of (9.2) in four steps. The heart of the paper is paragraph 2.1, where it is proved that if $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ $(0 < \lambda < 1)$ is a weak solution of the system (9.2) and some useful assumptions are satisfied, then $u \in H^{m+\vartheta}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, $\forall \vartheta \in \left(0, \frac{\lambda}{2}\right)$, $0 < \lambda < 1$. Using this result we obtain that $u \in H^{m+\vartheta'}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, $\forall \vartheta' \in \left(0, \vartheta + \frac{\lambda}{2}(1-\vartheta)\right)$

systems of order $2m$ in divergence form

$$\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}\,a^{\alpha}\,(x,Du)=0\,. \tag{6.0.1}$$

Concerning the differentiability, if $0<\lambda<1$ and $u\in H^{m}(\Omega,\mathbb{R}^{N})\cap C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^{N})$ is a solution of system (9.2), we answer to the question of what conditions are required for the vectors $a^{\alpha}(x,Du)$, in order that

$$u\in H^{m+1}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N}). \tag{6.0.2}$$

In this chapter, we consider solutions of class $C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^{N})$ because, as already known, if we take solutions $u\in H^{m}(\Omega,\mathbb{R}^{N})\cap H^{m-1,\infty}(\Omega,\mathbb{R}^{N})$, it is not possible in general to ensure differentiability (6.0.2) for nonlinear elliptic systems of order $2\,m$ even if the vectors $a^{\alpha}(x,Du)$ are smooth.

A first answer to the above problem has been given in [7] where the authors prove a result of local differentiability (6.0.2) for solutions of nonlinear elliptic systems of order $2m$ with quadratic growth.

The same hypotheses used in [7] are applied to second order ($m=1$) nonlinear *parabolic* systems of variational type by Fattorusso in 1987 in the note [12] and later by Marino and Maugeri in 1995 in [23] to extend the local differentiability by Campanato and Cannarsa from the elliptic case to the parabolic one. The goal is achieved making use of the interpolation theory in Besov spaces. Moreover, as differentiability achievements allow Campanato and Cannarsa to obtain partial Hölder continuity of the derivatives $D^{\alpha}\,u$, $|\alpha|=m$, similarly Marino and Maugeri obtain in [22] a result of partial Hölder continuity for spatial gradient of the solution to the parabolic system of second order.

We also mention the note [29] where comparable outcomes are obtained by Naumann and Wolf.

Similar results concerned with interior differentiability of weak solutions u to nonlinear parabolic systems of second order are obtained using more general hypotheses,

Chapter 6

Introduction to PART 2: Regularity properties of elliptic and parabolic systems

In the second part of this Ph.D thesis, the regularity properties for solutions of nonlinear elliptic and parabolic systems are studied. In particular, I continue the study started in my Master's degree thesis (see [15]), where the local differentiability and Hölder regularity for weak solutions of nonlinear elliptic systems of second order in divergence form were dealt with. Firstly, a generalization of the results contained in [15] from elliptic systems of the second order to nonlinear elliptic systems of order $2m$ in divergence form is presented. Secondly some results of [15] are extended from elliptic systems to nonlinear parabolic systems of order $2m$ in divergence form. The results contained in the second part of the present thesis, can also be seen in my papers [16], [18] and [17].

Now, we analyze in detail the contents of subsequent chapters of this thesis.

Firstly, in Chapter 7, some preliminary questions, useful later in the paper, are discussed.

Then, in Chapter 8, we investigate in an open bounded $\Omega \subset \mathbb{R}^n$ the problem of local differentiability and Hölder regularity for weak solutions u of nonlinear elliptic

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By Lemma 5.2.1 we have

$$\begin{aligned}\|u(t_2(s)+\tau,x)-v(t_2(s)+\tau,x)\| &\le C\tau^{\frac{1}{2}}e^{K(1+\alpha_3(\tau,s))\tau}s^{(1+\eta)\vartheta}\|u_d+\frac{\delta_{s^{1+\eta}}}{s^{1+\eta}}\|^{\vartheta}\\ &= C\tau^{\frac{1}{2}}e^{K(1+\alpha_3(\tau,s))\tau}s^{(1+\eta)\vartheta}s^{(1+\eta)\vartheta}\left(\|u_d\|^{\vartheta}+1\right)\\ &= c(\gamma_0,\vartheta)\tau^3 s^{-3(1+\eta)}\left(\|u_d\|^{\vartheta}+1\right).\end{aligned} \tag{5.3.21}$$

Then,

$$\begin{aligned}\|v(T_\varepsilon,x)-u_d\| = \|e^{\alpha_3\tau_\varepsilon}z(\tau_\varepsilon,\cdot)-u_d\| &= s_\varepsilon^{-(1+\eta)}\|z(\tau_\varepsilon,\cdot)-\ s_\varepsilon^{1+\eta}u_d\|\\ &\le s_\varepsilon^{-(1+\eta)}\left(\|\delta_{s_\varepsilon^{1+\eta}}\|+\frac{\varepsilon}{4}\,s_\varepsilon^{1+\eta}\right)<\frac{\varepsilon}{2}\,.\end{aligned}$$

Therefore

$$\|u(T_\varepsilon,x)\ -\ u_d\| \quad\le\quad \|u(T_\varepsilon,x)\ -\ v(T_\varepsilon,x)\|\ +\ \|v(T_\varepsilon,x)\ -\ u_d\| \quad<\quad \varepsilon. \tag{5.3.22}$$

$\square$

We can represent the solution of the linear problem (1.2.1) with $\alpha(t,x) = \alpha_3$ and $v_0 = s^{1+\eta}u_d + \delta_{s^{1+\eta}}$, by Fourier'series, in the following way

$$v(t_2(s)+\tau,x) = e^{\alpha_3\tau}\sum_{k=1}^{\infty} e^{-\mu_k\tau}\langle u(t_2(s),\cdot),P_k(\cdot)\rangle_{1,a}P_k(x). \tag{5.3.18}$$

Let us consider

$$z(\tau,x) := \sum_{k=1}^{\infty} e^{-\mu_k\tau}\langle u(t_2(s),\cdot),P_k(\cdot)\rangle_{1,a}P_k(x).$$

Then,

$$\begin{aligned} z(\tau,x) = &\sum_{k=1}^{\infty}\left(e^{-\mu_k\tau}-1\right)\left(\int_{-1}^{1}u(t_2(s),r)P_k(r)dr\right)P_k(x) + s^{1+\eta}u_d + \delta_{s^{1+\eta}} \\ &\xrightarrow{H_a^1} s^{1+\eta}u_d + \delta_{s^{1+\eta}} \ \text{ as } \tau \to 0^+ . \end{aligned} \tag{5.3.19}$$

Fix $0<\varepsilon<1$,

- $\exists s_\varepsilon \in (0,s_0)$ such that
$$\frac{\|\delta_{s_\varepsilon^{1+\eta}}\|_{1,a}}{s_\varepsilon^{1+\eta}} < \frac{\varepsilon}{4};$$
- $\exists \tau(s_\varepsilon) > 0$ such that
  - $C\tau_\varepsilon^{\frac{1}{2}} s_\varepsilon^{-K(1+\eta)} e^{K\tau}\left(\|u_d\|_{1,a}^{\vartheta}+1\right) < \frac{\varepsilon}{2}$
  - $\|z(\tau_\varepsilon,\cdot) - s_\varepsilon^{1+\eta}\, u_d\|_{1,a} \le \|\delta_{s_\varepsilon^{1+\eta}}\|_{1,a} + \frac{\varepsilon}{4}\, s_\varepsilon^{1+\eta}$.

Set $T_\varepsilon = t_2(s_\varepsilon) + \tau(s_\varepsilon)$. Let us define

$$\alpha(t,x) = \alpha_3(s_\varepsilon) = -\frac{\ln(s^{1+\eta})}{\tau} = -\frac{1+\eta}{\tau}\ln s, \forall t \in [t_2(s_\varepsilon),T_\varepsilon],\ \forall x \in (-1,1). \tag{5.3.20}$$

Thus, we have the following estimate

$$
\begin{aligned}
\|u(t_2(s),\cdot)-s^{1+\eta}u_d(\cdot)\|_{1,a} &\le \|u(t_2(s),\cdot)-v(t_2(s),\cdot)\|_{1,a}+\|v(t_2(s),\cdot)-s^{1+\eta}u_d(\cdot)\|_{1,a}\\
&\le C\,(t_2(s)-t_1(s))^{\frac{1}{2}}\,e^{K(t_2(s)-t_1(s))}\|su_0+\delta_s\|_{1,a}^{\vartheta}+Cs^{\frac{-\eta\lambda_2}{\beta}}s^{1+\eta}\\
&\le C\,(t_2(s)-t_1(s))^{\frac{1}{2}}\,s^{\frac{\eta K}{\beta}}s^{\vartheta}\left\|u_0+\frac{\delta_s}{s}\right\|_{1,a}^{\vartheta}+Cs^{\frac{-\eta\lambda_2}{\beta}}s^{1+\eta}\\
&=\le C\left((t_2(s)-t_1(s))^{\frac{1}{2}}\,s^{\frac{\eta K}{\beta}}s^{\vartheta-1-\eta}\left\|u_0+\frac{\delta_s}{s}\right\|_{1,a}^{\vartheta}+s^{\frac{-\eta\lambda_2}{\beta}}\|u_d\|_{1,a}\right)s^{1+\eta}\\
&\le C\left((t_2(s)-t_1(s))^{\frac{1}{2}}\,s^{\frac{\eta K}{\beta}+\vartheta-1-\eta}+s^{\frac{-\eta\lambda_2}{\beta}}\right)s^{1+\eta},
\end{aligned}
$$

$$\text{for every } s\in(0,s_0). \quad (5.3.17)$$

Now, we have

$$t_2(s)-t_1(s)=\frac{1}{\beta}\ln\left(\frac{s^{\eta}\,\|u_d\|_{1,a}^2}{\langle u_0+\frac{\delta_s}{s},u_d\rangle_{1,a}}\right)\longrightarrow+\infty,\quad \text{as } s\to 0^+.$$

Since $\frac{\eta K}{\beta}+\vartheta-1-\eta>0$ by the choice of β, we have

$$(t_2(s)-t_1(s))^{\frac{1}{2}}\,s^{\frac{\eta K}{\beta}+\vartheta-1-\eta}=\left(\frac{1}{\beta}\ln\left(\frac{s^{1+\eta}\|u_d\|_{1,a}}{\langle su_0+\delta_s,\omega_1\rangle}\right)\right)^{\frac{1}{2}}s^{\frac{\eta K}{\beta}+\vartheta-1-\eta}\longrightarrow 0,\qquad \text{as } s\to 0^+.$$

Defining

$$\delta_{s^{1+\eta}}(x):=u(t_2(s),\cdot)-s^{1+\eta}u_d(\cdot)\qquad x\in(-1,1),$$

estimate (5.3.17) yields

$$\frac{\|\delta_{s^{1+\eta}}(\cdot)\|_{1,a}}{s^{1+\eta}}\to 0,\ \text{ as } s\to 0^+.$$

STEP. 3 Let $\tau>0$. On the interval $(t_2(s),T(s))$, with $T(s)=t_2(s)+\tau$, we apply a positive constant control $\alpha_3(x)\equiv\alpha_3$ (its value will be chosen below).

that is, since $\omega_1 = \frac{u_d}{\|u_d\|}$,

$$t_2(s) = t_1(s) + \frac{1}{\beta}\ln\left(\frac{s^{\eta}\,\|u_d\|_{1,a}^2}{\langle u_0 + \frac{\delta_s}{s}, u_d\rangle_{1,a}}\right). \tag{5.3.13}$$

So, by (5.3.12) and the above estimates for $\|v(t_2(s),\cdot) - s^{1+\eta}\,u_d(\cdot)\|_{1,a}$ and $\|r_s(t_2(s),\cdot)\|_{1,a}$ we conclude that

$$\begin{aligned}&\|v(t_2(s),\cdot) - s^{1+\eta}u_d(\cdot)\|_{1,a} \le e^{(-\lambda_2+\beta)(t_2(s)-t_1(s))}\|su_0 + \delta_s\|_{1,a}\\ &= e^{-\lambda_2(t_2(s)-t_1(s))}\frac{s^{1+\eta}\|u_d\|_{1,a}}{u_1(s)}\|su_0+\delta_s\|_{1,a} = e^{-\lambda_2(t_2(s)-t_1(s))}\frac{s^{1+\eta}\|u_d\|_{1,a}}{z_1(s)}\left\|u_0 + \frac{\delta_s}{s}\right\|_{1,a}.\end{aligned} \tag{5.3.14}$$

Then, by (5.3.13), we deduce that $\exists s_0 \in (0, s^*)$ such that

$$e^{-\lambda_2(t_2(s)-t_1(s))}\frac{\|u_0+\frac{\delta_s}{s}\|_{1,a}}{z_1(s)} = \left(\frac{s^{\eta}\|u_d\|_{1,a}}{z_1(s)}\right)^{\frac{-\lambda_2}{\beta}}\frac{\|u_0+\frac{\delta_s}{s}\|_{1,a}}{z_1(s)} \le Cs^{\frac{-\eta\lambda_2}{\beta}}, \quad \forall s \in (0, s_0).$$

From the above, the inequality (5.3.14) becomes

$$\|v(t_2(s),\cdot) - s^{1+\eta}u_d(\cdot)\|_{1,a} \le cs^{\frac{-\eta\lambda_2}{\beta}}s^{1+\eta}, \quad \forall s \in (0, s_0). \tag{5.3.15}$$

Then, we can observe that

$$\alpha_2(t,x) = \alpha_*(x) + \beta < 0,\ \forall t \in [t_1(s), t_2(s)],\ \forall x \in (-1,1).$$

Thus, by Lemma 5.2.3, we deduce the following estimate

$$\|u(t_2(s),\cdot) - v(t_2(s),\cdot)\|_{1,a} \le C\,(t_2(s)-t_1(s))^{\frac{1}{2}}\,e^{K(t_2(s)-t_1(s))}\|su_0+\delta_s\|_{1,a}^{\vartheta}. \tag{5.3.16}$$

Then, by (5.3.13), we deduce that

$$e^{K(t_2(s)-t_1(s))} = \left(\frac{s^{\eta}\|u_d\|_{1,a}}{z_1(s)}\right)^{\frac{K}{\beta}} \le cs^{\frac{\eta K}{\beta}}, \quad \forall s \in (0, s_0).$$

The solution of (1.2.1), with $\alpha(t,x) = \alpha_*(x) + \beta,\ t > t_1(s),\ v_0 = u_0$, has the following representation in Fourier series ([3])

$$v(t,x) = \sum_{k=1}^{\infty} e^{(-\lambda_k+\beta)(t-t_1(s))} u_k(s)\omega_k(x)$$
$$= e^{\beta(t-t_1(s))} u_1(s)\omega_1(x) + \sum_{k>1} e^{(-\lambda_k+\beta)(t-t_1(s))} u_k(s)\omega_k(x)$$

Let

$$r_s(t,x) = \sum_{k>1} e^{(-\lambda_k+\beta)(t-t_1(s))} u_k(s)\omega_k(x)$$

where, $-\lambda_k < -\lambda_1 = 0$, for every $k \in \mathbb{N},\ k > 1$. Owing to (5.3.9),

$$\|v(t,\cdot) - s^{1+\eta}\, u_d\|_{1,a} \le \left\| e^{\beta(t-t_1(s))} u_1(s)\omega_1 - \|s^{1+\eta}\, u_d\|_{1,a}\omega_1 \right\|_{1,a} + \|r_s(t,x)\|_{1,a}$$
$$= \left| e^{\beta(t-t_1(s))} u_1(s) - s^{1+\eta}\, \|u_d\|_{1,a} \right| + \|r_s(t,x)\|_{1,a}\,.$$

Since $-\lambda_k < -\lambda_2,\ \forall k > 2$, applying Parseval's equality we have

$$\|r_s(t,x)\|^2_{1,a} \le e^{2(-\lambda_2+\beta)(t-t_1(s))} \sum_{k>1} |u_k(s)|^2 \|\omega_k(x)\|^2_{1,a}$$
$$= e^{2(-\lambda_2+\beta)(t-t_1(s))} \sum_{k>1} |\langle su_0 + \delta_s, \omega_k\rangle_{1,a}|^2 = e^{2(-\lambda_2+\beta)(t-t_1(s))} \|su_0 + \delta_s\|^2_{1,a}.$$

By (5.3.10) we obtain

$$\exists s^* \in (0,1) :\ u_1(s) = \langle su_0 + \delta_s, \omega_1\rangle_{1,a}, \forall s \in (0, s^*). \tag{5.3.11}$$

Then, we choose $t_2(s),\ t_2(s) > t_1(s)$ such that

$$e^{\beta(t_2(s)-t_1(s))} u_1(s) = s^{1+\eta}\, \|u_d\|_{1,a}\,, \tag{5.3.12}$$

[3] Observe that adding $\beta \in \mathbb{R}$ to the coefficient α_* there is a shift of the eigenvalues corresponding to α_* from $\{-\lambda_k\}_{k\in\mathbb{N}}$ to $\{-\lambda_k + \beta\}_{k\in\mathbb{N}}$, but the eigenfunctions remain the same for α_* and $\alpha_* + \beta$.

STEP. 2 Now, we will steer the system from the initial state

$$u(t_1(s),x) = s\,u_0(x) + \delta_s(x),\ x \in (-1,1),$$

to an arbitrarily small neighborhood of the target state

$$s^{1+\eta}\,u_d,\ \ \eta \in (0,\vartheta-1)\,.$$

at some time $t_2(s)$. For this purpose, define

$$\alpha_2(x) = \alpha_*(x) + \beta, \quad \forall x \in (-1,1),$$

with $\alpha_*(x) = -\frac{(a(x)u_{dx}(x))_x}{u_d(x)}$, $x \in (-1,1)$, and $\beta = \min\{-\|\alpha_*\|_{L^\infty(-1,1)}, -\frac{\eta K}{\vartheta-1-\eta}\} - 1$. We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad \text{and} \qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the spectral problem $A\omega = \lambda\omega$, with $A = A_0 + \alpha_* I$ and $D(A) = H^2_a(-1,1)$ (see Lemma 3.2.6).
Recalling Lemma 2.3.5, we can see that

$$\lambda_1 = 0 \qquad \text{and} \qquad \omega_1(x) = \frac{u_d(x)}{\|u_d\|_{1,a}} > 0,\ \forall x \in (-1,1)\,. \tag{5.3.9}$$

Set

$$u_k(s) := \langle u(t_1(s),\cdot), \omega_k(\cdot)\rangle_{1,a}, \quad \forall k \in \mathbb{N}.$$

Thus,

$$u_k(s) = s\,z_k(s), \quad \text{where } z_k(s) := \langle u_0 + \frac{\delta_s}{s}, \omega_k\rangle_{1,a},\ \forall k \in \mathbb{N}.$$

Then, by (5.3.9), we can observe that

$$z_1(s) \longrightarrow \frac{1}{\|u_d\|}\langle u_0, u_d\rangle_{1,a} > 0\,,\ \text{as } s \to 0. \tag{5.3.10}$$

$(-1,1)$. Now, we consider the linear problem (1.2.1) with $\alpha(t,x) \equiv \alpha_1(s)$, $\forall t \in [0, t_1(s)]$, and initial state $v_0 = u_0$. For $t = t_1(s)$, the solution $v(t,x)$ of the linear problem (1.2.1) has the following representation in Fourier's series

$$v(t_1(s), x) = e^{\alpha_1(s) t_1(s)} \sum_{k=1}^{\infty} e^{-\mu_k t_1(s)} \langle u_0, P_k \rangle_{1,a} P_k(x) = s\, z(t_1(s), x),\ \forall x \in (-1,1).$$

Therefore, by (5.3.2), we obtain

$$\|v(t_1(s), \cdot) - s u_0(\cdot)\|_{1,a} = s\, \|z(t_1(s), \cdot) - u_0(\cdot)\|_{1,a} \leq \frac{s^2}{2}, \tag{5.3.5}$$

Moreover, by Lemma 5.2.3, the choice of $t_1(s)$, and (5.3.3) we have

$$\|w(t_1(s), \cdot)\|_{1,a} = \|u(t_1(s), \cdot) - v(t_1(s), \cdot)\|_{1,a}$$
$$\leq \sqrt{t_1(s)} C e^{K\, t_1(s)} \|u_0\|_{1,a}^{\vartheta} \leq \frac{s^2}{2}. \tag{5.3.6}$$

From (5.3.5) and (5.3.6) we obtain

$$\|u(t_1(s), \cdot) - s u_0\|_{1,a}$$
$$\leq \|u(t_1(s), \cdot) - v(t_1(s), \cdot)\|_{1,a} + \|v(t_1(s), \cdot) - s u_0(\cdot)\|_{1,a} \leq s^2. \tag{5.3.7}$$

Let us define

$$\delta_s(x) := u(t_1(s), x) - s u_0(x), \quad \forall x \in (-1,1),$$

and observe that, in view of (5.3.7)

$$\frac{\|\delta_s(\cdot)\|_{1,a}}{s} \longrightarrow 0, \qquad \text{as } s \to 0. \tag{5.3.8}$$

In this way, we have steered the system from the initial state u_0 to the target state $s u_0 + \delta_s$ at time $t_1(s)$.

respectively, the eigenvalues and orthonormal eigenfunctions of the spectral problem $A_0\omega = \mu\omega$, with A_0 defined as in (3.1.4) (see Lemma 3.2.6) ([2]).

Set

$$z(t,x) := \sum_{k=1}^{\infty} e^{-\mu_k t} \langle u_0, P_k\rangle_{1,a} P_k(x).$$

Since $z \in \mathcal{H}(Q_T)$, one can observe that,

$$z(t,x) = \sum_{k=1}^{\infty} (e^{-\mu_k t} - 1)\langle u_0, P_k\rangle_{1,a} P_k(x)$$

$$+ u_0(x) \xrightarrow{H_a^1} u_0(x), \qquad \text{as } t \to 0.$$

Fix any $s \in (0,1)$. Thus,

$$\exists\ t^*(s) > 0 \ \text{ such that } \|z(t,\cdot) - u_0(\cdot)\|_{1,a} \le \frac{s}{2}, \ \ \forall t \le t^*(s). \tag{5.3.2}$$

Moreover,

$$\exists\ \bar{t}(s) > 0 \ \text{ such that } \ \sqrt{t} e^{K\,t} \le \frac{s^2}{2C\|u_0\|_{1,a}^{\vartheta}}, \quad \forall t \le \bar{t}(s),\ \forall \alpha_1 \in \mathbb{R}, \tag{5.3.3}$$

where $C = C(\alpha_1, \gamma_0, \theta, \nu, a)$, and $K = K(\gamma_0, \theta, \nu, a)$ are the constants of Lemma 5.2.3.

Now, set

$$t_1(s) = \min\{t^*(s), \bar{t}(s), 1\},$$

and observe that $t_1(s) \longrightarrow 0,$ as $s \to 0$.

We select the following negative constant bilinear control

$$\alpha(t,x) = \alpha_1(s) := \frac{\ln s}{t_1(s)} < 0, \ \ \forall t \in [0, t_1(s)], \forall x \in (-1,1), \tag{5.3.4}$$

that is, $\alpha_1(s)$ is such that $e^{\alpha_1(s)t_1(s)} = s$.

On the interval $(0, t_1(s))$, we apply the negative constant control $\alpha(t,x) = \alpha_1(s), \forall x \in$

[2] In the case $a(x) = 1 - x^2$, that is, where the principal part of the operator is that the Budyko-Sellers model, the orthonormal eigenfunctions are reduced to Legendre polynomials, and the eigenvalues are $\mu_k = (k-1)k, k \ge 1$.

From the above inequality, applying Gronwall's inequality we obtain

$$\int_{-1}^{1}(u^-(t,x))^2dx \le \nu_T^2 e^{2\|\alpha\|_\infty t} \int_{-1}^{1}(u^-(0,x))^2dx\,.$$

Since

$$u(0,x) = u_0(x) \ge 0\,,$$

we have

$$u^-(0,x) = 0.$$

Therefore,

$$u^-(t,x) = 0, \qquad\qquad \forall (t,x) \in Q_T.$$

From this, as we mentioned initially, it follows that

$$u(t,x) = u^+(t,x) \ge 0 \qquad \forall (t,x) \in Q_T.$$

$\square$

5.3 Proof of main results

Proof. (of Theorem 5.1.1) Let us consider any nonnegative initial state $u_0, u_d \in H^1_a(-1,1),\ u_d \ge 0,\quad \langle u_0, u_d\rangle >_{1,a} 0$ To prove Theorem 2.1 it is sufficient to consider the set of target states

$$u_d \in C^\infty([-1,1]), \qquad u_d > 0 \text{ on } [-1,1]. \tag{5.3.1}$$

Indeed, every function $u_d \in L^2(-1,1), u_d \ge 0$ can be approximated by a sequence of strictly positive functions of class $C^\infty([-1,1])$.

STEP. 1 We denote with

$$\{-\mu_k\}_{k\in\mathbb{N}} \qquad\qquad \text{and} \qquad\qquad \{P_k\}_{k\in\mathbb{N}},$$

we obtain

$$\int_{-1}^{1} \left[u_t u^- - (a(x)u_x)_x u^-\right] dx = \int_{-1}^{1} \left[\alpha u u^- + f(x,u)u^-\right] dx. \tag{5.2.4}$$

Recalling the definition u^+ and u^-, we have

$$\int_{-1}^{1} u_t u^- dx = \int_{-1}^{1} (u^+ - u^-)_t u^- dx = -\int_{-1}^{1} (u^-)_t u^- dx = -\frac{1}{2}\frac{d}{dt}\int (u^-)^2 dx\,.$$

Integrating by parts and recalling that $u^- \in H^1_a(-1,1)$, we obtain the following equality

$$\int_{-1}^{1} (a(x)u_x)_x u^-\, dx = [a(x)u_x u^-]_{-1}^{1} - \int_{-1}^{1} a(x)u_x(-u)_x\, dx = \int_{-1}^{1} a(x)u_x^2\, dx\,.$$

We also have

$$\int_{-1}^{1} \alpha u u^- dx = -\int_{-1}^{1} \alpha (u^-)^2 dx.$$

Moreover, using (5.1.4) we have

$$\begin{aligned}\int_{-1}^{1} f(x,u)u^-\, dx &= \int_{-1}^{1} f(x,u^+ - u^-)u^-\, dx \\ &= \int_{-1}^{1} f(x,-u^-)u^-\, dx = -\int_{-1}^{1} f(x,-u^-)\left(-u^-\right)\, dx \\ &\geq -\int_{-1}^{1} \nu\left(-u^-\right)^2\, dx = -\int_{-1}^{1} \nu\left(u^-\right)^2\, dx\end{aligned}$$

and therefore (5.2.4) becomes

$$-\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} (u^-)^2 dx + \int_{-1}^{1} \alpha(u^-)^2 dx + \int_{-1}^{1} \nu\left(u^-\right)^2\, dx \geq \int_{-1}^{1} a(x)u_x^2 \geq 0,$$

from which

$$\frac{d}{dt}\int_{-1}^{1} (u^-)^2 dx \leq 2\int_{-1}^{1} (\alpha+\nu)\,(u^-)^2 dx \leq 2\left(\|\alpha\|_\infty + \nu\right)\int_{-1}^{1} (u^-)^2 dx.$$

Proceeding as in the proof of Lemma 3.2.5, and applying Corollary 3.2.2 and Corollary 5.2.2 we obtain the following lemma.

Lemma 5.2.3. *Let* $T > 0, \vartheta > 1,\ \xi_a \in L^{2\vartheta-1}(-1,1), \alpha \in L^\infty(Q_T)$ *and* $u_0 \in H^1_a(-1,1)$. *Let* $u \in \mathcal{H}(Q_T)$ *be the solution of* (5.1.1) *and* $v \in \mathcal{H}(Q_T)$ *be the solution of* (1.2.1) *with the same coefficient* $\alpha \in L^\infty(Q_T)$ *and initial state* $v_0 = u_0$. *Then, the difference* $w = u - v$ *satisfies*

$$\|w(t,\cdot)\|_{1,a} \le \sqrt{T} C_T e^{K\|\alpha^+\|T} \|u_0\|^\vartheta_{1,a}, \qquad \forall t \in [0,T],$$

where $K = K(\gamma_0, \theta, \nu, a)$, $K = K(\alpha, \gamma_0, \theta, \nu, a)$ *and* $C_T \ge C_0 = 1\ \forall T \ge 0$.

Lemma 5.2.4. *Let* $T > 0$, $\alpha \in L^\infty(Q_T)$, *let* $u_0 \in H^1_a(-1,1)$, $u_0(x) \ge 0$ *a.e.* $x \in (-1,1)$ *and let* $u \in \mathcal{H}(Q_T)$ *be the solution to the semilinear system*

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(t,x)u + f(x,u) & \text{in } Q_T = (0,T) \times (-1,1) \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ u(0,x) = u_0(x) & x \in (-1,1)\ . \end{cases}$$

Then

$$u(t,x) \ge 0, \quad \forall (t,x) \in Q_T\,.$$

Proof. Let $u \in \mathcal{H}(Q_T)$ be the solution to the system (5.1.1). It is sufficient to prove that

$$u^-(t,x) \equiv 0 \qquad \text{in } Q_T\,.$$

Multiplying both members of the equation in (5.1.1) by u^- and integrating on $(-1,1)$

Then, we have

$$\|w\|^2_{B(Q_t)} \leq 2\|\alpha^+\|_\infty \int_0^t \|w\|^2_{B(Q_s)}\, ds \ + \frac{1}{2}\|w\|^2_{B(Q_t)}$$
$$+ c(\gamma_0, \vartheta, a)\, T^{\frac{3}{2}}\, \|u\|^{2\vartheta}_{\mathcal{H}(Q_T)}, \qquad t \in (0, T).$$

From wich, we deduce

$$\frac{1}{2}\|w\|^2_{B(Q_t)} \leq 2\|\alpha^+\|_\infty \int_0^t \|w\|^2_{B(Q_s)}\, ds$$
$$+ c(\gamma_0, \vartheta, a)\, T^{\frac{3}{2}}\, \|u\|^{2\vartheta}_{\mathcal{H}(Q_T)}, \qquad t \in (0, T).$$

Applying Gronwall's inequality we have

$$\|w\|^2_{B(Q_t)} \leq c(\gamma_0, \vartheta, a)\, T^{\frac{3}{2}}\, e^{4\|\alpha^+\|_\infty T}\|u\|^{2\vartheta}_{\mathcal{H}(Q_T)}, \qquad t \in (0, T).$$

$\square$

By the previous lemma and applying Lemma 3.2.5

Corollary 5.2.2. *Let* $T > 0, \vartheta > 1,\ \xi_a \in L^{2\vartheta-1}(-1, 1), \alpha \in L^\infty(Q_T)$ *and* $u_0 \in H^1_a(-1, 1)$. *Let* $u \in \mathcal{H}(Q_T)$ *be the solution of* (5.1.1) *and* $v \in \mathcal{H}(Q_T)$ *be the solution of* (1.2.1) *with the same coefficient* $\alpha \in L^\infty(Q_T)$ *and initial state* $v_0 = u_0$. *Then, the difference* $w = u - v$ *belongs to* $\mathcal{H}(Q_T)$ *and satisfies*

$$\|w\|_{\mathcal{B}(Q_T)} = \|u - v\|_{\mathcal{B}(Q_T)} \leq K_1(\|u_0\|_{1,a})\, T^{\frac{3}{4}}\, e^{K_2 T}\|u_0\|^{\vartheta}_{1,a},$$

where $K_1(\|u_0\|_{1,a}) = c(\alpha, \gamma_0, \theta, \nu, a)\, (k_1(\|u_0\|_{1,a}))^{\vartheta}$ *for some positive constant* $c(\alpha, \gamma_0, \theta, \nu, a)$, $k_1(\|u_0\|_{1,a})$ *is the constant given by Lemma 3.2.5, and* $K_2 = \frac{\vartheta}{2} + 2\|\alpha^+\|_\infty + (\nu + \|\alpha^+\|_\infty)\vartheta^2$ *(*α^+ *denotes the positive part of* α *).*

Hölder's inequality , we have

$$\int_0^t ds \int_{-1}^1 |f(x,u)||w|dx \le \gamma_0 \int_0^t ds \int_{-1}^1 |u|^{\vartheta}|w|dx$$

$$\le \gamma_0 \|u\|^{\vartheta}_{L^{2\vartheta}(Q_t)} \|w\|_{L^2(Q_t)}.$$

Thanks to the assumptions (5.1.5), i.e $\xi_a \in L^{2\vartheta-1}(-1,1) \subseteq L^1(-1,1)$, we can apply the Lemma 3.1.2, then we have

$$\|w\|_{L^2(Q_t)} \le c(a) t^{\frac{1}{4}} \|w\|_{B(Q_t)}. \tag{5.2.2}$$

Then, being $\xi_a \in L^{2\vartheta-1}(-1,1)$, by Corollary 3.1.4 we have

$$\|u\|^{\vartheta}_{L^{2\vartheta}(Q_t)} \le c(\vartheta, a)\, t^{\frac{1}{2}} \|u\|^{\vartheta}_{\mathcal{H}(Q_t)} \tag{5.2.3}$$

Then, by (5.2.2) and (5.2.3), applying *Young's* inequality, we obtain

$$\begin{aligned}
\int_0^t ds \int_{-1}^1 |f(x,u)||w|dx &\le \gamma_0 \|u\|^{\vartheta}_{L^{2\vartheta}(Q_t)} \|w\|_{L^2(Q_t)} \\
&\le \gamma_0 c(\vartheta, a)\, t^{\frac{1}{2}}\, t^{\frac{1}{4}} \|u\|^{\vartheta}_{\mathcal{H}(Q_t)} \|w\|_{B(Q_t)} \\
&\le \gamma_0 c(\vartheta, a)\, T^{\frac{3}{4}} \|u\|^{\vartheta}_{\mathcal{H}(Q_T)} \|w\|_{B(Q_t)} \\
&\le c(\gamma_0, \vartheta, a)\, T^{\frac{3}{2}} \|u\|^{2\vartheta}_{\mathcal{H}(Q_T)} + \frac{1}{4}\|w\|^2_{B(Q_t)}.
\end{aligned}$$

So, for every $t \in (0,T)$, we obtain

$$\begin{aligned}
&\|w(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^t ds \int_{-1}^1 a w_x^2\, dx \\
&\quad \le 2\|\alpha^+\|_\infty \int_0^t \|w(s,\cdot)\|^2_{L^2(-1,1)}\, ds + \frac{1}{2}\|w\|^2_{B(Q_t)} + c(\gamma_0, \vartheta, a)\, T^{\frac{3}{2}} \|u\|^{2\vartheta}_{\mathcal{H}(Q_T)} \\
&\quad \le 2\|\alpha^+\|_\infty \int_0^t \|w\|^2_{B(Q_s)}\, ds + \frac{1}{2}\|w\|^2_{B(Q_t)} + c(\gamma_0, \vartheta, a)\, T^{\frac{3}{2}} \|u\|^{2\vartheta}_{\mathcal{H}(Q_T)}.
\end{aligned}$$

Proof. Let us consider the difference between the solution u of (5.1.1) and the solution v of (1.2.1), with the same coefficient α and initial state $v_0 = u_0$.

Given

$$w(t,x) = u(t,x) - v(t,x) \quad \text{in} \;\; Q_T,$$

$w(t,x)$ is solution of the following system

$$\begin{cases} w_t - (aw_x)_x = \alpha w + f(x,u) \quad \text{in} \;\; Q_T \\ a(x)w_x(t,x)|_{x=\pm 1} = 0 \\ w(0,x) = 0 \end{cases} \qquad . \tag{5.2.1}$$

Multiplying by w both members of the equation in (5.2.1) we obtain

$$w_t w - (a(x)w_x)_x w = \alpha w^2 + f(x,u)w$$

and therefore integrating on $(-1,1)$, we deduce that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} w^2 dx + \int_{-1}^{1} a w_x^2 dx &= \int_{-1}^{1} \alpha w^2 dx + \int_{-1}^{1} f(x,u) w dx \\ &\le \int_{-1}^{1} \alpha^+ w^2 dx + \int_{-1}^{1} |f(x,u)||w| dx \\ &\qquad \le \|\alpha^+\|_\infty \int_{-1}^{1} w^2 dx + \int_{-1}^{1} |f(x,u)||w| dx \,. \end{aligned}$$

Fixing $t \in (0,T)$ and integrating on $(0,t)$, we obtain

$$\begin{aligned} &\|w(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^t ds \int_{-1}^{1} a w_x^2\, dx \\ &\qquad \le 2\|\alpha^+\|_\infty \int_0^t \|w(t,\cdot)\|^2_{L^2(-1,1)}\, ds + 2\int_0^t ds \int_{-1}^{1} |f(x,u)||w| dx \,. \end{aligned}$$

Since $u, v \in \mathcal{H}(Q_T)$ and therefore $w = u - v$ belongs to $\mathcal{H}(Q_T)$, by (5.1.2) and

In the following, we suppose that the semilinear system (5.1.1)

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(t,x)u + f(x,u) & \text{in } Q_T = (0,T)\times(-1,1) \\ \\ a(x)u_x(t,x)|_{x=\pm1} = 0 & t\in(0,T) \\ \\ u(0,x) = u_0(x) & x\in(-1,1)\ . \end{cases}$$

satisfies the assumptions $(A.1) - (A.4)$. We also recall the associated linear system (1.2.1)

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in } Q_T = (0,T)\times(-1,1) \\ \\ a(x)v_x(t,x)|_{x=\pm1} = 0 & t\in(0,T) \\ \\ v(0,x) = v_0(x) & x\in(-1,1)\ , \end{cases}$$

where $v_0 \in H^1_a(-1,1)$, $\alpha(t,x)$ and the diffusion coefficient $a(x)$ satisfy respectively the assumption $(A.2)$ and $(A.4)$. In particular, in the following we assume that the coefficient $a(x)$ of the associated linear system (1.2.1) is the same as the semilinear system (5.1.1).

Lemma 5.2.1. *Let $T>0, \vartheta>1, \xi_a \in L^{2\vartheta-1}(-1,1), \alpha\in L^\infty(Q_T)$ and $u_0\in H^1_a(-1,1)$. Let $u\in\mathcal{H}(Q_T)$ be the solution of* (5.1.1) *and $v\in\mathcal{H}(Q_T)$ be the solution of* (1.2.1) *with the same control $\alpha\in L^\infty(Q_T)$ and initial state $v_0=u_0$. Then, the difference $w=u-v$ belongs to $\mathcal{H}(Q_T)$ and satisfies*

$$\|w\|_{\mathcal{B}(Q_T)} = \|u-v\|_{\mathcal{B}(Q_T)} \le c(\gamma_0,\vartheta,a)\,T^{\frac{3}{4}}\,e^{2\|\alpha^+\|_\infty T}\|u\|^{\vartheta}_{\mathcal{H}(Q_T)},$$

where α^+denotes the positive part of α, and $c(\gamma_0,\vartheta,a)$ is a positive constant.

We are interested in studying the multiplicative controllability of (5.1.1) by the *bilinear control* $\alpha(t,x)$.

5.1.2 Main result

Let us start with the following definition.

Definition 5.1.1. We say that a function $\alpha \in L^\infty(Q_T)$ is piecewise static, if $\alpha(\cdot,x)$ is piecewise constant in t and $\alpha(t,\cdot) \in L^\infty(-1,1),\ t \in (0,T)$.

The *global approximate controllability* result is obtained for the semilinear system (5.1.1) in the following theorem.

Theorem 5.1.1. *For any $u_d \in H^1_a(-1,1), u_d \geq 0$ and any $u_0 \in H^1_a(-1,1)$ such that*

$$\langle u_0, u_d\rangle_{1,a} > 0, \tag{5.1.6}$$

for every $\varepsilon > 0$, there are $T = T(\varepsilon, u_0, u_d) \geq 0$ and a piecewise static bilinear control $\alpha(t,x) \in L^\infty(Q_T)$ such that

$$\|u(T,\cdot) - u_d\|_{1,a} \leq \varepsilon\,.$$

In the following, we will sometimes use $\|\cdot\|$ instead of $\|\cdot\|_{L^2(-1,1)}$, and $\|\cdot\|_\infty$ instead of $\|\cdot\|_{L^\infty(Q_T)}$.

5.2 Some useful lemmas

In this section I prove some useful results for the proof of the main theorem obtained in collaboration with P. Cannarsa in [10].

(A.4) $a \in C^1([-1,1])$ is such that

- $a(x) > 0 \; \forall\, x \in (-1,1), \quad a(-1) = a(1) = 0$
- the function $\xi_a(x) = \int_0^x \frac{ds}{a(s)}$ satisfies the following

$$\xi_a \in L^{2\vartheta - 1}(-1,1). \tag{5.1.5}$$

Remark 5.1.1.

- If $f(x,u)$ belongs to the space $C^1(\mathbb{R})$, with respect to u, a sufficient condition for the assumption (5.1.3) is that, for some $\vartheta > 1$ and $\gamma_1 > 0$,

$$|f_u(x,u)| \le \gamma_1 |u|^{\vartheta - 1} \qquad \text{for a.e. } x \in (-1,1),\ \forall u \in \mathbb{R}.$$

- The assumption (5.1.4) is more general than the classical sign assumption $\int_{-1}^{1} f(x,u) u \, dx \le 0$,[1] indeed the last condition is equivalent to $f(x,u)\, u \le 0$, for a.e. $x \in (-1,1), \quad \forall u \in \mathbb{R}$.

- $\frac{1}{a} \notin L^1(-1,1)$, so $a(\cdot)$ is strongly degenerate.

- The principal part of the operator in (5.1.1) coincides with that of the Budyko-Sellers model for $a(x) = 1 - x^2$. In this case, $\xi_a(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$,so $\xi_a \in L^p(-1,1)$, for every $p \ge 1$.

Example 5.1.1. *An example of function f that satisfies the assumptions (A.3) is the following*

$$f(x,u) = c(x) \min\{|u|^{\vartheta-1}, 1\} u - |u|^{\vartheta - 1} u,$$

where $c(\cdot) \in L^\infty(-1,1)$.

[1]This integral condition is used by A. Khapalov in [29], in the uniformly parabolic case, but also there it can be generalized by a condition similar to (5.1.4).

$(-1,1)$ by means of the *bilinear control* $\alpha(t,x)$)

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(t,x)u + f(x,u) & \text{in } \ Q_T \ := \ (0,T)\times(-1,1) \\ \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 & t\in(0,T) \\ \\ u(0,x) = u_0(x) & x\in(-1,1)\ . \end{cases} \tag{5.1.1}$$

under the following assumptions:

(A.1) $u_0 \in H^1_a(-1,1)$;

(A.2) $\alpha \in L^\infty(Q_T)$;

(A.3) $f : (-1,1)\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. f is Lebesgue measurable in x for every $u\in\mathbb{R}$, and continuous in u for almost every $x \in (-1,1)$) such that

- there exist $\vartheta > 1$, $\gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$|f(x,u)| \le \gamma_0\,|u|^\vartheta, \ \text{for a.e. } x\in(-1,1), \forall u\in\mathbb{R}\,, \tag{5.1.2}$$

$$\begin{aligned} &|f(x,u) - f(x,v)| \\ &\qquad \le \gamma_1\left(1+|u|^{\vartheta-1}+|v|^{\vartheta-1}\right)|u-v|, \ \text{for a.e. } x\in(-1,1), \ \forall u,v\in\mathbb{R}; \end{aligned} \tag{5.1.3}$$

- there exists a nonnegative constant ν such that

$$f(x,u)\,u \le \nu\,u^2, \qquad \text{for a.e. } x\in(-1,1), \quad \forall u\in\mathbb{R}\,. \tag{5.1.4}$$

Below we will put $\nu_T = e^{\nu T}$;

Chapter 5

Controllability of nonlinear problems

In this chapter we study the global approximate multiplicative controllability for a semilinear degenerate parabolic Cauchy-Neumann problem (see also [10]).
We will show that this system can be steered in $H^1_a(-1,1)$ from any nonzero, initial state $u_0 \in H^1_a(-1,1)$ into any neighborhood of any desirable nonnegative target-state $u_d \in H^1_a(-1,1)$ such that $\langle u_0, u_d\rangle_{1,a} > 0$, by bilinear piecewise static controls.

5.1 Notation and main results

5.1.1 Problem formulation

Given $T > 0$, let us consider the control system (*Cauchy-Neumann* strongly degenerate boundary semilinear problem in divergence form, governed in the bounded domain

Then, it is possible choose β_ε so that

$$e^{\beta_\varepsilon T_\varepsilon}\int_{-1}^{1} v_0\omega_1 dx = \|v_d\|\,,$$

that is, since $\omega_1 = \frac{v_d}{\|v_d\|}$,

$$\beta_\varepsilon = \frac{1}{T_\varepsilon}\ln\left(\frac{\|v_d\|^2}{\int_{-1}^{1} v_0 v_d dx}\right). \tag{4.2.9}$$

So, by (4.2.7), (4.2.9) and the above estimates for $\|v(T_\varepsilon,\cdot) - v_d(\cdot)\|$ and $\|r(T_\varepsilon,\cdot)\|$ we conclude that

$$\|v(T_\varepsilon,\cdot) - v_d(\cdot)\| \le e^{(-\lambda_2+\beta_\varepsilon)T_\varepsilon}\|v_0\| = e^{-\lambda_2 T_\varepsilon}\frac{\|v_d\|^2}{\int_{-1}^{1} v_0 v_d dx}\|v_0\| = \varepsilon\,.$$

From which we have the conclusion. □

Proof. (of Theorem 4.2.2) The proof of Theorem 4.2.1 can be adapted to Theorem 4.2.2, keeping in mind that, in STEP.3, inequality (4.2.8) continues to hold in this new setting. In fact we have

$$\int_{-1}^{1} v_0(x)\omega_1(x)dx = \int_{-1}^{1} v_0(x)\frac{v_d(x)}{\|v_d\|}dx =$$

$$= \frac{1}{\|v_d\|}\int_{-1}^{1} v_0 v_d dx > 0, \text{ by assumptions (5.1.6).}$$

From this point on, one can proceed as in the proof of Theorem 4.1.1.

□

the following Fourier series representation ([4])

$$v(t,x) = \sum_{k=1}^{\infty} e^{(-\lambda_k+\beta)t}\left(\int_{-1}^{1} v_0(s)\omega_k(s)ds\right)\omega_k(x) =$$

$$= e^{\beta t}\left(\int_{-1}^{1} v_0(s)\omega_1(s)ds\right)\omega_1(x) + \sum_{k>1} e^{(-\lambda_k+\beta)t}\left(\int_{-1}^{1} v_0(s)\omega_k(s)ds\right)\omega_k(x)$$

Let

$$r(t,x) = \sum_{k>1} e^{(-\lambda_k+\beta)t}\left(\int_{-1}^{1} v_0(s)\omega_k(s)ds\right)\omega_k(x)$$

where, recalling that $\lambda_k < \lambda_{k+1}$, we obtain

$$-\lambda_k < -\lambda_1 = 0 \qquad \text{for ever } \ k \in \mathbb{N},\ k > 1\,.$$

Owing to (5.3.9),

$$\|v(t,\cdot) - v_d\| \leq \left\| e^{\beta t}\left(\int_{-1}^{1} v_0(s)\omega_1(s)ds\right)\omega_1 - \|v_d\|\omega_1 \right\| + \|r(t,x)\| =$$

$$= \left| e^{\beta t}\left(\int_{-1}^{1} v_0(x)\omega_1(x)dx\right) - \|v_d\| \right| + \|r(t,x)\|$$

Since $-\lambda_k < -\lambda_2$, $\forall k > 2$, applying Parseval's equality we have

$$\|r(t,x)\|^2 \leq e^{2(-\lambda_2+\beta)t} \sum_{k>1} \left| \int_{-1}^{1} v_0 \omega_k ds \right|^2 \|\omega_k(x)\|^2 =$$

$$= e^{2(-\lambda_2+\beta)t} \sum_{k>1} \langle v_0, \omega_k \rangle^2 = e^{2(-\lambda_2+\beta)t} \|v_0\|^2.$$

Fixed $\varepsilon > 0$, we choose $T_\varepsilon > 0$ such that

$$e^{-\lambda_2 T_\varepsilon} = \varepsilon \frac{\int_{-1}^{1} v_0 v_d dx}{\|v_0\|\|v_d\|^2}\,. \tag{4.2.7}$$

Since $v_0 \in L^2(-1,1)$, $v_0 \geq 0$ and $v_0 \not\equiv 0$ in $(-1,1)$ and by (4.2.6), we obtain

$$\langle v_0, \omega_1 \rangle = \int_{-1}^{1} v_0(x)\omega_1(x)dx > 0. \tag{4.2.8}$$

[4]Observe that adding $\beta \in \mathbb{R}$ in the coefficient α_* there is a shift of the eigenvalues corresponding to α_* from $\{-\lambda_k\}_{k\in\mathbb{N}}$ to $\{-\lambda_k + \beta\}_{k\in\mathbb{N}}$, but the eigenfunctions remain the same for α_* and $\alpha_* + \beta$.

Indeed, regularizing by convolution, every function $v_d \in L^2(-1,1), v_d \geq 0$ can be approximated by a sequence of strictly positive $C^\infty([-1,1])-$ functions.

STEP.2 Taking any nonzero, nonnegative initial state $v_0 \in L^2(-1,1)$ and any target state v_d as described in (4.2.4) in STEP.1, let us set

$$\alpha_*(x) = -\frac{(a(x)v_{dx}(x))_x}{v_d(x)}, \qquad x \in (-1,1). \tag{4.2.5}$$

Then, by (5.3.1),

$$\alpha_*(x) \in L^\infty(-1,1)\,.$$

We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad\qquad \text{and} \qquad\qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions[3] of the spectral problem $A\omega + \lambda\omega = 0$, with $A = A_0 + \alpha_* I$ (see Lemma **??**).

We can see, by Lemma **??**, that

$$\lambda_1 = 0 \qquad \text{and} \qquad \omega_1(x) = \frac{v_d(x)}{\|v_d\|} > 0,\ \forall x \in (-1,1)\,. \tag{4.2.6}$$

STEP.3 Let us now choose the following static bilinear control

$$\alpha(x) = \alpha_*(x) + \beta,\ \forall x \in (-1,1), \quad \text{with } \beta \in \mathbb{R} \ (\beta \text{ to be determined below}).$$

The corresponding solution of (4.2.1), for this particular bilinear coefficient α, has

[3] As first eigenfunction we take the one which is positive in $(-1,1)$.

and therefore (4.2.3) becomes

$$-\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(v^-)^2dx + \int_{-1}^{1}\alpha(v^-)^2dx = \int_{-1}^{1}a(x)v_x^2 \geq 0,$$

from which

$$\frac{d}{dt}\int_{-1}^{1}(v^-)^2dx \leq 2\int_{-1}^{1}\alpha(v^-)^2dx \leq 2\|\alpha\|_\infty\int_{-1}^{1}(v^-)^2dx.$$

From the above inequality, applying Gronwall's lemma we obtain

$$\int_{-1}^{1}(v^-(t,x))^2dx \leq e^{2t\|\alpha\|_\infty}\int_{-1}^{1}(v^-(0,x))^2dx\,.$$

Since

$$v(0,x) = v_0(x) \geq 0\,,$$

we have

$$v^-(0,x) = 0.$$

Therefore,

$$v^-(t,x) = 0, \qquad\qquad \forall(t,x) \in Q_T.$$

From this, as we mentioned initially, it follows that

$$v(t,x) = v^+(t,x) \geq 0 \qquad \forall(t,x) \in Q_T.$$

□

We are now ready to prove our main result.

Proof. (of Theorem 4.2.1)

<u>STEP.1</u> To prove Theorem 4.2.1 it is sufficient to consider the set of target states

$$v_d \in C^\infty([-1,1]), \qquad v_d > 0 \text{ on } [-1,1]. \tag{4.2.4}$$

$(-1,1)$ and let $v \in \mathcal{B}(0,T)$ be the solution to the linear system

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & in \quad Q_T = (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ v(0,x) = v_0(x) & x \in (-1,1) \ . \end{cases}$$

Then

$$v(t,x) \geq 0, \quad \forall (t,x) \in Q_T \, .$$

Proof. Let $v \in \mathcal{B}(0,T)$ be the solution to the system (4.2.1), and we consider the positive-part and the negative-part. It is sufficient to prove that

$$v^-(t,x) \equiv 0 \qquad \text{in } Q_T \, .$$

Multiplying both members equation of the problem (4.2.1) by v^- and integrating it on $(-1,1)$ we obtain

$$\int_{-1}^{1} \left[v_t v^- - (a(x)v_x)_x v^- - \alpha v v^- \right] dx = 0. \tag{4.2.3}$$

Recalling the definition v^+ and v^-, we obtain

$$\int_{-1}^{1} v_t v^- dx = \int_{-1}^{1} (v^+ - v^-)_t v^- dx = -\int_{-1}^{1} (v^-)_t v^- dx = -\frac{1}{2}\frac{d}{dt}\int (v^-)^2 dx \, .$$

Integrating by parts and applying Theorem 2.1.1, we obtain $v^- \in H^1_a(-1,1)$ and the following equality

$$\int_{-1}^{1} (a(x)v_x)_x v^- \, dx = [a(x)v_x v^-]_{-1}^{1} - \int_{-1}^{1} a(x)v_x(-v)_x \, dx = \int_{-1}^{1} a(x)v_x^2 \, dx \, .$$

We also have

$$\int_{-1}^{1} \alpha v v^- dx = -\int_{-1}^{1} \alpha (v^-)^2 dx$$

Theorem 4.2.1. *The linear system (4.2.1) is nonnegatively approximately controllable in $L^2(-1,1)$ by means of static controls in $L^\infty(-1,1)$. Moreover, the corresponding solution to (4.2.1) remains nonnegative at all times.*

Then the results present in Theorem 4.2.1 can be extended to a larger class of initial states.

Theorem 4.2.2. *For any $v_d \in L^2(-1,1), v_d \geq 0$ and any $v_0 \in L^2(-1,1)$ such that*

$$\int_{-1}^{1} v_0 v_d dx > 0, \tag{4.2.2}$$

for every $\varepsilon > 0$, there are $T = T(\varepsilon, v_0, v_d) \geq 0$ and a static bilinear control, $\alpha = \alpha(x)$, $\alpha \in L^\infty(-1,1)$ such that

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

Remark 4.2.2. The solution $v(t,x)$ of the problem (4.2.1) in the assumptions of Theorem 4.1.2 does not remain nonnegative in Q_T, like in Theorem 4.1.1, but it can also assume negative values.

4.2.3 Proofs of main results.

For the proof of Theorem 4.2.1 the following Lemma is necessary.

Lemma 4.2.3. *Let $T > 0$, $\alpha \in L^\infty(Q_T)$, let $v_0 \in L^2(-1,1)$, $v_0(x) \geq 0$ a.e. $x \in$*

Remark 4.2.1. We observe that

1. $\frac{1}{a} \notin L^1(-1,1)$, so $a(x)$ is strongly degenerate

2. the principal part of the operator in (4.2.1) coincides with that of the Budyko-Sellers model for $a(x) = 1 - x^2$. In this case $A(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \in L^1(-1,1)$

3. a sufficient condition for 3.b) is that $a'(\pm 1) \neq 0$ (if $a \in C^2([-1,1])$ the above condition is also necessary).

We are interested in studying the multiplicative controllability of problem (4.2.1) by the *bilinear control* $\alpha(t,x)$. In particular, for the above linear problem, we will discuss results guaranteeing global nonnegative approximate controllability in large time (for multiplicative controllability see [29, 32, 13]).
Now we recall one definition from control theory.

Definition 4.2.1. We say that the system (4.2.1) is *nonnegatively* globally approximately controllable in $L^2(-1,1)$, if for every $\varepsilon > 0$ and for every *nonnegative* $v_0(x)$, $v_d(x) \in L^2(-1,1)$ with $v_0 \not\equiv 0$ there are a $T = T(\varepsilon, v_0, v_d)$ and a bilinear control $\alpha(t,x) \in L^\infty(Q_T)$ such that for the corresponding solution $v(t,x)$ of (4.2.1) we obtain

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

In the following, we will sometimes use $\|\cdot\|$ instead of $\|\cdot\|_{L^2(-1,1)}$.

4.2.2 Main goals.

In this work at first the *nonnegative global approximate controllability* result is obtained for the linear system (4.2.1) in the following theorem.

with the bilinear control $\alpha(t,x) \in L^\infty(Q_T)$. The problem is strongly degenerate in the sense that $a \in C^1([-1,1])$, positive on $(-1,1)$, is allowed to vanish at ± 1 provided that a certain integrability condition is fulfilled. We will show that the above system can be steered in $L^2(-1,1)$ from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear static controls. Moreover, we extend the above result relaxing the sign constraint on v_0.

4.2.1 Problem formulation

Let us consider the following *Cauchy-Neumann* strongly degenerate boundary linear problem in divergence form, governed in the bounded domain $(-1,1)$ by means of the *bilinear control* $\alpha(t,x)$

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in} \quad Q_T = (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ v(0,x) = v_0(x) & x \in (-1,1)\ . \end{cases} \tag{4.2.1}$$

We assume that

1. $v_0 \in L^2(-1,1)$
2. $\alpha \in L^\infty(Q_T)$
3. $a \in C^1([-1,1])$ satisfies
   (a) $a(x) > 0\ \forall\, x \in (-1,1), \quad a(-1) = a(1) = 0$
   (b) $A \in L^1(-1,1)$, where $A(x) = \int_0^x \frac{ds}{a(s)}$.

Then, it is possible choose δ_ε so that

$$e^{\delta_\varepsilon T_\varepsilon}\langle v_0, \omega_1\rangle = \|v_d\|,$$

that is, since $\omega_1 = \frac{v_d}{\|v_d\|}$,

$$\delta_\varepsilon = \frac{1}{T_\varepsilon}\ln\left(\frac{\|v_d\|^2}{\langle v_0, v_d\rangle}\right). \tag{4.1.12}$$

So, by (4.1.8) – (4.1.10) and (5.3.13) we conclude that

$$\|v(T_\varepsilon,\cdot) - v_d(\cdot)\| \le e^{(-\lambda_2+\delta_\varepsilon)T_\varepsilon}\|v_0\| = e^{-\lambda_2 T_\varepsilon}\frac{\|v_d\|^2}{\langle v_0, v_d\rangle}\|v_0\| = \varepsilon\,.$$

From which we have the conclusion. □

Proof. (of Theorem 4.1.2) The proof of Theorem 4.1.1 can be adapted to Theorem 4.1.2, keeping in mind that, in STEP.3, inequality (5.3.11) continues to hold in this new setting. In fact we have

$$\langle v_0, \omega_1\rangle = \frac{1}{\|v_d\|}\langle v_0, v_d\rangle > 0, \text{ by assumptions (5.1.6).}$$

From this point on, one can proceed as in the proof of Theorem 4.1.1.

□

4.2 Strongly degenerate problems

In this section we study the global approximate multiplicative controllability for the linear degenerate parabolic Cauchy-Neumann problem

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in} \quad Q_T = (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm1} = 0 & t\in(0,T) \\ v(0,x) = v_0(x) & x\in(-1,1)\ , \end{cases}$$

and $\alpha_* + \delta$.

The corresponding solution of (5.1.1), for this particular bilinear coefficient α, has the following Fourier series representation

$$v(t,x) = \sum_{k=1}^{\infty} e^{(-\lambda_k+\delta)t}\langle v_0, \omega_k\rangle \omega_k(x)$$
$$= e^{\delta t}\langle v_0, \omega_1\rangle \omega_1(x) + \sum_{k>1} e^{(-\lambda_k+\delta)t}\langle v_0, \omega_k\rangle \omega_k(x)\,.$$

Let

$$r(t,x) = \sum_{k>1} e^{(-\lambda_k+\delta)t}\langle v_0, \omega_k\rangle \omega_k(x)$$

where, recalling that $\lambda_k < \lambda_{k+1}$, we obtain

$$-\lambda_k < -\lambda_1 = 0 \quad \text{ for ever } \; k \in \mathbb{N},\, k > 1\,.$$

Owing to (4.1.7),

$$\|v(t,\cdot) - v_d\| \leq \left\| e^{\delta t}\langle v_0, \omega_1\rangle \omega_1 - \|v_d\|\omega_1 \right\| + \|r(t,x)\|$$
$$= \left| e^{\delta t}\langle v_0, \omega_1\rangle - \|v_d\| \right| + \|r(t,x)\|\,. \quad (4.1.8)$$

Since $-\lambda_k < -\lambda_2$, $\forall k > 2$, applying Bessel's inequality we have

$$\|r(t,x)\|^2 \leq e^{2(-\lambda_2+\delta)t} \sum_{k>1} |\langle v_0, \omega_k\rangle|^2 \|\omega_k(x)\|^2$$
$$= e^{2(-\lambda_2+\delta)t} \sum_{k>1} \langle v_0, \omega_k\rangle^2 \leq e^{2(-\lambda_2+\delta)t}\|v_0\|^2. \quad (4.1.9)$$

Fixed $\varepsilon > 0$, we choose $T_\varepsilon > 0$ such that

$$e^{-\lambda_2 T_\varepsilon} = \varepsilon \frac{\langle v_0, v_d\rangle}{\|v_0\|\|v_d\|^2}. \quad (4.1.10)$$

Since $v_0 \in L^2(-1,1)$, $v_0 \geq 0$ and $v_0 \not\equiv 0$ in $(-1,1)$ and by (4.1.7), we obtain

$$\langle v_0, \omega_1\rangle = \int_{-1}^{1} v_0(x)\omega_1(x)dx > 0. \quad (4.1.11)$$

Finally, since $\frac{(a(x)\bar{\omega}_{1x}(x))_x}{\bar{\omega}_1(x)} = -\bar{\lambda}_1 \ \forall x \in (-1,1)$ ([2]), we have

$$\frac{(a\,\bar{v}^{\varepsilon}_{dx})_x}{\bar{v}^{\varepsilon}_d} \in L^{\infty}(-1,1)\,.$$

<u>STEP.2</u> Taking any nonzero, nonnegative initial state $v_0 \in L^2(-1,1)$ and any target state v_d as described in (5.3.1) in STEP.1, let us set

$$\alpha_*(x) = -\frac{(a(x)v_{dx}(x))_x}{v_d(x)}, \qquad x \in (-1,1). \tag{4.1.6}$$

Then, by (5.3.1),

$$\alpha_* \in L^{\infty}(-1,1)\,.$$

We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad\qquad \text{and} \qquad\qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the spectral problem $A\omega + \lambda\omega = 0$, with $A = A_0 + \alpha_* I$ (see Lemma 3.2.6), where as first eigenfunction we take the one which is positive in $(-1,1)$.

We can see, by Lemma 2.3.5, that

$$\lambda_1 = 0 \qquad \text{and} \qquad \omega_1(x) = \frac{v_d(x)}{\|v_d\|} > 0,\ \forall x \in (-1,1)\,. \tag{4.1.7}$$

<u>STEP.3</u> Let us now choose the following static bilinear control

$$\alpha(x) = \alpha_*(x) + \delta,\ \forall x \in (-1,1), \quad \text{with } \delta \in \mathbb{R} \ (\delta \text{ to be determined below}).$$

Adding $\delta \in \mathbb{R}$ in the coefficient α_* there is a shift of the eigenvalues corresponding to α_* from $\{-\lambda_k\}_{k\in\mathbb{N}}$ to $\{-\lambda_k + \delta\}_{k\in\mathbb{N}}$, but the eigenfunctions remain the same for α_*

[2] $-\bar{\lambda}_1$ is the first eigenvalue of the Sturm-Liouville problem (4.1.5).

Now, let us consider $\bar{\omega}_1$, the first positive eigenfunction of A_0 with norm 1. Note that $\bar{\omega}_1$ is a solution of the following Sturm-Liouville problem

$$\begin{cases} (a(x)\omega_x)_x + \lambda\,\omega = 0 & \text{in} \quad (-1,1) \\ \begin{cases} \beta_0\omega(-1) + \beta_1 a(-1)\omega_x(-1) = 0 \\ \gamma_0\,\omega(1) \;+\; \gamma_1\,a(1)\,\omega_x(1) = 0 \end{cases} \end{cases} \tag{4.1.5}$$

Define

$$\bar{v}_d^{\varepsilon}(x) = \xi_\sigma(x)\,\bar{\omega}_1(x) + (1-\xi_\sigma(x))\;v_d^{\varepsilon}(x), \qquad x \in [-1,1],$$

where $\xi_\sigma \in C^\infty([-1,1])$ (σ is a positive real number) is a symmetrical cut-off function

- $\xi_\sigma(-x) = \xi_\sigma(x), \qquad \forall x \in [-1,1]$
- $0 \le \xi_\sigma(x) \le 1, \qquad \forall x \in [0,1]$
- $\xi_\sigma(x) = 0, \qquad \forall x \in [0, 1-\sigma]$
- $\xi_\sigma(x) = 1, \qquad \forall x \in [1-\frac{\sigma}{2}, 1].$

Then,

$$\bar{v}_d^{\varepsilon} \in H_a^2(-1,1), \bar{v}_d^{\varepsilon} > 0 \text{ in } (-1,1) \text{ and } \begin{cases} \beta_0\bar{v}_d^{\varepsilon}(-1) + \beta_1 a(-1)\bar{v}_{dx}^{\varepsilon}(-1) = 0 \\ \gamma_0\,\bar{v}_d^{\varepsilon}(1) \;+\; \gamma_1\,a(1)\,\bar{v}_{dx}^{\varepsilon}(1) = 0 \end{cases}$$

Moreover, taking into account that there is $\sigma > 0$ such that

$$\|v_d^{\varepsilon} - \bar{v}_d^{\varepsilon}\|^2 \le \int_{-1}^{-1+\sigma} \left(\bar{\omega}_1(x) - v_d^{\varepsilon}(x)\right)^2\,dx + \int_{1-\sigma}^{1} \left(\bar{\omega}_1(x) - v_d^{\varepsilon}(x)\right)^2\,dx \le \frac{\varepsilon^2}{4},$$

we have

$$\|v_d - \bar{v}_d^{\varepsilon}\| \le \|v_d - v_d^{\varepsilon}\| + \|v_d^{\varepsilon} - \bar{v}_d^{\varepsilon}\| \le \varepsilon\,.$$

Since

$$v(0,x) = v_0(x) \geq 0\,,$$

we have

$$v^-(0,x) = 0.$$

Therefore,

$$v^-(t,x) = 0, \qquad\qquad \forall (t,x) \in Q_T.$$

From this, as we mentioned initially, it follows that

$$v(t,x) = v^+(t,x) \geq 0 \qquad \forall (t,x) \in Q_T.$$

□

4.1.3 Proofs of main results.

We are now ready to prove our main result.

Proof. (of Theorem 4.1.1)

STEP.1 Let A_0 be the operator defined in (2.2.1), to prove Theorem 4.1.1 it is sufficient to consider the set of target states

$$v_d \in D(A_0),\ v_d > 0 \text{ on } (-1,1) \text{ such that } \frac{(a\, v_{dx})_x}{v_d} \in L^\infty(-1,1)\,. \tag{4.1.4}$$

Indeed, regularizing by convolution, every function $v_d \in L^2(-1,1), v_d \geq 0$ can be approximated by a sequence of strictly positive $C^\infty([-1,1])-$ functions.

Then, fixing $\varepsilon > 0$, we can find a function $v_d^\varepsilon \in C^\infty([-1,1]), v_d^\varepsilon > 0$ in $[-1,1]$ such that $\|v_d - v_d^\varepsilon\| \leq \frac{\varepsilon}{2}$.

Recalling the definition v^+ and v^-, we obtain

$$\int_{-1}^{1} v_t v^- dx = \int_{-1}^{1} (v^+ - v^-)_t v^- dx = -\int_{-1}^{1} (v^-)_t v^- dx = -\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} (v^-)^2 dx\,.$$

Integrating by parts and applying Theorem 2.1.1 (see Appendix), we obtain $v^- \in H_a^1(-1,1)$ and the following equality

$$\int_{-1}^{1} (a(x)v_x)_x v^- \, dx = [a(x)v_x v^-]_{-1}^{1} - \int_{-1}^{1} a(x)v_x(-v)_x \, dx$$

If $\beta_1\gamma_1 \neq 0$, using the Robin boundary conditions and the sign assumptions, we have

$$[a(x)v_x v^-]_{-1}^{1} = a(1)v_x(t,1)v^-(t,1) - a(-1)v_x(t,-1)v^-(t,-1) =$$

$$= -\frac{\gamma_0}{\gamma_1} v(t,1)v^-(t,1) + \frac{\beta_0}{\beta_1} v(t,-1)v^-(t,-1) \geq 0\,.$$

If $\beta_1\gamma_1 = 0$([1]), proceeding similarly, we obtain

$$[a(x)v_x v^-]_{-1}^{1} \geq 0.$$

We also have

$$\int_{-1}^{1} \alpha v v^- dx = -\int_{-1}^{1} \alpha (v^-)^2 dx$$

and therefore (5.2.4) becomes

$$-\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} (v^-)^2 dx + \int_{-1}^{1} \alpha (v^-)^2 dx = [a(x)v_x v^-]_{-1}^{1} + \int_{-1}^{1} a(x)v_x^2 \geq 0,$$

from which

$$\frac{d}{dt}\int_{-1}^{1} (v^-)^2 dx \leq 2\int_{-1}^{1} \alpha (v^-)^2 dx \leq 2\|\alpha\|_\infty \int_{-1}^{1} (v^-)^2 dx.$$

From the above inequality, applying Gronwall's lemma we obtain

$$\int_{-1}^{1} (v^-(t,x))^2 dx \leq e^{2t\|\alpha\|_\infty} \int_{-1}^{1} (v^-(0,x))^2 dx\,.$$

[1] In the particular case $\beta_1 = \gamma_1 = 0$ we have $[a(x)v_x v^-]_{-1}^{1} = 0$. Indeed, in this case the problem (5.1.1) is reduced to a Cauchy-Dirichlet problem.

In the following, we will sometimes use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ instead of $\|\cdot\|_{L^2(-1,1)}$ and $\langle\cdot,\cdot\rangle_{L^2(-1,1)}$.

For the proof of Theorem 4.1.1 the following Lemma is necessary.

Lemma 4.1.3. *Let $T>0$, $\alpha\in L^\infty(Q_T)$, let $v_0\in L^2(-1,1)$, $v_0(x)\geq 0$ a.e. $x\in(-1,1)$ and let $v\in\mathcal{B}(0,T)$ be the solution to the linear system*

$$\begin{cases} v_t-(a(x)v_x)_x=\alpha(t,x)v & in \quad Q_T\,=\,(0,T)\times(-1,1)\\ \begin{cases}\beta_0 v(t,-1)+\beta_1 a(-1)v_x(t,-1)=0 & t\in(0,T)\\ \gamma_0\, v(t,1)\,+\,\gamma_1\, a(1)\, v_x(t,1)=0 & t\in(0,T)\end{cases}\\ v(0,x)=v_0(x) & x\in(-1,1)\ , \end{cases}$$

Then

$$v(t,x)\geq 0,\quad \forall (t,x)\in Q_T\,.$$

Proof. Let $v\in\mathcal{B}(0,T)$ be the solution to the system (5.1.1), and we consider the positive-part and the negative-part (see Appendix). It is sufficient to prove that

$$v^-(t,x)\equiv 0 \qquad \text{in } Q_T\,.$$

Multiplying both members equation of the problem (5.1.1) by v^- and integrating it on $(-1,1)$ we obtain

$$\int_{-1}^{1}\left[v_t v^- -(a(x)v_x)_x v^- -\alpha v v^-\right]dx=0. \tag{4.1.3}$$

Definition 4.1.1. We say that the system (5.1.1) is nonnegatively globally approximately controllable in $L^2(-1,1)$, if for every $\varepsilon > 0$ and for every nonnegative $v_0(x)$, $v_d(x) \in L^2(-1,1)$ with $v_0 \not\equiv 0$ there are a $T = T(\varepsilon, v_0, v_d)$ and a bilinear control $\alpha(t,x) \in L^\infty(Q_T)$ such that for the corresponding solution $v(t,x)$ of (5.1.1) we obtain

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

In this work at first the *nonnegative global approximate controllability* result is obtained for the linear system (5.1.1) in the following theorem.

Theorem 4.1.1. *The linear system (5.1.1) is nonnegatively approximately controllable in $L^2(-1,1)$ by means of static controls in $L^\infty(-1,1)$. Moreover, the corresponding solution to (5.1.1) remains nonnegative at all times.*

Then, the results present in Theorem 4.1.1 can be extended to a larger class of initial states.

Theorem 4.1.2. *For any $v_d \in L^2(-1,1), v_d \geq 0$ and any $v_0 \in L^2(-1,1)$ such that*

$$\langle v_0, v_d \rangle_{L^2(-1,1)} > 0, \tag{4.1.2}$$

for every $\varepsilon > 0$, there are $T = T(\varepsilon, v_0, v_d) \geq 0$ and a static bilinear control, $\alpha = \alpha(x)$, $\alpha \in L^\infty(-1,1)$ such that

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

Remark 4.1.1. The solution $v(t,x)$ of the problem (5.1.1) in the assumptions of Theorem 4.1.2 does not remain nonnegative in Q_T, like in Theorem 4.1.1, but it can also assume negative values.

$$
\begin{cases}
v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in} \quad Q_T = (0,T)\times(-1,1) \\
\begin{cases}
\beta_0 v(t,-1) + \beta_1 a(-1) v_x(t,-1) = 0 & t \in (0,T) \\
\gamma_0\, v(t,1) + \gamma_1\, a(1)\, v_x(t,1) = 0 & t \in (0,T)
\end{cases} \\
v(0,x) = v_0(x) & x \in (-1,1)
\end{cases}
\tag{4.1.1}
$$

We assume that

i. $v_0 \in L^2(-1,1)$

ii. $\alpha \in L^\infty(Q_T)$

iii. $a \in C^0([-1,1]) \cap C^1(-1,1)$ fulfills the following properties

  (a) $a(x) > 0 \ \forall\, x \in (-1,1), \quad a(-1) = a(1) = 0$

  (b) $\frac{1}{a} \in L^1(-1,1)$

iv. $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R},\ \beta_0^2 + \beta_1^2 > 0,\ \gamma_0^2 + \gamma_1^2 > 0$, satisfy the sign condition

  (a) $\beta_0\beta_1 \leq 0$ and $\gamma_0\gamma_1 \geq 0$.

Under the assumptions *iii.*) we say that the problem (5.1.1) is weakly degenerate.

4.1.2 Main goals.

We are interested in studying the multiplicative controllability of problem (5.1.1) by the *bilinear control* $\alpha(t,x)$. In particular, for the above linear problem, we will discuss results guaranteeing global nonnegative approximate controllability in large time (for multiplicative controllability see [29], [32], [13], [11]).

Now we recall one definition from control theory.

Chapter 4

Controllability of linear problems

4.1 Weakly degenerate problems

In this work we study the global approximate multiplicative controllability for a weakly degenerate parabolic Cauchy-Robin problem. The problem is weakly degenerate in the sense that the diffusion coefficient is positive in the interior of the domain and is allowed to vanish at the boundary, provided the reciprocal of the diffusion coefficient is summable. In this paper, we will show that the above system can be steered, in the space of square-summable functions, from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear static controls. Moreover, we extend the above result relaxing the sign constraint on the initial-state.

4.1.1 Problem formulation.

Let us consider the following *Cauchy-Robin* weakly degenerate boundary linear problem in divergence form, governed in the bounded domain $(-1,1)$ by means of the *bilinear control* $\alpha(t,x)$

In the case $a(x) = 1 - x^2$, so that $A_0 = ((1-x^2)u_x)_x$, then the orthonormal eigenfunctions are reduced to Legendre's polynomials $P_k(x)$, and the eigenvalues are $\mu_k = (k-1)k, k \in \mathbb{N}$. $P_k(x)$ is equal to $\sqrt{\frac{2}{2k+1}}L_k(x)$, where $L_k(x)$ is assigned by *Rodrigues's formula*:

$$L_k(x) = \frac{1}{2^{k-1}(k-1)!}\frac{d}{dx^{k-1}}(x^2-1)^{k-1} \qquad (k \geq 1).$$

By Lemma 3.2.1 we deduce

$$\int_{Q_t} \|f(x,u)\|^2\,dx\,ds \le \gamma_0 \int_{Q_t} |u|^{2\vartheta} dx\,ds$$
$$\le c(\gamma_0,\vartheta,a)T\|u\|_{H^1(0,T;L^2(-1,1))}\|u\|^{2\vartheta-1}_{L^\infty(0,T;H^1_a(-1,1))}$$
$$\le c(\gamma_0,\vartheta,a)T\left(\int_0^t \|u_t(s,\cdot)\|^2\,ds\right)^{\frac{1}{2}}\left(\sup_{t\in[0,T]}\|u(s,\cdot)\|_{1,a}\right)^{2\vartheta-1}$$
$$\le c(\alpha,\gamma_0,\vartheta,\nu,a)T\left(\chi_T^2\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)\|u_0\|^2_{1,a}\right)^{\frac{1}{2}}\left(\chi_T\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{\frac{1}{2}}\|u_0\|_{1,a}\right)^{2\vartheta-1}$$
$$\le c(\alpha,\gamma_0,\vartheta,\nu,a)T\chi_T^{2\vartheta}\left[1+\|u_0\|^{\vartheta-1}_{1,a}\right]^{\vartheta}\|u_0\|^{2\vartheta}_{1,a}$$

From which the conclusion

$$\|u\|^2_{\mathcal{H}(Q_T)} \le c(\alpha,\gamma_0,\vartheta,\nu,a)\ \Big[\max\{T,1\}\chi_T^2\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)\|u_0\|^2_{1,a}$$
$$+T\chi_T^{2\vartheta}\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{\vartheta}\|u_0\|^{2\vartheta}_{1,a}\Big]$$
$$\le c(\alpha,\gamma_0,\vartheta,\nu,a)\ \max\{T,1\}\chi_T^{2\vartheta}\left[1+\|u_0\|^{\vartheta-1}_{1,a}+\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{\vartheta}\right]\left(\|u_0\|^2_{1,a}+\|u_0\|^{2\vartheta}_{1,a}\right)$$
$$\le c(\alpha,\gamma_0,\vartheta,\nu,a)\,e^T\chi_T^{2\vartheta}\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{\vartheta}\left(1+\|u_0\|^{2\vartheta-2}_{1,a}\right)\|u_0\|^2_{1,a}$$
$$\le c(\alpha,\gamma_0,\vartheta,\nu,a)\,e^T e^{2(\nu+\|\alpha^+\|_\infty)\vartheta T}\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{\vartheta}\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{2}\|u_0\|^2_{1,a}$$
$$\le c(\alpha,\gamma_0,\vartheta,\nu,a)\,e^{[1+2(\nu+\|\alpha^+\|_\infty)\vartheta]T}\left(1+\|u_0\|^{\vartheta-1}_{1,a}\right)^{2+\vartheta}\|u_0\|^2_{1,a}.$$

3.2.1 Spectral properties of A

Let $A = A_0 + \alpha I$, where the operator A_0 is defined in (3.1.4) and $\alpha \in L^\infty(-1,1)$. Since A is self-adjoint and $D(A) \hookrightarrow L^2(-1,1)$ is compact, we have the following (see also [6]).

Lemma 3.2.6. *There exists an increasing sequence $\{\lambda_k\}_{k\in\mathbb{N}}$, with $\lambda_k \longrightarrow +\infty$ as $k \to \infty$, such that the eigenvalues of A are given by $\{-\lambda_k\}_{k\in\mathbb{N}}$, and the corresponding eigenfunctions $\{\omega_k\}_{k\in\mathbb{N}}$ form a complete orthonormal system in $L^2(-1,1)$.*

Thus,

$$
\begin{aligned}
&\int_0^t \|u_t(\cdot,s)\|^2\,ds + \|\sqrt{a}u_x(t,\cdot)\|^2\,ds \\
&\quad \le (\|\alpha\|_\infty + \nu)\,\|u(t,\cdot)\|^2 + \|\sqrt{a}u_{0x}\|^2 \; + \|\alpha^-\|_\infty \|u_0\|^2 + 2\int_{-1}^1 |F(x,u_0(x))|\,dx \\
&\qquad \le (\|\alpha\|_\infty + \nu)\,\|u(t,\cdot)\|^2 + |u_0|^2_{1,a} + \|\alpha^-\|_\infty \|u_0\|^2 + c(\gamma_0,\vartheta)\,\|u_0\|^{\vartheta+1}_{1,a}.
\end{aligned}
$$

Let us consider for simplicity $\chi_T := e^{(\nu+\|\alpha^+\|_\infty)T}$. By Lemma 3.2.3, we deduce

$$
\begin{aligned}
&\|u(t,\cdot)\|^2 + \|\sqrt{a}u_x(t,\cdot)\|^2 + \int_0^t \|u_t(\cdot,s)\|^2 \\
&\quad \le (\|\alpha\|_\infty + \nu + 1)\,\|u(t,\cdot)\|^2 + |u_0|^2_{1,a} + \|\alpha^-\|_\infty \|u_0\|^2 + c(\gamma_0,\vartheta)\,\|u_0\|^{\vartheta+1}_{1,a} \\
&\quad \le (\|\alpha\|_\infty + \nu + 1)\,\|u\|^2_{\mathcal{B}(Q_t)} + |u_0|^2_{1,a} + \|\alpha^-\|_\infty \|u_0\|^2 + c(\gamma_0,\vartheta,a)\|u_0\|^{\vartheta+1}_{1,a} \\
&\quad \le \left(c(\alpha,\nu)\nu_T^2\, e^{2\|\alpha^+\|_\infty T} + \|\alpha^-\|_\infty\right)\,\|u_0\|^2 + |u_0|^2_{1,a} + c(\gamma_0,\vartheta,a)\|u_0\|^{\vartheta+1}_{1,a} \\
&\quad \le \max\left\{c(\alpha,\nu)\nu_T^2\, e^{2\|\alpha^+\|_\infty T} + \|\alpha^-\|_\infty, c(\gamma_0,\vartheta,a), 1\right\}\,\left[\|u_0\|^2_{1,a} + \|u_0\|^{\vartheta+1}_{1,a}\right] \\
&\qquad \le c(\alpha,\gamma_0,\vartheta,\nu,a)\,\chi_T^2\,\left[1 + \|u_0\|^{\vartheta-1}_{1,a}\right]\|u_0\|^2_{1,a}\,.
\end{aligned}
$$

Moreover, by (5.1.1), we have

$$
(a(x)u_x(t,x))_x = u_t(t,x) - \alpha(x)u(t,x) - f(x,u),
$$

then, for every $t \in (0,T)$, we obtain

$$
\begin{aligned}
&\int_0^t \|\,(a(\cdot)u_x(s,\cdot))_x\,\|^2\,ds \\
&\quad \le 2\int_0^t \|u_t(s,\cdot)\|^2\,ds + 2\|\alpha\|_\infty \int_0^t \|u(s,\cdot)\|^2\,ds + 2\int_{Q_t} \|f(x,u)\|^2\,dx\,ds \\
&\quad \le c(\alpha,\gamma_0,\vartheta,\nu,a)\,\max\{T,1\}\chi_T^2\left[1 + \|u_0\|^{\vartheta-1}_{1,a}\right]\|u_0\|^2_{1,a} + 2\int_{Q_t} \|f(x,u)\|^2\,dx\,ds.
\end{aligned}
$$

Then, by Lemma 3.1.1, we deduce that

$$\int_{-1}^{1} |F(x,u_0(x))|\, dx \le c(\gamma_0,\vartheta)\|u_0\|^{\vartheta+1}_{L^{\vartheta+1}(-1,1)} \le c(\gamma_0,\vartheta,a)\|u_0\|^{\vartheta+1}_{1,a}. \tag{3.2.7}$$

Finally, we observe the following property of the function F :

keeping in mind that, by (5.1.4), for almost every $x \in (-1,1)$, we obtain

- $f(x,u) \le \nu u$, for every $u \in \mathbb{R}, u \ge 0$
- $f(x,u) \ge \nu u$, for every $u \in \mathbb{R}, u < 0$,

then, for almost every $x \in (-1,1)$, we have

- for every $u \in \mathbb{R}, u \ge 0$, $F(x,u) = \int_0^u f(x,\zeta)d\zeta \le \nu \int_0^u \zeta d\zeta \le \frac{\nu}{2}u^2$
- for every $u \in \mathbb{R}, u < 0$, $F(x,u) = -\int_u^0 f(x,\zeta)d\zeta \le -\nu \int_0^u \zeta d\zeta \le \frac{\nu}{2}u^2$.

Then

$$F(x,u) \le \frac{\nu}{2}u^2\,, \quad \forall (x,u) \in (-1,1)\times\mathbb{R}. \tag{3.2.8}$$

By (3.2.6), we obtain

$$\int_{-1}^{1} u_t^2(t,x)dx + \frac{1}{2}\frac{d}{dt}\int_{-1}^{1}\left\{a(x)u_x^2(t,x) \,-\alpha(x)u^2(t,x) - 2F(x,u)\right\}\, dx = 0. \tag{3.2.9}$$

Fix $t \in (0,T)$ and integrate on $(0,t)$, to have

$$\int_0^t\int_{-1}^{1} u_t^2(s,x)dx\, ds + \frac{1}{2}\int_{-1}^{1}\left\{a(x)u_x^2(t,x) \,-\alpha(x)u^2(t,x)\right\}\, dx$$
$$= \int_{-1}^{1} F(x,u(t,x))\, dx + \frac{1}{2}\int_{-1}^{1}\left\{a(x)u_{0x}^2(x) \,-\alpha(x)u_0^2(x)\right\}\, dx - \int_{-1}^{1} F(x,u_0(x))\, dx$$

Lemma 3.2.5. *Let* $T > 0$, $u_0 \in H^1_a(-1,1)$ *and* $\alpha \in L^\infty(Q_T)$. *The solution* $u \in \mathcal{H}(Q_T)$ *of system* (5.1.1) *satisfies the following estimate*

$$\|u\|_{\mathcal{H}(Q_T)} \leq k_1(\|u_0\|_{1,a}) e^{k_2 T} \|u_0\|_{1,a},$$

where $k_1(\|u_0\|_{1,a}) = c(\alpha, \gamma_0, \theta, \nu, a) \left(1 + \|u_0\|_{1,a}^{\vartheta-1}\right)^{1+\frac{\vartheta}{2}}$, $c(\alpha, \gamma_0, \theta, \nu, a)$ *is a positive constant, and* $k_2 = \frac{1}{2} + (\nu + \|\alpha^+\|_\infty)\vartheta$.

Proof. Multiplying by u_t both members of the equation in (5.1.1) and integrating on $(-1,1)$ we obtain

$$\int_{-1}^{1} u_t^2(t,x)dx - \int_{-1}^{1} \left(a(x)u_x(t,x)\right)_x u_t(t,x)dx$$
$$= \int_{-1}^{1} \alpha(x)u(t,x)\, u_t(t,x)dx + \int_{-1}^{1} f(x,u)\, u_t(t,x)\, dx,$$

thus,

$$\int_{-1}^{1} u_t^2(t,x)dx + \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} a(x)u_x^2(t,x)\, dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} \alpha(x)u^2(t,x)dx + \int_{-1}^{1} f(x,u)\, u_t(t,x)\, dx.$$

Now, let us consider the following function $F : (-1,1) \times \mathbb{R} \longrightarrow \mathbb{R}$,

$$F(x,u) := \int_0^u f(x,\zeta)\, d\zeta\,,\ \forall (x,u) \in (-1,1) \times \mathbb{R}.$$

Then, we observe that

$$\frac{\partial F(x,u(t,x))}{\partial t} = f(x,u(t,x))u_t(t,x)\,,\ \forall (t,x) \in Q_T. \tag{3.2.6}$$

Moreover, by (5.1.2) (see assumptions $(A.3)$), we have

$$F(x,u_0(x)) = \int_0^{u_0} f(x,\zeta)d\zeta \leq \gamma_0 \int_0^{u_0} |\zeta|^\vartheta\, d\zeta = \frac{\gamma_0}{\vartheta+1}|u_0|^{\vartheta+1},\ \forall x \in (-1,1).$$

Moreover, applying the inequality (5.1.3) (see assumptions ($A.3$)) and Hölder inequality we obtain

$$
\begin{aligned}
&\int_{Q_T} |f(x,u)-f(x,v)|^2\,dx\,dt \\
&\qquad\le \gamma_1^2 \int_{Q_T} \left(1+|u|^{\vartheta-1}+|v|^{\vartheta-1}\right)^2 |u-v|^2\,dx\,dt \\
&\le c(\gamma_1)\left(\int_{Q_T}\left(1+|u|^{2(\vartheta-1)}+|v|^{2(\vartheta-1)}\right)^{\frac{\vartheta}{\vartheta-1}}dxdt\right)^{\frac{\vartheta-1}{\vartheta}}\left(\int_{Q_T}|u-v|^{2\vartheta}\,dxdt\right)^{\frac{1}{\vartheta}} \\
&\qquad\le c(\gamma_1,\vartheta)\left(T^{1-\frac{1}{\vartheta}}+\|u\|^{2(\vartheta-1)}_{L^{2\vartheta}(Q_T)}+\|v\|^{2(\vartheta-1)}_{L^{2\vartheta}(Q_T)}\right)\|u-v\|^2_{L^{2\vartheta}(Q_T)},
\end{aligned}
\tag{3.2.5}
$$

Then, by (3.2.4) and (3.2.5), applying Corollary 3.1.4 we have

$$
\begin{aligned}
&\|W\|^2_{\mathcal{H}(Q_T)} \\
&\quad\le c(\gamma_1,\vartheta,a)C_0^2(1)T^{1-\frac{1}{\vartheta}}\left(1+\|u\|^{2(\vartheta-1)}_{\mathcal{H}(Q_T)}+\|v\|^{2(\vartheta-1)}_{\mathcal{H}(Q_T)}\right)T^{\frac{1}{\vartheta}}\|u-v\|^2_{\mathcal{H}(Q_T)} \\
&\qquad\le c(\gamma_1,\vartheta,a)\left(1+2R^{2(\vartheta-1)}\right)T\|u-v\|^2_{\mathcal{H}(Q_T)}.
\end{aligned}
$$

Let

$$
T_1(R)=\frac{1}{2c(\gamma_1,\vartheta,a)\left(1+2R^{2(\vartheta-1)}\right)},
$$

and define $T_R=\min\{T_0(R),T_1(R)\}$. Then, Λ is a contraction map. Therefore, Λ has a unique fix point in $\mathcal{H}_R(Q_{T_R})$, from which the conclusion follows.

■

Now, thanks to a classical result (see e.g. [34] and [36]), the following lemma assures the global existence of the solution of (3.2.1).

Now, we fix $T_0(R) = \min\left\{\frac{1}{C_0^{2\vartheta}(1)c^2(\gamma_0,\vartheta,a)R^{2(\vartheta-1)}}, 1\right\}$. Then we have

$$\begin{aligned}\|\Lambda(u)\|_{\mathcal{H}(Q_T)} = \|U\|_{\mathcal{H}(Q_T)} \\ \le C_0(1)\left(2^{\vartheta}c(\gamma_0,\vartheta,a)R^{\vartheta}T^{\frac{1}{2}} + R\right) \\ \le 2C_0(1)R, \quad \forall T \in [0, T_0(R)].\end{aligned}$$

Thus, $\Lambda u \in \mathcal{H}_R(Q_T)$, $\forall T \in [0, T_0(R)]$.

STEP.2 We prove that exists $T_R \le T_0(R)$ such that the map Λ is a contraction.

Let T, $0 < T \le T_0(R)$ (T will be fix below). Fix $u, v \in \mathcal{H}_R(Q_T)$, and set $W = \Lambda(u) - \Lambda(v)$, W is solution of the following problem

$$\begin{cases} W_t - (a\,W_x)_x = \alpha W + f(x,u) - f(x,v) \quad \text{in } Q_T \\ a(x)W_x(t,x)|_{x=\pm 1} = 0 \\ W(0,x) = 0 \quad . \end{cases} \tag{3.2.3}$$

By Lemma 3.2.1 $f(\cdot,u) \in L^2(Q_T)$ and applying Proposition 3.1.5 we deduce that a unique solution $W \in \mathcal{H}(Q_T)$ of (3.2.3) exists and we have

$$\|W\|_{\mathcal{H}(Q_T)} \le C_0(T)\|f(\cdot,u) - f(\cdot,v)\|_{L^2(Q_T)}. \tag{3.2.4}$$

where $C_0(1)$ is the constant $C_0(T)$ (nondecreasing in T) defined in Proposition 3.1.5 and valued in 1. Then, let us define the following map

$$\Lambda : \mathcal{H}_R(Q_T) \longrightarrow \mathcal{H}_R(Q_T),$$

such that

$$\Lambda(u)(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}\phi(u(s))\,ds\,,\ \forall t \in [0,T].$$

Step. 1 We prove that the map Λ is well defined for some T.

Fix $u \in \mathcal{H}_R(Q_T)$. Let us consider $U(t,x) = \Lambda\,(u)\,(t,x)$, then U is solution of the following linear problem

$$\begin{cases} U_t - (a\,U_x)_x = \alpha U + f(x,u) \quad \text{in } \ Q_T \\ a(x)U_x(t,x)|_{x=\pm 1} = 0 \\ U(0,x) = u_0 \qquad\qquad\qquad . \end{cases} \tag{3.2.2}$$

By Lemma 3.2.1 $f(\cdot,u) \in L^2(Q_T) = L^2(0,T;L^2(-1,1))$, then applying Proposition 3.1.5 we deduce that a unique solution $U \in \mathcal{H}(Q_T)$ of (3.2.2) exists and we have

$$\|U\|_{\mathcal{H}(Q_T)} \leq C_0(T)\left(\|f(\cdot,u)\|_{L^2(Q_T)} + \|u_0\|_{1,a}\right).$$

Thus, keeping in mind that $C_0(T) \leq C_0(1)$, by our choice of T, and applying Lemma 3.2.1 we obtain

$$\begin{aligned} \|U\|_{\mathcal{H}(Q_T)} &\leq C_0(1)\left(\|f(\cdot,u)\|_{L^2(Q_T)} + \|u_0\|_{1,a}\right) \\ &\leq c_0(1)\left(\gamma_0\|u\|^{\vartheta}_{L^{2\vartheta}(Q_T)} + \|u_0\|_{1,a}\right) \\ &\leq C_0(1)\left(c(\gamma_0,\vartheta,a)T^{\frac{1}{2}}\|u\|^{\vartheta}_{\mathcal{H}(Q_T)} + \|u_0\|_{1,a}\right) \\ &\leq C_0(1)\left(c(\gamma_0,\vartheta,a)T^{\frac{1}{2}}(2C_0(1)R)^{\vartheta} + R\right) \\ &\leq C_0(1)\left(c(\gamma_0,\vartheta,a)(C_0(1))^{\vartheta}R^{\vartheta}T^{\frac{1}{2}} + R\right). \end{aligned}$$

Then we obtain

$$
\begin{aligned}
&\|u(t,\cdot)\|^2_{L^2(\Omega)} + 2\int_0^t\int_{-1}^1 a(x)u_x^2(t,x)dx\,ds \\
&\qquad \le \|u_0\|^2_{L^2(\Omega)} + \int_0^t 2\left(\|\alpha^+\|_\infty + \nu\right)\,\|u(s,\cdot)\|^2_{L^2(\Omega)}\,ds \\
&\qquad\qquad \le \|u_0\|^2_{L^2(\Omega)} \\
&+\int_0^t 2\left(\|\alpha^+\|_\infty + \nu\right)\left(\|u(s,\cdot)\|^2_{L^2(\Omega)} + 2\int_0^s\int_{-1}^1 a(x)u_x^2(\tau,x)dx\,d\tau\right)ds,\ \forall t\in[0,T].
\end{aligned}
$$

Applying Gromwall's lemma we have

$$
\|u(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^t\int_{-1}^1 a(x)u_x^2(t,x)dx\,ds \le e^{2\|\alpha^+\|_\infty t + 2\int_0^t \nu\,ds}\|u_0\|^2_{L^2(-1,1)}\,.
$$

Therefore

$$
\|u\|^2_{B(Q_T)} \le \nu_T^2\, e^{2\|\alpha^+\|_\infty T}\|u_0\|^2_{L^2(-1,1)}.
$$

■

Now, the following result assures the local existence and uniqueness of the solution of (3.2.1).

Theorem 3.2.4. *For every $R>0$, there is $T_R>0$ such that for all $u_0\in H^1_a(-1,1)$ with $\|u_0\|_{1,a}\le R$ there is a unique solution $u\in\mathcal{H}(Q_{T_R})$ to (5.1.1).*

Proof. Let us fix $R>0$, $u_0\in H^1_a(-1,1)$ such that $\|u_0\|_{1,a}\le R$. Let $0<T\le 1$ (further constraints on T will be imposed below). We define

$$
\mathcal{H}_R(Q_T) := \{u\in\mathcal{H}(Q_T)\,:\,\|u\|_{\mathcal{H}(Q_T)}\le 2C_0(1)R\},
$$

for some positive constant $c(\gamma_0, \vartheta, a)$.

For the sequel, the next lemma is necessary.

Lemma 3.2.3. *Let* $T > 0$, $u_0 \in H^1_a(-1,1)$ *and* $\alpha \in L^\infty(Q_T)$. *A solution* $u \in \mathcal{H}(Q_T)$ *of system* (5.1.1) *satisfies the following a priori estimate*

$$\|u\|_{\mathcal{B}(Q_T)} \le \nu_T \, e^{\|\alpha^+\|_\infty T} \, \|u_0\|_{L^2(-1,1)} \,,$$

where α^+ *denotes the positive part of* α *.* ([7])

Proof. Multiplying by u both members of the equation present in (5.1.1) and integrating on $(-1,1)$ and applying Lemma 3.2.1 and assumption (5.1.4) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2(t,x)dx + \int_{-1}^{1} a(x)u_x^2(t,x)dx \\ = \int_{-1}^{1} \alpha(x)u^2 dx + \int_{-1}^{1} f(x,u)u\,dx \le \int_{-1}^{1} \alpha^+(t,x)u^2 + \int_{-1}^{1} \nu u^2 dx.\end{aligned}$$

Integrating on $(0,t)$, we have

$$\begin{aligned}\frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \int_{-1}^{1} a(x)u_x^2(t,x)dx\,ds \\ \le \frac{1}{2}\|u_0\|^2_{L^2(-1,1)} + \|\alpha^+\|_\infty \int_0^t \|u(s,\cdot)\|^2_{L^2(\Omega)}\,ds + \int_0^t \nu\|u(s,\cdot)\|^2_{L^2(\Omega)}\,ds\,.\end{aligned}$$

[7] *We recall that* $\nu_T = e^{\nu T}$.

where A is the operator defined in (3.1.5), $\alpha \in L^\infty(-1,1)$, $u_0 \in H^1_a(-1,1)$, and, for every $u \in H^1_a(-1,1)$,

$$\phi(u)(x) := f(x, u(x)), \ \ \forall x \in (-1,1).$$

We assume hereafter that assumptions $(A.3)$ and $(A.4)$ are enforced.

The next lemma shows that $\phi : H^1_a(-1,1) \longrightarrow L^2(-1,1)$.

Lemma 3.2.1. *Let $T > 0, \vartheta > 1, \xi_a \in L^{2\vartheta-1}(-1,1)$, and let $u \in \mathcal{H}(Q_T)$. Let $f : (-1,1)\times\mathbb{R} \to \mathbb{R}$ be a function that satisfies assumptions $(A.3)$. Then, the function $(t,x) \longmapsto f(x,u(t,x))$ belongs to $L^2(Q_T)$ and the following estimate holds*

$$\int_{Q_T} |f(x,u(t,x))|^2 \, dx\, dt \le c(\gamma_0,\vartheta,a)T \, \|u\|_{H^1(0,T;L^2(-1,1))} \, \|u\|^{2\vartheta-1}_{L^\infty(0,T;H^1_a(-1,1))},$$

for some positive constant $c(\gamma_0,\vartheta,a)$.

Proof. By Lemma 3.1.3, since $\xi_a \in L^{2\vartheta-1}(-1,1)$ then $u \in L^{2\vartheta}(Q_T)$. By (5.1.2) we obtain

$$\begin{aligned}\int_{Q_T} |f(x,u(t,x))|^2 \, dx\, dt &\le \gamma_0^2 \int_{Q_T} |u|^{2\vartheta} \, dx\, dt \\ &\le c(\gamma_0,\vartheta,a)T \, \|u\|_{H^1(0,T;L^2(-1,1))} \, \|u\|^{2\vartheta-1}_{L^\infty(0,T;H^1_a(-1,1))} < +\infty,\end{aligned}$$

from wich the conclusion follows. ∎

Corollary 3.2.2. *Let $T > 0, \vartheta > 1, \xi_a \in L^{2\vartheta-1}(-1,1)$, and let $u \in \mathcal{H}(Q_T)$. Let $f : (-1,1)\times\mathbb{R} \to \mathbb{R}$ be a function that satisfies assumptions $(A.3)$. Then, we have the following estimate*

$$\int_{Q_T} |f(x,u(t,x))|^2 \, dx\, dt \le c(\gamma_0,\vartheta,a)T \, \|u\|^{2\vartheta}_{\mathcal{H}(Q_T)},$$

for almost all $t \in [0,T]$ (see [2]). [2]

For every $\alpha \in L^\infty(-1,1)$(^{3}) and every $u_0 \in L^2(-1,1)$, there exists a unique weak solution of (3.1.6), which is given by the following representation $e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s)\,ds,\ t \in [0,T]$ (see also [?]).

Now, using a *maximal regularity* result in the Hilbert space $L^2(-1,1)$[4] (see [4] and [?]), we deduce the following result

Proposition 3.1.5. *Given $T > 0$ and $g \in L^2(0,T;L^2(-1,1))$.*[5] *For every $\alpha \in L^\infty(-1,1)$ (^{6}) and every $u_0 \in H^1_a(-1,1)$, there exists a unique solution $u \in \mathcal{H}(Q_T)$ of (3.1.6). Moreover, a positive constant $C_0(T)$ exists (nondecreasing in T), such that the following inequality holds*

$$\|u\|_{\mathcal{H}(Q_T)} \le C_0(T)\left[\|u_0\|_{1,a} + \|g\|_{L^2(Q_T)}\right].$$

3.2 Existence and uniqueness of the solution of semilinear problems

Observe that problem (5.1.1) can be recast in the Hilbert space $L^2(-1,1)$ as

$$\begin{cases} u'(t) = A\,u(t) + \phi(u)\,, & t > 0 \\ \\ u(0) = u_0\ . \end{cases} \tag{3.2.1}$$

---

[2] A^* denotes the adjoint of A.

[3] See also note (g). The same remark applies to the present context.

[4] By *Maximal regularity* we mean that u' and Au have the same regularity of g.

[5] *We observe that $L^2(0,T;L^2(-1,1)) = L^2(Q_T)$.*

[6] *By repeated applications of Proposition 3.1.5, one can obtain an existence and uniqueness result when α is piecewise static ($\alpha(\cdot,x)$ piecewise constant in t, and $\alpha(t,\cdot) \in L^\infty(-1,1), \forall t \in (0,T)$). The same result holds for $\alpha \in L^\infty(Q_T)$, but for the purposes of the present paper the piecewise static case will suffice.*

Observe that A_0 is a closed, self-adjoint, dissipative operator with dense domain in $L^2(-1,1)$. Therefore, A_0 is the infinitesimal generator of a C_0 − semigroup of contractions in $L^2(-1,1)$.

Next, given $\alpha \in L^\infty(-1,1)$, let us introduce the operator

$$\begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha I \ . \end{cases} \tag{3.1.5}$$

For such an operator we have that

- $D(A)$ is compactly embedded and dense in $L^2(-1,1)$ (see [**?**]).
- $A : D(A) \longrightarrow L^2(-1,1)$ is the infinitesimal generator of a strongly continuous semigroup, e^{tA}, of bounded linear operators on $L^2(-1,1)$.

We consider the following linear problem in the Hilbert space $L^2(-1,1)$

$$\begin{cases} u'(t) = A\,u(t) \ + g(t), \qquad t > 0 \\ \\ u(0) = u_0 \ \ , \end{cases} \tag{3.1.6}$$

where A is the operator in (3.1.5), $g \in L^1(0,T;L^2(-1,1))$, $u_0 \in L^2(-1,1)$.

We recall that a *weak solution* of (3.1.6) is a function $u \in C^0([0,T];L^2(-1,1))$ such that for every $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle \ + \langle g(t), v\rangle,$$

Corollary 3.1.4. *Let $T > 0$, $p \geq 1$. If $\xi_a \in L^{2p-1}(-1,1)$, then*

$$\mathcal{H}(Q_T) \subset L^{2p}(Q_T)$$

and

$$\|u\|_{L^{2p}(Q_T)} \leq c(a,p)\, T^{\frac{1}{2p}}\, \|u\|_{\mathcal{H}(Q_T)},$$

where $c = c(a,p)$ is a positive constant.

3.1.1 Existence and uniqueness of the solution of linear problems

In this section, we recall the existence and uniqueness result, obtained in [9] (see also [1] and [**?**]), for the linear problem

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in } Q_T = (0,T) \times (-1,1) \\ a(x)v_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ v(0,x) = v_0(x) & x \in (-1,1)\,, \end{cases} \tag{3.1.3}$$

where $v_0 \in L^2(-1,1)$, $\alpha(t,x)$ and the diffusion coefficient $a(x)$ satisfy respectively the assumption $(A.2)$ and $(A.4)$ [1].

First, let us consider the operator $(A_0, D(A_0))$ defined by

$$\begin{cases} D(A_0) = H^2_a(-1,1) \\ A_0 u = (au_x)_x\,, \quad \forall\, u \in D(A_0)\,. \end{cases} \tag{3.1.4}$$

[1] In this section, it is sufficient that $a(\cdot)$ satisfies assumption $(A.4)$ with $\xi_a \in L^1(-1,1)$, instead of the condition (5.1.5).

and

$$\|u\|_{L^{2p}(Q_T)} \le c(a,p)\, T^{\frac{1}{2p}}\, \|u\|_{H^1(0,T;L^2(-1,1))}^{\frac{1}{2p}}\, \|u\|_{L^\infty(0,T;H^1_a(-1,1))}^{1-\frac{1}{2p}},$$

where $c = c(a,p)$ is a positive constant.

Proof. For every $u \in H^1(0,T;L^2(-1,1)) \cap L^\infty(0,T;H^1_a(-1,1))$ we have

$$\begin{aligned}\int_{Q_T} |u|^{2p}\, dx\, dt &= \int_0^T \int_{-1}^1 |u|\, |u|^{2p-1}\, dx\, dt \\ &\le \int_0^T \left(\int_{-1}^1 |u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{-1}^1 |u|^{4p-2}\, dx\right)^{\frac{1}{2}} dt\,.\end{aligned}$$

Recalling that $u \in H^1(0,T;L^2(-1,1))$, by the previous Lemma 3.1.1 and since $\xi^a \in L^{2p-1}(-1,1)$, we obtain

$$\begin{aligned}\int_{Q_T} |u|^{2p}\, dx\, dt &\le \|u\|_{H^1(0,T;L^2(-1,1))} \int_0^T \|u\|_{L^{4p-2}(-1,1)}^{2p-1}\, dt \\ &\le c(a,p)\, \|u\|_{H^1(0,T;L^2(-1,1))} \int_0^T \|u\|_{H^1_a(-1,1)}^{2p-1}\, dt\,.\end{aligned}$$

From the last inequality, it follows that

$$\begin{aligned}&\int_{Q_T} |u|^{2p}\, dx\, dt \\ &\qquad \le c(a,p)\, T\, \|u\|_{H^1(0,T;L^2(-1,1))}\, \|u\|_{L^\infty(0,T;H^1_a(-1,1))}^{2p-1}.\end{aligned}$$

■

By Lemma 3.1.3 one directly obtains the following.

Proof. For every $u \in L^2(0,T;H^1_a(-1,1))$ we have

$$\int_{Q_T} |u|^{2p}\,dx\,dt = \int_0^T \int_{-1}^1 |u|^p\,|u|^p\,dx\,dt$$
$$\leq \int_0^T \left(\int_{-1}^1 |u|^2\,dx\right)^{\frac{p}{2}} \left(\int_{-1}^1 |u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{2}} dt\,.$$

Recalling that $u \in L^\infty(0,T;L^2(-1,1))$, by Lemma 3.1.1 we obtain

$$\int_{Q_T} |u|^{2p}\,dx\,dt \leq \|u\|^p_{L^\infty(0,T;L^2(-1,1))} \int_0^T \|u\|^p_{L^{\frac{2p}{2-p}}(-1,1)}\,dt$$
$$\leq c(a,p)\,\|u\|^p_{L^\infty(0,T;L^2(-1,1))} \int_0^T \|u\|^p_{H^1_a(-1,1)}\,dt\,.$$

Moreover, using Hölder's inequality , we have

$$\int_0^T \|u\|^p_{H^1_a(-1,1)}\,dt \leq \left(\int_0^T dt\right)^{1-\frac{p}{2}} \left(\int_0^T \|u\|^2_{H^1_a(-1,1)}\,dt\right)^{\frac{p}{2}}$$
$$\leq T^{1-\frac{p}{2}}\|u\|^p_{L^2(0,T;H^1_a(-1,1))}\,.$$

From the last two inequalities, it follows that

$$\int_{Q_T} |u|^{2p}\,dx\,dt$$
$$\leq c(a,p)\,T^{1-\frac{p}{2}}\,\|u\|^p_{L^2(0,T;H^1_a(-1,1))}\,\|u\|^p_{L^\infty(0,T;L^2(-1,1))}$$
$$\leq c(a,p)\;T^{1-\frac{p}{2}}\|u\|^{2p}_{B(Q_T)}.$$

■

Lemma 3.1.3. *Let* $T > 0$, $p \geq 1$. *If* $\xi_a \in L^{2p-1}(-1,1)$, *then*

$$H^1(0,T;L^2(-1,1)) \cap L^\infty(0,T;H^1_a(-1,1)) \subset L^{2p}(Q_T)$$

Moreover,

$$\|u\|_{L^{2p}(-1,1)} \leq c(a,p)\,\|u\|_{1,a},$$

where $c = c(a,p)$ is a positive constant.

Proof. Let $u \in H^1_a(-1,1)$. First, for every $x \in (-1,1)$, we have the following estimate

$$\begin{aligned}|u(x) - u(0)| &= \left|\int_0^x u'(s)\,ds\right| \\ &\leq \left(\int_0^x a(s)|u'(s)|^2\,ds\right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{a(s)}\,ds\right)^{\frac{1}{2}} = \sqrt{\xi_a(x)}|u|_{1,a}. \quad (3.1.1)\end{aligned}$$

Moreover, keeping in mind that $\xi_a \in L^p(-1,1)$, we have

$$\begin{aligned}\int_{-1}^1 |u(0)|\,dx &\leq \int_{-1}^1 |u(x) - u(0)|\,dx + \int_{-1}^1 |u(x)|\,dx \\ &\leq |u|_{1,a} \int_{-1}^1 \sqrt{\xi_a(x)}\,dx + \sqrt{2}\|u\|.\end{aligned}$$

Thus,

$$|u(0)| \leq c_a\,|u|_{1,a} + \|u\|, \qquad \text{where } c_a = \frac{1}{2}\int_{-1}^1 \sqrt{\xi_a(x)}\,dx. \quad (3.1.2)$$

Finally, by (3.1.1) and (3.1.2) we have

$$\begin{aligned}\int_{-1}^1 |u(x)|^{2p}\,dx &\leq c(p)\int_{-1}^1 \left(|u(x) - u(0)|^{2p} + |u(0)|^{2p}\right)\,dx \\ &\leq c(p)|u|_{1,a}^{2p} \int_{-1}^1 \xi_a^p(x)\,dx + c(a,p)\,\left(|u|_{1,a} + \|u\|\right)^{2p} \leq c(a,p)\,\|u\|_{1,a}^{2p}.\end{aligned}$$

■

Lemma 3.1.2. *Let $T > 0$. If $\xi_a \in L^{\frac{p}{2-p}}(-1,1)$ for some $p \in \left[\frac{1}{2}, 2\right)$, then*

$$L^2(0,T; H^1_a(-1,1)) \cap L^\infty(0,T; L^2(-1,1)) \subset L^{2p}(Q_T)$$

and

$$\|u\|_{L^{2p}(Q_T)} \leq c(a,p)\,T^{\frac{1}{2p}\left(1-\frac{p}{2}\right)}\|u\|_{B(Q_T)}\,,$$

where $c = c(a,p)$ is a positive constant.

where $|u|_{1,a} := \|\sqrt{a}u_x\|_{L^2(-1,1)}$.

In [?] we proved that the imbedding $H^1_a(-1,1) \hookrightarrow L^2(-1,1)$ is compact.

3.0.3 The function spaces $\mathcal{B}(Q_T)$ and $\mathcal{H}(Q_T)$

Given $T > 0$, let us define the function spaces:

$$\mathcal{B}(Q_T) := C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1))$$

with the following norm

$$\|u\|^2_{\mathcal{B}(Q_T)} = \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^T \int_{-1}^1 a(x)u_x^2 dx\, dt\,,$$

and

$$\mathcal{H}(Q_T) := L^2(0,T; H^2_a(-1,1)) \cap H^1(0,T; L^2(-1,1)) \cap C([0,T]; H^1_a(-1,1))$$

with the following norm

$$\|u\|^2_{\mathcal{H}(Q_T)} = \sup_{t\in[0,T]} \left(\|u\|^2 + \|\sqrt{a}u_x\|^2\right) + \int_0^T \left(\|u_t\|^2 + \|(au_x)_x\|^2\right)\, dt.$$

3.1 Some embedding theorems for weighted Sobolev spaces

Let $\xi_a(x) = \int_0^x \frac{1}{a(s)}\, ds$, then we have the following

Lemma 3.1.1. *If $\xi_a \in L^p(-1,1)$, for some $p \geq 1$, then*

$$H^1_a(-1,1) \hookrightarrow L^{2p}(-1,1)\,.$$

Chapter 3

Well-posedness for nonlinear problems

Weighted Sobolev spaces

In order to deal with the well-posedness of problem (5.1.1), it is necessary to introduce the following weighted Sobolev spaces $H^1_a(-1,1)$ and $H^2_a(-1,1)$.
We denote by $H^1_a(-1,1)$ the space of all functions $u \in L^2(-1,1)$ such that u is locally absolutely continuous in $(-1,1)$ and $\sqrt{a}\, u_x \in L^2(-1,1)$.
Moreover, we define

$$\begin{aligned} H^2_a(-1,1) &:= \{u \in H^1_a(-1,1) |\, au_x \in H^1(-1,1)\} \\ &= \{u \in L^2(-1,1) | au \in H^1_0(-1,1),\, au_x \in H^1(-1,1) \text{ and } (a\, u_x)(\pm 1) = 0\} \end{aligned}$$

$H^1_a(-1,1)$ and $H^2_a(-1,1)$ are Hilbert spaces with their natural scalar products and the associated norms

$$\|u\|^2_{1,a} := \|u\|^2_{L^2(-1,1)} + |u|^2_{1,a}$$

and

$$\|u\|^2_{2,a} := \|u\|^2_{1,a} + \|(au_x)_x\|^2_{L^2(-1,1)},$$

doesn't change sign in $(-1,1)$.

STEP.2 Let us now prove that

$$k_* = 1\,, \tag{2.3.15}$$

that is, $\lambda_1 = 0$. By a well-known variational characterization of the first eigenvalue, we have

$$\lambda_1 = \inf_{u\in H^1_a(-1,1)} \frac{\int_{-1}^1 (a\,u_x^2\; - \alpha_*\,u^2)\;dx}{\int_{-1}^1 u^2\,dx}\;.$$

By Lemma 3.2.6, since $\lambda_{k_*} = 0$, it is sufficient to prove that $\lambda_1 \geq 0$, or

$$\int_{-1}^1 \alpha_*\,u^2\,dx \leq \int_{-1}^1 a\,u_x^2\,dx, \qquad \forall\,u \in H^1_a(-1,1) \tag{2.3.16}$$

Integrating by parts, we have

$$\int_{-1}^1 \alpha_*\,u^2\,dx = -\int_{-1}^1 \frac{(a\,v_x)_x}{v}\,u^2\,dx = \int_{-1}^1 a\,v_x \left(\frac{u^2}{v}\right)_x\,dx =$$
$$= \int_{-1}^1 a\,v_x \frac{2uu_x}{v}\,dx - \int_{-1}^1 a\,v_x^2 \left(\frac{u^2}{v^2}\right)\,dx =$$
$$= 2\int_{-1}^1 \sqrt{a}\,\frac{v_x}{v} u\sqrt{a}u_x\;dx - \int_{-1}^1 a\,v_x^2 \left(\frac{u^2}{v^2}\right)\;dx \leq$$
$$\leq \int_{-1}^1 a\,\left(\frac{v_x u}{v}\right)^2\,dx + \int_{-1}^1 au_x^2\,dx - \int_{-1}^1 a\,v_x^2\left(\frac{u^2}{v^2}\right)\,dx = \int_{-1}^1 au_x^2\;dx\,,$$

from which (2.3.16). $\square$

Moreover, $\frac{v}{\|v\|}$ *and* $-\frac{v}{\|v\|}$ *are the only eigenfunctions of* A *with norm* 1 *that do not change sign in* $(-1,1)$.

Remark 2.3.1. Problem (2.3.10) is equivalent to the following differential problem

$$\begin{cases} (a(x)\omega_x)_x + \alpha_*(x)\omega + \lambda\,\omega = 0 \qquad \text{in} \qquad (-1,1) \\ \\ a(x)\omega_x(x)|_{x=\pm 1} = 0 \quad . \end{cases} \tag{2.3.11}$$

Proof. (of Lemma 2.3.5) STEP.1 We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad\qquad \text{and} \qquad\qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the operator (2.3.10) (see Lemma 3.2.6). Therefore,

$$\langle \omega_k, \omega_h\rangle_{L^2(-1,1)} = \int_{-1}^{1} \omega_k(x)\omega_h(x)dx = 0, \qquad \text{if } h \neq k\,. \tag{2.3.12}$$

We can see, by easy calculations, that an eigenfunction of the operator defined in (2.3.10) is the function

$$\frac{v(x)}{\|v\|}\,,$$

associated with the eigenvalue $\lambda = 0$. Taking into account the above and considering that $v(x) > 0$, $\forall x \in (-1,1)$

$$\exists\, k_* \in \mathbb{N}\ : \omega_{k_*}(x) = \frac{v(x)}{\|v\|} > 0 \ \text{ or } \ \omega_{k_*}(x) = -\frac{v(x)}{\|v\|} < 0,\ \forall x \in (-1,1)\,. \tag{2.3.13}$$

Writing (6) with $k = k_*$ we obtain

$$\langle \omega_{k_*}, \omega_h\rangle_{L^2(-1,1)} = \int_{-1}^{1} \omega_{k_*}(x)\omega_h(x)dx = 0, \qquad \forall h \neq k_*\,. \tag{2.3.14}$$

Therefore, considering (2.3.14) and keeping in mind that $\omega_{k_*} > 0$ or $\omega_{k_*} < 0$ in $(-1,1)$, we observe that ω_{k_*} is the only eigenfunction of the operator defined in (2.3.10) that

In the space

$$\mathcal{B}(Q_T) = C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1))$$

let us define the following norm

$$\|u\|^2_{\mathcal{B}(Q_T)} = \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^T \int_{-1}^1 a(x)u_x^2 dx\, dt,\ \forall u \in \mathcal{B}(Q_T)\,. \qquad (2.3.9)$$

2.3.2 New results for singular Sturm-Liouville problems

Let $A = A_0 + \alpha I$, where the operator A_0 is defined in (3.1.4) and $\alpha \in L^\infty(-1,1)$. Since A is self-adjoint and $D(A) \hookrightarrow L^2(-1,1)$ is compact (see Proposition 2.2.2), we have the following (see also [6]).

Lemma 2.3.4. *There exists an increasing sequence $\{\lambda_k\}_{k\in\mathbb{N}}$, with $\lambda_k \longrightarrow +\infty$ as $k \to \infty$, such that the eigenvalues of A are given by $\{-\lambda_k\}_{k\in\mathbb{N}}$, and the corresponding eigenfunctions $\{\omega_k\}_{k\in\mathbb{N}}$ form a complete orthonormal system in $L^2(-1,1)$.*

In this note we obtain the following result

Lemma 2.3.5. *Let $v \in C^\infty([-1,1]), v > 0$ on $[-1,1]$, let $\alpha_*(x) = -\frac{(a(x)v_x(x))_x}{v(x)}$, $x \in (-1,1)$. Let A be the operator defined in (3.1.5) with $\alpha = \alpha_*$*

$$\begin{cases} D(A) = H^2_a(-1,1) \\ \\ A = A_0 + \alpha_* I\ , \end{cases} \qquad (2.3.10)$$

and let $\{\lambda_k\}, \{\omega_k\}$ be the eigenvalues and eigenfunctions of A, respectively, given by Lemma 3.2.6. Then

$$\lambda_1 = 0 \quad and \quad |\omega_1| = \frac{v}{\|v\|}.$$

Next, given $\alpha \in L^\infty(-1,1)$, let us introduce the operator

$$\begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha I \ . \end{cases} \tag{2.3.7}$$

For such an operator we have the following proposition.

Proposition 2.3.2. • *$D(A)$ is compactly embedded and dense in $L^2(-1,1)$.*

- *$A : D(A) \longrightarrow L^2(-1,1)$ is the infinitesimal generator of a strongly continuous semigroup, e^{tA}, of bounded linear operators on $L^2(-1,1)$.*

Observe that problem (1.2.1) can be recast in the Hilbert space $L^2(-1,1)$ as

$$\begin{cases} u'(t) = A\,u(t)\,, \qquad t > 0 \\ \\ u(0) = u_0 \ . \end{cases} \tag{2.3.8}$$

where A is the operator in (3.1.5).

We recall that a *weak solution* of (3.1.6) is a function $u \in C^0([0,T]; L^2(-1,1))$ such that for every $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle\,,$$

for almost $t \in [0,T]$ (see [2]).

Theorem 2.3.3. *For every $\alpha \in L^\infty((0,T)\times(-1,1))$ and every $u_0 \in L^2(-1,1)$, there exists a unique*

$$u \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1))$$

weak solution to (1.2.1), which coincides with $e^{tA}u_0$.

By integrating on $[0,1]$, we obtain

$$|u(0)| \leq \int_0^1 |u(x)|\,dx + |u|_{1,a}\int_0^1 \sqrt{A(x)}\,dx \leq$$

$$\leq \|u\|_{L^2(-1,1)} + |u|_{1,a}\int_0^1 \sqrt{A(x)}\,dx \leq C\|u\|_{1,a}\,.$$

Then,

$$|u(0)| \leq C\,R\,. \tag{2.3.4}$$

Now, it follows that

$$|u(x)|^2 \leq 2|u(0)|^2 + 2A(x)|u|^2_{1,a} \leq C\,R^2 + 2A(x)R^2\,.$$

Finally, since $A \in L^1(-1,1)$, by integrating on $[1-h,1]$ we obtain

$$\int_{1-h}^1 |u(x)|^2\,dx \leq C\,hR^2 + 2R^2\int_{1-h}^1 A(x)\,dx \longrightarrow 0, \qquad \text{as } h \to 0^+.$$

Similarly, we can prove that

$$\sup_{\|u\|_{1,a}\leq R}\int_{-1}^{-1+h} |u(x)|^2\,dx \longrightarrow 0, \qquad \text{as } h \to 0^+. \tag{2.3.5}$$

By (2.3.2), (2.3.3) and (2.3.5) we obtain (2.3.1). $\square$

We now recall the existence and uniqueness result for system (1.2.1) obtained in [9] (see also [1]). Let us consider, first, the operator $(A_0, D(A_0))$ defined by

$$\begin{cases} D(A_0) = H^2_a(-1,1) \\ \\ A_0u = (au_x)_x\,, \ \ \forall\, u \in D(A_0)\,. \end{cases} \tag{2.3.6}$$

Observe that A_0 is a closed, self-adjoint, dissipative operator with dense domain in $L^2(-1,1)$. Therefore, A_0 is the infinitesimal generator of a $C_0 -$ semigroup of contractions in $L^2(-1,1)$.

$$= \int_{-1-h}^{-1} |u(x+h)|^2 \, dx \; + \int_{-1}^{1-h} |u(x+h) - u(x)|^2 \, dx \; + \int_{1-h}^{1} |u(x)|^2 \, dx =$$

$$= \int_{-1}^{-1+h} |u(x)|^2 \, dx \; + \int_{-1}^{1-h} |u(x+h) - u(x)|^2 \, dx \; + \int_{1-h}^{1} |u(x)|^2 \, dx$$

First, let us prove that

$$\sup_{\|u\|_{1,a} \le R} \int_{-1}^{1-h} |u(x+h) - u(x)|^2 \, dx \longrightarrow 0, \qquad \text{as } h \to 0^+. \tag{2.3.2}$$

Recalling that $A(x) = \int_0^x \frac{ds}{a(s)}$, we have

$$|u(x+h) - u(x)| \le \int_x^{x+h} \sqrt{a(s)} |u'(s)| \frac{1}{\sqrt{a(s)}} \, ds \le$$

$$\le \left(\int_{-1}^{1} a(s) |u'(s)|^2 \, ds \right)^{\frac{1}{2}} \left(\int_x^{x+h} \frac{ds}{a(s)} \right)^{\frac{1}{2}} = |u|_{1,a} \left[A(x+h) - A(x) \right]^{\frac{1}{2}} .$$

By integrating on $[-1, 1-h]$, since $A \in L^1(-1,1)$ (by assumption 3.b)), we obtain

$$\int_{-1}^{1-h} |u(x+h) - u(x)|^2 \, dx \le |u|_{1,a}^2 \int_{-1}^{1-h} \left(A(x+h) - A(x) \right) \, dx \le$$

$$\le R^2 \left[\int_{-1+h}^{1} A(x) \, dx \; - \int_{-1}^{1-h} A(x) \, dx \right] =$$

$$= R^2 \left[\int_{1-h}^{1} A(x) \, dx \; - \int_{-1}^{-1+h} A(x) \, dx \right] \longrightarrow 0, \qquad \text{as } h \to 0^+.$$

Now, let us prove that

$$\sup_{\|u\|_{1,a} \le R} \int_{1-h}^{1} |u(x)|^2 \, dx \longrightarrow 0, \qquad \text{as } h \to 0^+. \tag{2.3.3}$$

We have

$$|u(0)| \le |u(x)| + \int_0^x \sqrt{a(s)} |u'(s)| \frac{1}{\sqrt{a(s)}} \, ds \le$$

$$\le |u(x)| + \left(\int_{-1}^{1} a(s) |u'(s)|^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^x \frac{ds}{a(s)} \right)^{\frac{1}{2}} \le |u(x)| + |u|_{1,a} \sqrt{A(x)} \, .$$

and

$$H_a^2(-1,1) := \{u \in H_a^1(-1,1)|\, au_x \in H^1(-1,1)\} =$$

$$= \{u \in L^2(-1,1)|u \text{ locally absolutely continuous in } (-1,1),$$

$$au \in H_0^1(-1,1),\ au_x \in H^1(-1,1) \text{ and } (a\,u_x)(\pm 1) = 0\}$$

respectively with the following norms

$$\|u\|^2_{H_a^1} := \|u\|^2_{L^2(-1,1)} + |u|^2_{1,a} \text{ and } \|u\|^2_{H_a^2} := \|u\|^2_{H_a^1} + \|(au_x)_x\|^2_{L^2(-1,1)};$$

where $|u|_{1,a} = \|\sqrt{a}u_x\|_{L^2(-1,1)}$ is a seminorm.

In this note we obtain the following result.

Lemma 2.3.1. *Assume that $\xi_a \in L^1(-1,1)$.*

$$H_a^1(-1,1) \hookrightarrow L^2(-1,1) \quad \textit{with compact embedding.}$$

Proof. Given $u \in H_a^1(-1,1)$, let

$$\bar{u}(x) = \begin{cases} u & \text{if } x \in [-1,1] \\ \\ 0 & \text{elsewere} \end{cases}.$$

It is sufficient to prove that, for every $R > 0$,

$$\sup_{\|u\|_{1,a} \le R} \int_{\mathbb{R}} |\bar{u}(x+h) - \bar{u}(x)|^2\, dx \longrightarrow 0, \qquad \text{as } h \to 0 \tag{2.3.1}$$

Let $h > 0$([2]) and let $u \in H_a^1(-1,1)$ be such that $\|u\|_{1,a} \le R$, we have the following equality

$$\int_{\mathbb{R}} |\bar{u}(x+h) - \bar{u}(x)|^2\, dx =$$

[2] In the case $h < 0$ we proceed similarly.

Integrating by parts, keeping in mind that $\beta_1\gamma_1 \neq 0$, we have

$$\int_{-1}^{1} \alpha_* \, u^2 \, dx = -\int_{-1}^{1} \frac{(a\, v_x)_x}{v} \, u^2 \, dx = -\left[a\, v_x \frac{u^2}{v}\right]_{-1}^{1} + \int_{-1}^{1} a\, v_x \left(\frac{u^2}{v}\right)_x \, dx$$

$$= -a(1)\, v_x(1) \frac{u^2(t,1)}{v(1)} + a(-1)\, v_x(-1) \frac{u^2(t,-1)}{v(-1)}$$

$$+ \int_{-1}^{1} a\, v_x \frac{2u u_x}{v} \, dx - \int_{-1}^{1} a\, v_x^2 \left(\frac{u^2}{v^2}\right) dx$$

$$= \frac{\gamma_0}{\gamma_1} v(1) \frac{u^2(t,1)}{v(1)} - \frac{\beta_0}{\beta_1} v(-1) \frac{u^2(t,-1)}{v(-1)}$$

$$+ 2 \int_{-1}^{1} \sqrt{a}\, \frac{v_x}{v} u \sqrt{a} u_x \, dx - \int_{-1}^{1} a\, v_x^2 \left(\frac{u^2}{v^2}\right) \, dx$$

$$\leq \frac{\gamma_0}{\gamma_1} u^2(t,1) - \frac{\beta_0}{\beta_1} u^2(t,-1)$$

$$+ \int_{-1}^{1} a \, \left(\frac{v_x u}{v}\right)^2 \, dx + \int_{-1}^{1} a u_x^2 \, dx - \int_{-1}^{1} a\, v_x^2 \left(\frac{u^2}{v^2}\right) \, dx$$

$$= -\left[a u_x \, u\right]_{-1}^{1} + \int_{-1}^{1} a u_x^2 \, dx \, ,$$

from which (2.2.8) follows. In fact, (2.2.8) holds true even for $\beta_1\gamma_1 = 0$, as one can show by similarly argument. □

2.3 Well-posedness in weighted Sobolev spaces: strongly degenerate case

2.3.1 Weighted Sobolev spaces

In order to deal with the well-posedness of problem (1.2.1), it is necessary to introduce the following weighted Sobolev spaces

$$H^1_a(-1,1) :=$$

$$:= \{u \in L^2(-1,1) : u \text{ locally absolutely continuous in } (-1,1), \sqrt{a} u_x \in L^2(-1,1)\}$$

Therefore, considering (2.2.6) and (2.2.7), we observe that ω_{k_*} is the only eigenfunction of the operator defined in (2.2.4) that doesn't change sign in $(-1,1)$.

STEP.2 Let us now prove that

$$k_* = 1\,,$$

that is, $\lambda_1 = 0$. Recall that

$$\lambda_1 = \min_{u\in D(A)\setminus\{0\}} \frac{-\langle Au, u\rangle}{\|u\|^2},$$

where

$$\langle Au, u\rangle = \int_{-1}^{1} \left((a\,u_x)_x\,u + \alpha_*\,u^2\right)\,dx = [au_x\,u]_{-1}^{1} - \int_{-1}^{1} a\,u_x^2\,dx + \int_{-1}^{1} \alpha_*\,u^2\,dx\,.$$

By Lemma 2.2.4, since $\lambda_{k_*} = 0$, it is sufficient to prove that $\lambda_1 \geq 0$, or

$$\int_{-1}^{1} \alpha_*\,u^2\,dx\, + [au_x\,u]_{-1}^{1} \leq \int_{-1}^{1} a\,u_x^2\,dx, \qquad \forall\, u \in H_a^1(-1,1)\,. \tag{2.2.8}$$

If $\beta_1\gamma_1 \neq 0$, using the Robin boundary conditions, we have

$$\begin{aligned}[au_x\,u]_{-1}^{1} &= a(1)u_x(t,1)\,u(t,1) - a(-1)u_x(t,-1)\,u(t,-1)\\ &= \frac{-\gamma_0}{\gamma_1}u^2(t,1) + \frac{\beta_0}{\beta_1}\,u^2(t,-1).\end{aligned}$$

$L^\infty(-1,1)$. *Then*

$$\lambda_1 = 0 \quad and \quad |\omega_1| = \frac{v}{\|v\|}.$$

Moreover, $\frac{v}{\|v\|}$ *and* $-\frac{v}{\|v\|}$ *are the only eigenfunctions of* A *with norm* 1 *that do not change sign in* $(-1,1)$.

Remark 2.2.1. Problem (2.2.4) is equivalent to the following Sturm-Liouville system

$$\begin{cases} (a(x)\omega_x)_x + \alpha_*(x)\omega + \lambda\,\omega = 0 \qquad \text{in} \qquad (-1,1) \\ \begin{cases} \beta_0\omega(-1) + \beta_1 a(-1)\omega_x(-1) = 0 \\ \gamma_0\,\omega(1) \,+\, \gamma_1\,a(1)\,\omega_x(1) = 0 \end{cases} \end{cases}.$$

Proof. (of Lemma 2.2.5)

<u>STEP.1</u> We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad\qquad \text{and} \qquad\qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the operator (2.2.4) (see Lemma 2.2.4). Therefore,

$$\langle \omega_k, \omega_h \rangle = 0, \qquad \text{if } h \neq k\,. \tag{2.2.5}$$

One can check, by easy calculations, that $\frac{v(x)}{\|v\|}$ is an eigenfunction of A associated with the eigenvalue $\lambda = 0$. Since $\frac{v}{\|v\|}$ has norm 1 and $v(x) > 0$ on $(-1,1)$, we have that

$$\exists\, k_* \in \mathbb{N}\, : \omega_{k_*}(x) = \frac{v(x)}{\|v\|} > 0 \ \text{ or } \ \omega_{k_*}(x) = -\frac{v(x)}{\|v\|} < 0, \ \forall x \in (-1,1)\,. \tag{2.2.6}$$

Writing (2.2.5) with $k = k_*$ we obtain

$$\langle \omega_{k_*}, \omega_h \rangle = \int_{-1}^{1} \omega_{k_*}(x)\omega_h(x)dx = 0, \qquad \forall h \neq k_*\,. \tag{2.2.7}$$

for almost $t \in [0,T]$ (see [2]).

Theorem 2.2.3. *For every $\alpha \in L^\infty(-1,1)$([1]) and every $u_0 \in L^2(-1,1)$, there exists a unique weak solution $u \in \mathcal{B}(Q_T)$ to (**??**), which coincides with $e^{tA}u_0$.*

2.2.2 Sturm-Liouville systems.

Let $A = A_0 + \alpha I$, where the operator A_0 is defined in (2.2.1) and $\alpha \in L^\infty(-1,1)$. Since A is self-adjoint and $D(A) \hookrightarrow L^2(-1,1)$ is compact (see Proposition 2.2.2), we have the following (see also [6]).

Lemma 2.2.4. *There exists an increasing sequence with $\{\lambda_k\}_{k\in\mathbb{N}}$, $\lambda_k \longrightarrow +\infty$ as $k \to \infty$, such that the eigenvalues of A are given by $\{-\lambda_k\}_{k\in\mathbb{N}}$, and the corresponding eigenfunctions $\{\omega_k\}_{k\in\mathbb{N}}$ form a complete orthonormal system in $L^2(-1,1)$.*

In this thesis we obtain the following result (see also [12]).

Lemma 2.2.5. *Let A be the operator defined in (2.2.2) with $\alpha = \alpha_*$*

$$\begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha_* I \ , \end{cases} \tag{2.2.4}$$

and let $\{\lambda_k\}, \{\omega_k\}$ be the eigenvalues and eigenfunctions of A, respectively, given by Lemma 2.2.4. Let $v \in D(A)$ be such that $v > 0$ on $(-1,1)$, and $\alpha_*(x) = -\frac{(a(x)v_x(x))_x}{v(x)} \in$

[1] *By repeated applications of Theorem 2.2.3, one can obtain an existence and uniqueness result when α is piecewise static ($\alpha(\cdot,x)$ piecewise constant in t, and $\alpha(t,\cdot) \in L^\infty(-1,1), \forall t \in (0,T)$). The same result holds for $\alpha \in L^\infty(Q_T)$, but for the purposes of the present thesis the piecewise static case will suffice.*

where $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R},\ \beta_0^2 + \beta_1^2 > 0,\ \gamma_0^2 + \gamma_1^2 > 0$, satisfy the sign condition

$$\beta_0 \beta_1 \leq 0 \text{ and } \gamma_0 \gamma_1 \geq 0.$$

Observe that A_0 is a closed, self-adjoint, dissipative operator with dense domain in $L^2(-1,1)$. Therefore, A_0 is the infinitesimal generator of a C_0- semigroup of contractions in $L^2(-1,1)$.

Next, given $\alpha \in L^\infty(-1,1)$, let us introduce the operator

$$\begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha I \ . \end{cases} \tag{2.2.2}$$

For such an operator we have the following proposition.

Proposition 2.2.2. *$D(A)$ is compactly embedded and dense in $L^2(-1,1)$.*

$A : D(A) \longrightarrow L^2(-1,1)$ is the infinitesimal generator of a strongly continuous semigroup, e^{tA}, of bounded linear operators on $L^2(-1,1)$.

Observe that problem (**??**) can be recast in the Hilbert space $L^2(-1,1)$ as

$$\begin{cases} u'(t) = A\,u(t)\,, \qquad t > 0 \\ \\ u(0) = u_0 \ . \end{cases} \tag{2.2.3}$$

where A is the operator in (2.2.2).

We recall that a *weak solution* of (2.2.3) is a function $u \in C^0([0,T]; L^2(-1,1))$ such that for every $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle \,,$$

and

$$H_a^2(-1,1) := \{u \in H_a^1(-1,1) |\, au_x \in H^1(-1,1)\},$$

respectively with the following norms

$$\|u\|_{1,a}^2 := \|u\|_{L^2(-1,1)}^2 + |u|_{1,a}^2 \text{ and } \|u\|_{2,a}^2 := \|u\|_{1,a}^2 + \|(au_x)_x\|_{L^2(-1,1)}^2,$$

where $|u|_{1,a} := \|\sqrt{a}u_x\|_{L^2(-1,1)}$ is a seminorm.

In this paper we consider the following space

$$\mathcal{B}(Q_T) = C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H_a^1(-1,1)),$$

where let us define the following norm

$$\|u\|_{\mathcal{B}(Q_T)}^2 := \sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^2(-1,1)}^2 + 2\int_0^T \int_{-1}^1 a(x)u_x^2 dx\, dt,\ \forall u \in \mathcal{B}(Q_T)\,.$$

In [1] the following result is obtained.

Lemma 2.2.1. $H_a^1(-1,1) \hookrightarrow L^2(-1,1)$ *with compact embedding.*

A similar result is obtained in [11], in cooperation with P. Cannarsa, in the strongly degenerate case (see also Section 2.3).

We now recall the existence and uniqueness result for system (**??**) obtained in [9] (see also [1]). Let us consider, first, the operator $(A_0, D(A_0))$ defined by

$$\begin{cases} D(A_0) = \left\{ u \in H_a^2(-1,1) \,\middle|\, \begin{cases} \beta_0 u(-1) + \beta_1 a(-1) u_x(-1) = 0 \\ \gamma_0\, u(1) \,+\, \gamma_1\, a(1)\, u_x(1) = 0 \end{cases} \right\} \\ \\ A_0 u = (au_x)_x\,, \ \ \forall\, u \in D(A_0)\,, \end{cases} \tag{2.2.1}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$.

Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

for all $0 \leq t \leq T$.

In particular, if $\psi(t) \equiv 0$ in (2.1.1), i.e. $\eta' \leq \phi\,\eta$ for a.e. $t \in [0,T]$, and $\eta(0) = 0$, then

$$\eta \equiv 0 \qquad \textit{in } [0,T].$$

2.2 Well-posedness in weighted Sobolev spaces: weakly degenerate case

In order to deal with the well-posedness of problem (**??**), it is necessary to introduce the weighted Sobolev spaces $H^1_a(-1,1)$ and $H^2_a(-1,1)$.

2.2.1 Weighted Sobolev spaces

Let us consider the function $a \in C^0([-1,1]) \cap C^1(-1,1)$ such that $a(\cdot)$ fulfills the following properties

$a(x) > 0\ \forall\, x \in (-1,1), \quad a(-1) = a(1) = 0,$

$\frac{1}{a} \in L^1(-1,1).$

Let us define the following weighted Sobolev spaces

$$H^1_a(-1,1) := \{u \in L^2(-1,1) : u \text{ absolutely continuous in } [-1,1],$$
$$\sqrt{a}u_x \in L^2(-1,1)\}$$

and the negative-part function

$$v^-(x) = \max(0, -v(x)), \qquad\qquad \forall x \in \Omega\,.$$

Then we have the following equality

$$v = v^+ - v^- \qquad\qquad \text{in } \Omega\,.$$

For the functions v^+ and v^- the following result of regularity in Sobolev's spaces will be useful (see [33], Appendix A).

Theorem 2.1.1. *Let* $\Omega \subset \mathbb{R}^n$, $u : \Omega \longrightarrow \mathbb{R}$, $u \in H^{1,s}(\Omega)$, $1 \leq s \leq \infty$. *Then*

$$u^+,\, u^- \in H^{1,s}(\Omega)$$

and for $1 \leq i \leq n$

$$(u^+)_{x_i} = \begin{cases} u_{x_i} & \textit{in } \{x \in \Omega : u(x) > 0\} \\ \\ 0 & \textit{in } \{x \in \Omega : u(x) \leq 0\}\ , \end{cases}$$

and

$$(u^-)_{x_i} = \begin{cases} -u_{x_i} & \textit{in } \{x \in \Omega : u(x) < 0\} \\ \\ 0 & \textit{in } \{x \in \Omega : u(x) \geq 0\}\ . \end{cases}$$

2.1.2 Gronwall's inequalities

We look at the differential form of Gronwall's inequality (see [21]).

Lemma 2.1.2. *(Gronwall's inequality: differential form). Let* $\eta(t)$ *be a nonnegative, absolutely continuous function on* $[0, T]$*, which satisfies for a.e.* $t \in [0, T]$ *the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \tag{2.1.1}$$

Chapter 2

Well-posedness for linear problems

In this chapter we start by defining the weighted Sobolev spaces $H^1_a(-1,1)$ and $H^2_a(-1,1)$, then we give the proof of some results, obtained in collaboration with P. Cannarsa, that will be useful in the following chapters. In particular, we obtain several results for regular and singular Sturm-Liouville systems (see also [12] and [11]). The main results of this chapter is to study the well-posedness for linear systems weakly degenerate (Section 2.2) and strongly degenerate (Section 2.3).

2.1 Preliminaries

We start by introducing the positive and negative part and by recalling a result which deals with their regularity.

2.1.1 Positive and negative part

Given $\Omega \subseteq \mathbb{R}^n$, $v : \Omega \longrightarrow \mathbb{R}$ we consider the positive-part function

$$v^+(x) = \max\left(v(x), 0\right), \qquad \forall x \in \Omega\,,$$

Once well-posedness is established in Chapter 4, we turn to the analysis of the approximate controllability of (1.0.1) via bilinear controls. We show that any initial state $u_0 \in H^1_a(-1,1)$, can be steered in a sufficiently large time into any nonnegative neighborhood of any nonnegative target state $u_d \in H^1_a(-1,1)$ satisfying the following

$$\langle u_0, u_d \rangle_{1,a} > 0\,.$$

The main technical difficulty to overcome with respect to the uniformly parabolic case treated in [29], is the fact that functions in $H^1_a(-1,1)$ need not be necessarily bounded when the operator is strongly degenerate.

Moreover, unlike [29] where specific growth bounds were assumed for f, here we are interested in studying general polynomial nonlinearities (see (1.0.2) and (1.0.3)), under the sign condition (1.0.4), in order to be able to cover not only Budyko's model but also Sellers's. The way we propose in the thesis to make the above program work consists in taking initial and target states little more regular than in [29], that is, in $H^1_a(-1,1)$. Although in this thesis we propose a complete solution of the approximate controllability problem for (1.0.1), we believe that our methodology could be extended other interesting related questions. For instance as mentioned above, we would to derive similar results for semilinear weakly degenerate control systems. Moreover, in the future we intend to investigate problems in higher space dimensions on domains with specific geometries.

Finally, once the above two issues have been addressed, we would like to extend our approach to other nonlinear systems of parabolic type, such as the equations of fluid dynamics.

can be steered, in the space of square-summable functions, from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear static controls. Moreover, we extend the above result relaxing the sign constraint on the initial-state.

On the other hand, in the SD case (Section 2.3 and Section 4.2) one is forced to restrict to the Neumann type boundary conditions (as in the Budyko-Sellers model)

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in} \quad Q_T \;=\; (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm 1} = 0 & t\in(0,T) \\ v(0,x) = v_0(x) & x\in(-1,1)\;. \end{cases} \tag{1.2.1}$$

Even in this case, we establish the global approximate multiplicative controllability in $L^2(-1,1)$ (Section 4.2), after proving the compact embedding in $L^2(-1,1)$ of the weighted Sobolev space $H^1_a(-1,1)$ under the assumption $\xi_a \in L^1(-1,1)$, where $\xi_a(x) = \int_0^x \frac{ds}{a(s)}$.

The nonlinear problem (1.0.1) is treated in Chapter 3 (Well-posedness) and Chapter 5 (Controllability) of this thesis.

For brevity, we focus just on strongly degenerate problems, thus including the Budyko-Sellers model, but we are confident that this approach also applies to semilinear weakly degenerate equations with general Robin type boundary conditions. We will consider such generalizations in future works.

We begin by establishing the existence and uniqueness of solutions to (1.0.1). We follow the classical method which consists in obtaining a local result by fixed point arguments, and then show that the solution in global in time by proving an a priori estimate.

1.2 Structure and contents

In the first part of this thesis we study the approximate controllability of (1.0.1) by bilinear controls.

First, we consider the linear problem (i.e., when $f \equiv 0$) in two distinct kinds of set-up (In Chapter 2 the *Well-posedness* and in Chapter 4 the *Controllability*), namely

- weakly degenerate problems (WD), that is, when $\frac{1}{a} \in L^1(-1,1)$, (Section 2.2 and Section 4.1)
- strongly degenerate problems (SD), that is, when $\frac{1}{a} \notin L^1(-1,1)$. (Section 2.3 and Section 4.2)

Observe that the Budyko-Sellers model is an example of SD operator.

The WD case is somewhat similar to the uniformly parabolic case. Indeed, it turns out that all functions in the domain of the corresponding differential operator possess a trace on the boundary, in spite of the fact that the operator degenerates at such points. Indeed, in the WD case we are able to study the equation

$$v_t - (a(x)v_x)_x = \alpha(t,x)v \qquad \text{in} \quad Q_T \; = \; (0,T) \times (-1,1)\,,$$

with general Robin boundary conditions

$$\begin{cases} \beta_0 v(t,-1) + \beta_1 a(-1) v_x(t,-1) = 0 & t \in (0,T) \\ \gamma_0\, v(t,1) \; + \; \gamma_1\, a(1)\, v_x(t,1) = 0 & t \in (0,T)\,. \end{cases}$$

For this Cauchy-Robin problem we obtain an result of global approximate multiplicative controllability in $L^2(-1,1)$ (Section 4.1). Indeed we show that the above system

Additive control problems for the Budyko-Sellers model have been studied by J.I.Diaz, in the work [18]. Even in Budyko-Sellers model, modeling the control action by an additive term would require huge amounts of energy, which may not always be realistic to afford. On the other hand, one could imagine to influence the so-called *albedo* by some kind of device as predicted by J. Von Neumann

"*Microscopic layers of colored matter spread on an icy surface, or in the atmosphere above one, could inhibit the reflection-radiation process, melt the ice and change the local climate.*" (J. von Neumann, *Nature*, 1955)

and

"*Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present*" (J. von Neumann, *Nature*, 1955) .

From the mathematical view point such a control action would take the form of a bilinear control, that is, a control given by a multiplicative coefficient.
This explains the growing interest in *multiplicative controllability.* General references for *multiplicative controllability* are, e.g., [27], [28], [30], [31], [32], [3].
Our approach is inspired by [29] and [13]. In [29] A.Y. Khapalov studied the global nonnegative approximate controllability of the one dimensional *non-degenerate* semilinear convection-diffusion-reaction equation governed in a bounded domain via the bilinear control $\alpha \in L^\infty(Q_T)$. In [13] P. Cannarsa and A.Y. Khapalov derived the same approximate controllability property in suitable classes of functions that change sign.

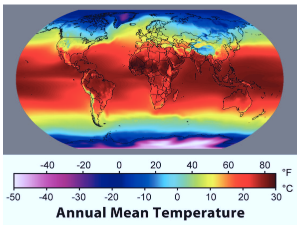

Figure 1.3: www.globalwarmingart.com (copyrighted by Global Warming Art)

In the one-dimensional Budyko-Sellers we take the average of the temperature at $x = \cos\phi$. In such a model, the sea level mean zonally averaged temperature $u(t,x)$ on the Earth, where t denotes time satisfies the following degenerate *Cauchy-Neumann* problem (1.1.1) in the bounded domain $(-1,1)$

$$\begin{cases} u_t - \big((1-x^2)u_x\big)_x = g(t,x)\, h(x,u) + f(t,x), \qquad x \in (-1,1) \\ (1-x^2)u_{x|x=\pm1} = 0 \ . \end{cases} \tag{1.1.1}$$

Observe that the leading part of the differential operator in (1.1.1) satisfies assumptions $(A.4)$.

1.1.2 Mathematical motivations

In Control theory, boundary and interior locally distributed controls are usually employed (see, e.g., [14], [15], [16], [22], [23], [25]), [4] and [5]. These controls are additive terms in the equation and have localized support.
However, such models are unfit to study several interesting applied problems such as chemical reactions controlled by catalysts, and also smart materials, which are able to change their principal parameters under certain conditions.

- Budyko

$$\beta(u) = \begin{cases} \beta_0 & u < -10 \\ [\beta_0, \beta_1] & u = -10 \\ \beta_1 & u > -10 \ , \end{cases}$$

- Sellers

$$\beta(u) = \begin{cases} \beta_0 & u < u_- \\ \text{line} & u_- \leq u \leq u_+ \\ \beta_1 & u > u_+ \ , \end{cases}$$

where $u_\pm = -10 \pm \delta, \delta > 0$.

Moreover, in Budyko we have

$$R_e(t, X, u) = A(t, X) + B(t, X)u,$$

while in Sellers

$$R_e(t, X, u) \simeq C\, u^4 \,.$$

On $\mathcal{M} = \Sigma^2$ the Laplace-Beltrami operator is

$$\Delta_{\mathcal{M}} = \frac{1}{\sin\phi}\Big\{\frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial u}{\partial\phi}\Big) + \frac{1}{\sin\phi}\,\frac{\partial^2 u}{\partial\lambda^2}\Big\}$$

where ϕ is the *colatitude* and λ is the *longitude*.

1-*albedo* function).

Albedo is the reflecting power of a surface. It is defined as the ratio of reflected radiation from the surface to incident radiation upon it. It may also be expressed as a percentage, and is measured on a scale from zero for no reflecting power of a perfectly black surface, to 1 for perfect reflection of a white surface.

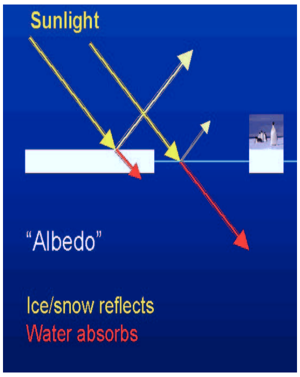

Figure 1.2: www.esr.org (copyrighted by ESR)

The main difference between Budyko's model and the one by Sellers, is that in the former the coalbedo function is discontinuous, while in the latter it is a continuous function. In fact we have

The effect of solar radiation on climate can be summarized in the following figure

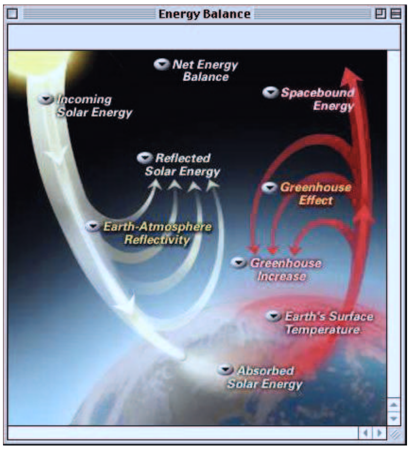

Figure 1.1: www.edu-design-principles.org (copyrighted by DPD)

We have the following *energy balance* :

Heat variation $= R_a - R_e + D$

- R_a = absorbed energy
- R_e = emitted energy
- D = diffusion

The general formulation of the Budyko-Sellers model on a compact surface $\mathcal{M}$ without boundary is as follows

$$u_t - \Delta_{\mathcal{M}} u = R_a(t, X, u) - R_e(t, X, u)$$

where $u(t, X)$ is the distribution of temperature and $\Delta_{\mathcal{M}}$ is the classical Laplace-Beltrami operator. Moreover,

$$R_a(t, X, u) = Q(t, X)\beta(X, u).$$

In the above, Q is the *insolation* function, and β is the *coalbedo* function (that is,

1.1 Motivations

1.1.1 Physical motivations: Climate models and degenerate parabolic equations

Climate depends on various parameters such as temperature, humidity, wind intensity, the effect of greenhouse gases, and so on. It is also affected by a complex set of interactions in the atmosphere, oceans and continents, that involve physical, chemical, geological and biological processes.

One of the first attempts to model the effects of interaction between large ice masses and solar radiation on climate is the one due, independently, by Budyko [7, 8] and Sellers [37] (see also [17, 18, 19], [20], [26] and the references therein). Such a model studies how extensive the climate response is to an event such as a sharp increase in greenhouse gases; in this case we talk about climate sensitivity. A process that changes climate sensitivity is called *feedback*. If the process increases the intensity of response we say that it has *positive feedback*, whereas it has *negative feedback* if it reduces the intensity of response.

The Budyko-Sellers model studies the role played by continental and oceanic areas of ice on climate change.

(A.3) $f : (-1,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

there exist $\vartheta > 1$, $\gamma_0 > 0$, and $\gamma_1 > 0$ such that

$$|f(x,u)| \leq \gamma_0 \, |u|^{\vartheta}, \text{ for a.e. } x \in (-1,1), \forall u \in \mathbb{R}\,, \tag{1.0.2}$$

and

$$\begin{aligned}&|f(x,u) - f(x,v)| \\ &\quad \leq \gamma_1 \left(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1}\right) |u - v|, \text{ for a.e. } x \in (-1,1),\ \forall u, v \in \mathbb{R};\end{aligned} \tag{1.0.3}$$

there exists a nonnegative constant ν such that

$$f(x,u)\, u \leq \nu\, u^2, \qquad \text{for a.e. } x \in (-1,1), \quad \forall u \in \mathbb{R}\,; \tag{1.0.4}$$

(A.4) $a \in C^1([-1,1])$ is such that

$$a(x) > 0 \;\forall\, x \in (-1,1), \quad a(-1) = a(1) = 0,$$

and, the function $\xi_a(x) = \int_0^x \frac{ds}{a(s)}$ satisfies the following

$$\xi_a \in L^{2\vartheta-1}(-1,1).$$

The equation in the *Cauchy-Neumann* problem (1.0.1) is a degenerate parabolic equation because the diffusion coefficient, positive on $(-1,1)$, is allowed to vanish at the extreme points of $[-1,1]$.
Interest in degenerate parabolic equation dates back by almost a century. Significant contributions are due to G. Fichera (see e.g. [24]) and Oleinik (see e.g. [35]).
The main physical motivations for studying degenerate parabolic problems with the structure described above come from mathematical model in climate science as we explain below.

Chapter 1

Introduction to PART 1: Approximate multiplicative controllability for degenerate parabolic problems

This thesis is concerned with the analysis of linear and semilinear parabolic control systems in one space dimension, governed in the bounded domain $(-1,1)$ by means of the *bilinear control* $\alpha(t,x)$, of the form

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(t,x)u + f(x,u) & \text{in } \; Q_T \; := \; (0,T)\times(-1,1) \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ u(0,x) = u_0(x) & x \in (-1,1) \; . \end{cases} \tag{1.0.1}$$

under the following assumptions:

(A.1) $u_0 \in H^1_a(-1,1) := \{u \in L^2(-1,1) : u$ locally absolutely continuous in $(-1,1)$, $\sqrt{a}u_x \in L^2(-1,1)\}$;

(A.2) $\alpha \in L^\infty(Q_T)$;

Introduction

Paris, France
November 25, 2011

Giuseppe Floridia

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Preface

This thesis consists of two parts, both related to the theory of parabolic equations and systems. The first part is devoted to control theory which studies the possibility of influencing the evolution of a given system by an external action called control. Here we address approximate controllability problems via multiplicative controls, motivated by our interest in some differential models for the study of climatology.
In the second part of the thesis we address regularity issues on the local differentiability and Hölder regularity for weak solutions of nonlinear systems in divergence form.
In order to improve readability, the two parts have been organized as completely independent chapters, with two separate introductions and bibliographies.
All the new results of this thesis have been presented at conferences and workshops, and most of them appeared or are to appear as research articles in international journals. Related directions for future research are also outlined in body of the work.

Rome, Italy — Giuseppe Floridia
December 8, 2011

List of Figures

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To my Master,

Professor Piermarco Cannarsa,

for all the time that he dedicated to my beginnings

in scientific research in Mathematical Analysis.

To my father,

who gave me the passion for mathematics

and for logical deductive reasoning.

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Author: **Giuseppe Floridia**

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Signature of Author

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The undersigned hereby certify that they have read and recommend to the University of Catania for acceptance a thesis entitled "**Approximate multiplicative controllability for degenerate parabolic problems and regularity properties of elliptic and parabolic systems**" by **Giuseppe Floridia** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Dated: December 2011

Coordinator: ______________________
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APPROXIMATE MULTIPLICATIVE CONTROLLABILITY FOR DEGENERATE PARABOLIC PROBLEMS AND REGULARITY PROPERTIES OF ELLIPTIC AND PARABOLIC SYSTEMS

Giuseppe Floridia

UNIVERSITÀ DEGLI STUDI DI CATANIA

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

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