Graph designs

PHD thesis of **Giorgio Ragusa**
Supervisor: Prof. **Mario Gionfriddo**
University of Catania

Chapter 1

Introduction and applications

1.1 Definitions

The modern study of block designs is often said to have begun with the publication in 1936 of a paper by the statistician F. Yates. In that paper he considered collections of subsets of a set with certain balance properties, that are now known as *balanced incomplete block designs (BIBD).*Using k-subset as an abbreviation for k-element subset, we make the definition:

Definition 1. *A* $(v, k, \lambda)-$*BIBD* $(S, \mathcal{B})$ *is a collection of* k*-subsets called blocks of a* v*-set* S*,* $k < v$*, such that each pair of elements of* S *occur together in exactly* λ *of the blocks.*

Definition 2. *A finite projective plane of order* $n > 1$ *is a* $(n^2 + n + 1, n + 1, 1)-$*BIBD.*

Definition 3. *A finite affine plane of order* $n > 1$ *is a* $(n^2, n, 1)-$*BIBD.*

Definition 4. *A Triple System of order* v *and index* λ*,* $TS(v, \lambda)$*, is a* $(v, 3, \lambda)-$*BIBD. A triple system of index 1 is called a Steiner Tripe system,* $STS(v)$*.*

Definition 5. *A Quadruple System of order* v *and index* λ*,* $S_\lambda(2, 4, v)$*, is a* $(v, 4, \lambda)-$*BIBD. A quadruple system of index 1 is called a Steiner Quadruple System,* $SQS(v)$*.*

Figure 1.1 shows the minimum projective plane, an $STS(7)$ called the Fano's Plane.

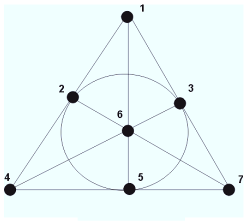


Figure 1.1: The Fano's plane

Definition 6. *A BIBD is resolvable if the blocks can be arranged into r groups so that the blocks of each group are disjoint and contain in their union each element exactly once. The groups are called the resolution classes.*

It is not difficult to prove that every affine plane of order n is resolvable with $n+1$ resolution classes. For every finite affine plane of order n $(A, \mathcal{R})$ there exists a finite projective plane of order n $(P, \mathcal{R}')$ and an injective function $f : A \to P$ such that for all $R \in \mathcal{R}$ there exists a $R' \in \mathcal{R}'$ with $f(R) \subseteq R'$. For this it is sufficient to adjoin to every block of a resolution class C_i a new point ∞_i and the new block $\{\infty_1, \infty_2, \dots, \infty_{n+1}\}$. We say that $(A, \mathcal{R})$ is *embedded* into $(P, \mathcal{R}')$.

In the last part of the previous century a new approch of design theory appeared: the designs are considered as decompositions of graphs. This approch is more usefully for applications where the position of the elements is important, as we show in the next section.

Denote by

- $H = (V(H), E(H))$ a graph having vertex set $V(H)$ and edge-set $E(H)$;
- λH the graph H in which every edge has multiplicity λ.
- K_n the complete undirected graph on n vertices;
- G a subgraph of K_n having nonisolated vertices;

Definition 7. *A* G-decomposition *of* λH *(or a* $(\lambda H, G)$-decomposition*) is a partition of the edges of* λH *into subgraphs (*G-blocks*) each of which is isomorphic to* G.

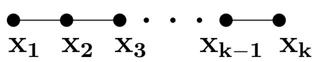
A $(\lambda H, G)$-decomposition is denoted by $(V, \mathcal{C})$, where $V = V(H)$ is the vertex set of λH, and $\mathcal{C}$ is the G-block-set.

A $(\lambda K_n, G)$-decomposition is called a *G-design of order n and index λ*.

A G-design of order v and index λ is called a

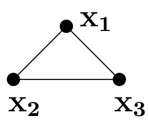
- $(v, k, \lambda)-$BIBD if $G = K_k$;
- path design $P(v, k, \lambda)$ if $G = P_k$, the path of length $k-1$ (k vertices);
- m-cycle system if $G = C_m$, the cycle of length m;
- E_2-design if $G = E_2$, the graph with four vertices and two disjoint edges;
- a Kite System $KS(v, \lambda)$ if $G = K_3 + e$, the simple graph on 4 vertices consisting of a triangle and a single edge (tail) sharing one common vertex.

The following figures show some graphs.



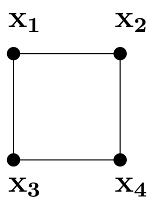
$\mathbf{P_k = [x_1, x_2, \ldots, x_k]}$

(path of length k − 1)

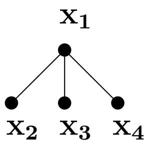


$\mathbf{K_3 = (x_1, x_2, x_3)}$

(complete graph on 3 vertices)

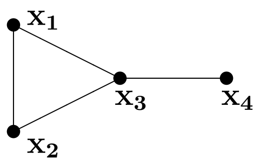


$\mathbf{C_4 = (x_1, x_2, x_3, x_4)}$ **(4-cycle)**



$\mathbf{K_{1,3} = [x_1; x_2, x_3, x_4]}$

(star with 3 pendant vertices)



$\mathbf{D = (x_1, x_2, x_3) - x_4}$

($\mathbf{K_3 + e}$ or kite)

Figure 1.2 shows a K_3-decomposition of K_7, i.e. an $STS(7)$ isomorphic to the Fano Plane.

It is well-known [5, 6, 7, 8, 37, 38, 74] that:

1. a $TS(n, \lambda)$ exists if and only if $\lambda(n-1) \equiv 0 \pmod 2$ and $\lambda n(n-1) \equiv 0 \pmod 6$;

2. an $S_\lambda(2, 4, n)$ exists if and only if $\lambda n(n-1) \equiv 0 \pmod{12}$ and $\lambda(n-1) \equiv 0 \pmod 3$;

3. a λ-fold C_4-system of order n exists if and only if $\lambda n(n-1) \equiv 0 \pmod 8$ and $\lambda(n-1) \equiv 0 \pmod 2$;

4. a λ-fold kite-system of order n if and only if $\lambda n(n-1) \equiv 0 \pmod 8$.

If $G = K_s$ and H is a complete multipartite graph with h_1 parts of size g_1, h_2 parts of size g_2, ... h_r parts of size g_r, an G-decomposition of H is well known as an $s - GDD$ of type $g_1^{h_1} g_2^{h_2} \dots g_r^{h_r}$. Figure 1.3 shows a $K_{2,2,3}$, i. e. a complete multipartite graph with 2 parts of size 2 and 1 part of size 3. Trivially a $3 - GDD$ of type $2^2 3^1$ can't exist because the multipartite graph has 16 edges and $16 \equiv 1 \pmod 3$.

Definition 8. *We say that a G-design $(W, \mathcal{B})$ is a subdesign of $(V, \mathcal{C})$ if $W \subseteq V$ and $\mathcal{B} \subseteq \mathcal{C}$.*

Definition 9. *A $(\lambda H, G)$-decomposition $(V, \mathcal{C})$ is balanced if each vertex belongs to the same number of blocks. An H_s-design is a balanced P_s-design.*

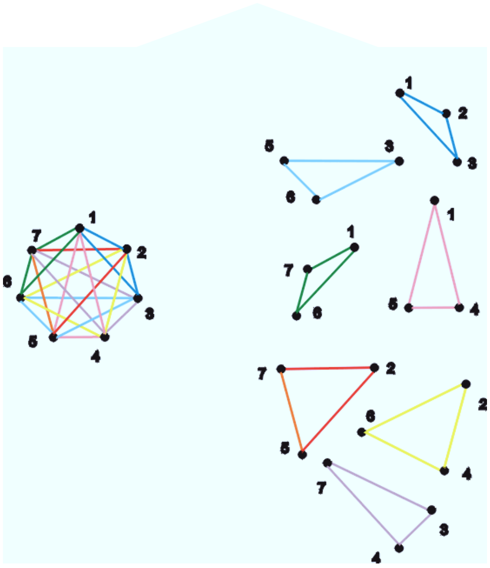


Figure 1.2: An K_3-decomposition of K_7

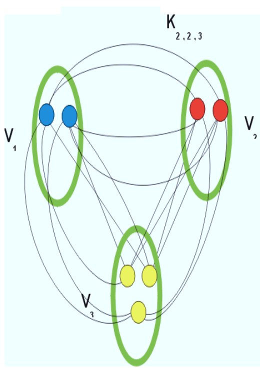


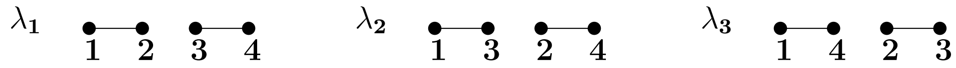
Figure 1.3: A $K_{2,2,3}$

A *packing* of λH with copies of G is a triple $(X, \mathcal{B}, L)$, where X is the vertex set of H, $\mathcal{B}$ is a collection of copies of G from the edge set of λH and L is the graph generated by the set of edges of λH not belonging to a graph of $\mathcal{B}$. The graph L is called the *leave*. If $|\mathcal{B}|$ is as large as possible, the packing $(X, \mathcal{B}, L)$ is said to be maximum ([49]). When the leave L is empty, a maximum packing of λH with copies of G coincides with a λ-fold G-decomposition of λH.

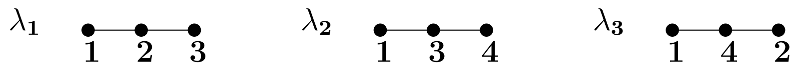
1.2 An application: the networks

Traffic grooming is the generic term for packing low rate signals into higher speed streams. By using traffic grooming, one can bypass the electronics in the nodes which are not sources or destinations of traffic, and therefore reduce the cost of the network. When we consider unidirectional SONET/WDM ring networks, the routing is unique and we have to assign to each request between two nodes a wavelength and some bandwidth on this wavelength. If the traffic is uniform and if a given wavelength can carry at most C requests, we can assign to each request at most 1C of the bandwidth. C is known as the *grooming ratio* or the *grooming factor*. Furthermore if the traffic requirement is symmetric, it can be easily shown (by exchanging wavelengths) that there always exists an optimal solution in which the same wavelength is given to each pair of symmetric requests. Thus without loss of generality we assign to each pair of symmetric requests, called a circle, the same wavelength. Then each circle uses 1C of the bandwidth in the whole ring. If the two end-nodes of a circle are i and j, we need one ADM at node i and one at node j. The main point is that if two requests have a common end-node, they can share an ADM if they are assigned the same wavelength. For example, suppose that we have symmetric requests between nodes 1 and 2, and also between 2 and 3. If they are assigned two different wavelengths, then we need 4 ADMs, whereas if they are assigned the same wavelength we need only 3 ADMs. The so called *traffic grooming problem* consists in minimizing the total number of ADMs to be used, in order to reduce the overall cost of the network. Suppose we have a ring with 4 nodes $0, 1, 2, 3$ and all-to-all uniform traffic. There are therefore 6 circles (pairs of symmetric requests) $\{i, j\}$ for $0 \leq i < j \leq 3$. If there is no grooming we need 6 wavelengths (one per circle) and a total of 12 ADMs. If we have a grooming factor C = 2, we can put on the same wavelength two circles, using 3 (assignement 1) or 4 (assignment 2) ADMs according to whether they share an end-node or not. For example we can put together $\{1,2\}$ and $\{2,3\}$ on one wavelength; $\{1,3\}$ and $\{3,4\}$ on a second wavelength, and $\{1,4\}$ and $\{2,4\}$ on a third one, for a total of 9 ADMs.

In terms of design theory assignment 1 is a E_2 design of order 4:



whereas assignment 2 is a P_3 design of order 4:



The problem for a unidirectional SONET ring with n nodes, grooming ratio C, and all-to-all uniform unitary traffic has been modeled as a graph partition problem in both [3] and [34]. In the all-to-all case the set of requests is modelled by the complete graph K_n. To a wavelength k is associated a subgraph B_k in which each edge corresponds to a pair of symmetric requests (that is, a circle) and each node to an ADM. The grooming constraint, i.e. the fact that a wavelength can carry at most C requests, corresponds to the fact that the number of edges $|E(B_k)|$ of each subgraph B_k is at most C. The cost corresponds to the total number of vertices used in the subgraphs, and the objective is therefore to minimize this number.

TRAFFIC GROOMING IN THE RING
INPUT: Two integers n and C.
OUTPUT: Partition $E(K_n)$ into subgraphs B_k, $1 \leq k \leq s$, s.t. $|E(B_k)| \leq C$ for all k.
OBJECTIVE: Minimize $\Sigma_{k=1}^{s}|V(B_k)|$.

With the all-to-all set of requests, optimal constructions for a given grooming ratio C have been obtained using tools of graph and design theory, in particular for grooming ratio $C \leq 7$ and $C \geq N(N-1)/6$. For example, two different optimal networks with 8 nodes and $C = 4$ can be obtained by:

- a $(K_3 + e)$-design $(V, \mathcal{B})$, with $V = \{1, 2, \ldots, 8\}$ and $\mathcal{B} = \{(7, 2, 1) - 3,$

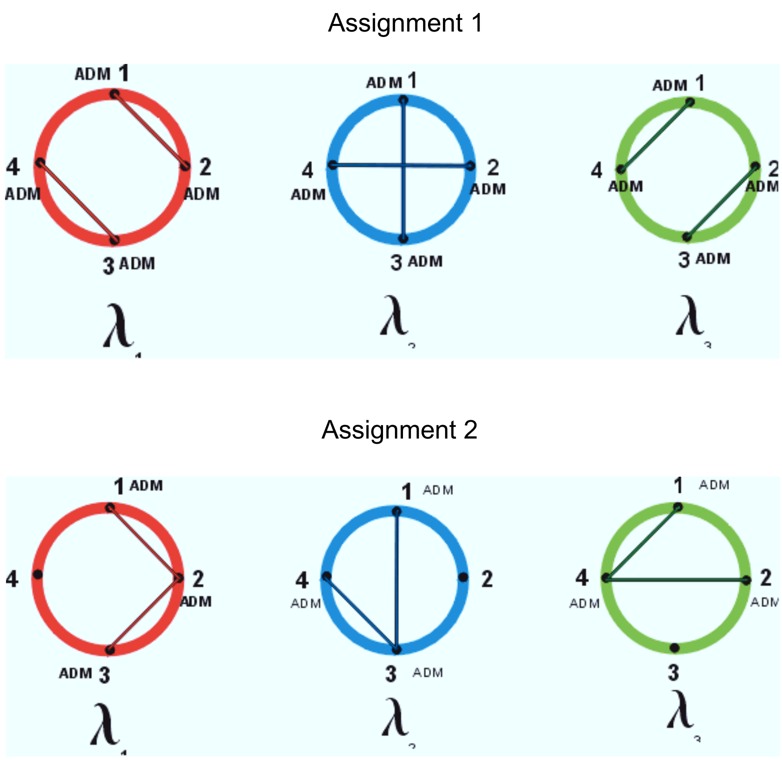
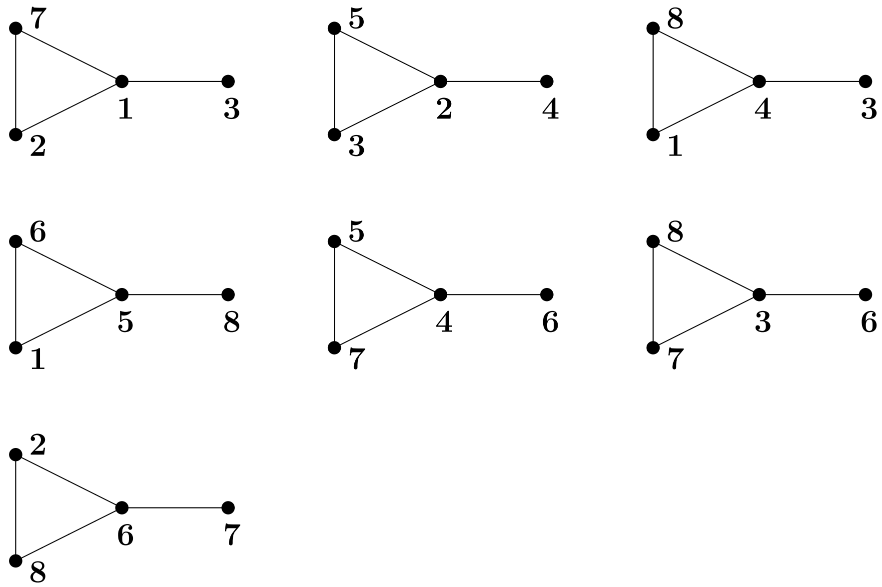
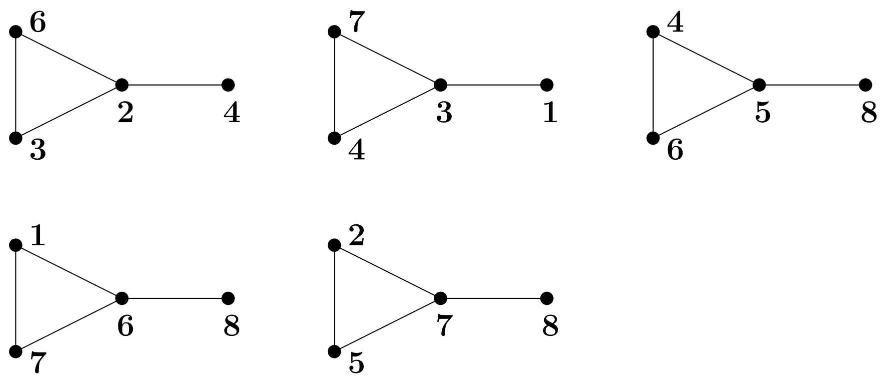


Figure 1.4: Two different assegnment

$(5,3,2)-4$, $(8,1,4)-3$, $(6,1,5)-8$, $(5,7,4)-6$, $(8,7,3)-6$, $(2,8,6)-7\}$.



- a $\{K_3+e, C_4\}$-decomposition $(V,\mathcal{B})$, with $V=\{1,2,\dots,8\}$ and $\mathcal{B}=\{(6,3,2)-4$, $(7,4,3)-1$, $(4,6,5)-8$, $(1,7,6)-8$, $(2,5,7)-8$, $(1,4,8,2),(1,5,3,8)\}$



Most of the papers on grooming deal with a single (static) traffic matrix. Some articles consider variable (dynamic) traffic, such as finding a solution which works for the maximum traffic demand or for all request graphs with a given maximum degree, but all keep a fixed grooming factor. In [24] an interesting variation of the traffic grooming problem, grooming for two-period optical networks, has been introduced in order to capture some dynamic nature of the traffic. Informally, in the two-period grooming problem each time period supports different traffic requirements. During the first period of time there is all-to-all uniform traffic among n nodes, each request using $1/C$ of the bandwidth; but during the second period there is all-to-all traffic only among a subset V of v nodes, each request now being allowed to use a larger fraction of the bandwidth, namely $1/C_0$ where $C_0 < C$. Denote by X the subset of n nodes. Therefore the two-period grooming problem can be expressed as follows:

TWO-PERIOD GROOMING IN THE RING
INPUT: Four integers n, v, C, C_0.
OUTPUT: A partition of $E(K_n)$ into subgraphs $B_k, 1 \leq k \leq s$, such that for all $k, |E(B_k)| \leq C$, and $|E(B_k) \cap (V \times V)| \leq C_0$, with $V \subseteq X$, $|V| = v$.
OBJECTIVE: Minimize $\Sigma_{k=1}^{s} |V(B_k)|$.

A grooming of a two-period network $N(n, v; C, C_0)$ with grooming ratios (C, C_0) coincides with a graph decomposition $(X, \mathcal{B})$ of K_n such that $(X, \mathcal{B})$ is a grooming $N(n, C)$ in the first time period, and $(X, \mathcal{B})$ embeds a graph decomposition of K_v such that $(V, \mathcal{D})$ is a grooming $N(v, C_0)$ in the second time period. In [4] this problem is solved for $C = 4$.

Chapter 2

Simultaneous metamorphoses of K_4–designs

2.1 Preliminaries

Definition 10. Let $(X, \mathcal{B})$ be a λ-fold G-decomposition of λH. Let G_i, $i = 1, \ldots, \mu$, be non isomorphic proper subgraphs of G, each without isolated vertices. Put $\mathcal{B}_i = \{B_i \mid B \in \mathcal{B}\}$, where B_i is a subgraph of B isomorphic to G_i. A $\{G_1, G_2, \ldots, G_\mu\}$-*metamorphosis* of $(X, \mathcal{B})$ is a rearrangement, for each $i = 1, \ldots, \mu$, of the edges of $\bigcup_{B \in \mathcal{B}}(E(B) \setminus E(B_i))$ into a family $\mathcal{B}'_i$ of copies of G_i with a leave L_i, such that $(X, \mathcal{B}_i \cup \mathcal{B}'_i, L_i)$ is a maximum packing of λH with copies of G_i.

For $\mu = 1$, the above definition coincides with the first definition of metamorphosis given by C. C. Lindner and A. Street in [53]. For this reason a $\{G_i, \ldots, G_\mu\}$-metamorphosis is a *simultaneous metamorphosis* introduced by P. Adams, E. Billington, E. S. Mahmoodian in [1].

In this chapter, we study the simultaneous metamorphosis of an $S_\lambda(2, 4, n)$ when it is $G = K_4, G_1 = C_4, G_2 = K_3 + e$. In the following we always denote the sets $\mathcal{B}'_1$, $\mathcal{B}'_2$, L_1, L_2 by $\mathcal{C}$, $\mathcal{K}$, $L_\mathcal{C}$ and $L_\mathcal{K}$, respectively.

Necessary and sufficient conditions for the existence of an $S_\lambda(2, 4, n)$ having a metamorphosis into a maximum packing of λK_n with 4-cycles (with kites) are given in [46] ([44]). See the following table, where $\emptyset$ denotes the empty graph.

$\lambda \pmod{12}$	$n \geq 4$	$L_{\mathcal{C}}$	$L_{\mathcal{K}}$
1,5,7,11	$1 \pmod{24}$	$\emptyset$	$\emptyset$
	$4 \pmod{24}$	1-factor	P_3 or, if $n > 4$, E_2
	$13 \pmod{24}$	C_6 or 2 K_3s	P_3 or E_2
	$16 \pmod{24}$	1-factor	$\emptyset$
2,10	$1, 4 \pmod{12}$	$\emptyset$	$\emptyset$
	$7, 10 \pmod{12}$	$2P_2$	P_3 or $2P_2$ or E_2
3,9	$1 \pmod{8}$	$\emptyset$	$\emptyset$
	$0 \pmod{8}$	1-factor	$\emptyset$
	$4 \pmod{8}$	1-factor	P_3 or $2P_2$ or E_2
	$5 \pmod{8}$	$2P_2$	P_3 or $2P_2$ or E_2
4,8	$1 \pmod{3}$	$\emptyset$	$\emptyset$
6	$0, 1 \pmod{4}$	$\emptyset$	$\emptyset$
	$2, 3 \pmod{4}$	$2P_2$	P_3 or $2P_2$ or E_2
0	$\forall n \geq 4$	$\emptyset$	$\emptyset$

Pairing [44] and [46] it is easy to check that in some cases C_4-metamorphoses and $(K_3 + e)$-metamorphoses follow from a *same* starting $S_\lambda(2, 4, n)$. Collecting these results we get our first result.

Theorem 2.1.1. *[44, 46] If $\lambda = 1$ and $n \equiv 4, 13 \pmod{24}$, $\lambda = 2$ and $n = 7, 10, 19$, $\lambda = 3$ and $n \equiv 4, 5 \pmod{8}$, $\lambda = 6$ and $n \equiv 2, 3 \pmod{4}$, then there exists an $S_\lambda(2, 4, n)$ having a $\{C_4, K_3 + e\}$-metamorphosis.*

Theorem 2.1.2. *[Weighting construction] . Suppose there exist:*

1. *an $\{r, s\}$-GDD of type $g_1^{u_1} g_2^{u_2} \ldots g_h^{u_h}$;*
2. *an $S_\lambda(2, 4, \alpha + wg_i), i = 1, \ldots, h$, with $\alpha = 0, 1$, having a $\{C_4, K_3 + e\}$-metamorphosis;*
3. *a 4-GDD of index λ and type w^r, having a $\{C_4, K_3 + e\}$-metamorphosis;*
4. *a 4-GDD of index λ and type w^s, having a $\{C_4, K_3 + e\}$-metamorphosis.*

Then there is an $S_\lambda(2, 4, w(g_1u_1 + \ldots + g_hu_h) + \alpha)$ having a $\{C_4, K_3 + e\}$-metamorphosis.

Proof The proof follows easily from the well-known Wilson fundamental construction [11]. □

2.2 $\lambda = 1$

Lemma 2.2.1. *There exists a 4-GDD of type* $(2t)^4$, *with* $t \geq 2, t \neq 3$, *having a* $\{C_4, K_3 + e\}$*-metamorphosis.*

Proof For $t \geq 2, t \neq 3$, let $X = \mathbb{Z}_{2t} \times \mathbb{Z}_4$, $\mathcal{G} = \{\mathbb{Z}_{2t} \times \{k\}, k \in \mathbb{Z}_4\}$ and $\mathcal{B} = \{\{(i,1),(j,2),(i \circ_1 j, 3),(i \circ_2 j, 0)\} \mid i,j \in \mathbb{Z}_{2t}\}$, where $(\mathbb{Z}_{2t}, \circ_1)$ and $(\mathbb{Z}_{2t}, \circ_2)$ are two orthogonal quasigroups of order $2t$ [2]. Then $\Gamma = (X, \mathcal{G}, \mathcal{B})$ is the 4-GDD of type $(2t)^4$.

Remove from each block the edges $\{(i,1),(j,2)\}$, $\{(i \circ_1 j, 3),(i \circ_2 j, 0)\}$. These edges cover two complete bipartite graphs $K_{2t,2t}$, then we can rearrange them into the set $\mathcal{C}$ of 4-cycles [39].

For each $0 \leq i \leq 2t-1$ and for each $0 \leq j \leq t-1$, remove the edges $\{(i,1),(j,2)\}$, $\{(j,2),(i \circ_1 j, 3)\}$, $\{(i,1),(i \circ_1 (j+t), 3)\}$, $\{(i \circ_1 (j+t), 3),(i \circ_2 (j+t), 0)\}$. Since $\{(j,2),(i \circ_1 j, 3) \mid 0 \leq i \leq 2t-1, 0 \leq j \leq t-1\} = \{(j,2),(i \circ_1 (j+t), 3) \mid 0 \leq i \leq 2t-1, 0 \leq j \leq t-1\}$, the removed edges can be assembled into the set $\mathcal{K} = \{((i,1),(j,2),(i \circ_1 (j+t), 3)) - (i \circ_2 (j+t), 0) \mid 0 \leq i \leq 2t-1,\ 0 \leq j \leq t-1\}$. □

In order to give a $\{G_1, G_2, \ldots, G_\mu\}$-metamorphosis, it is sufficient, for $\lambda = 1$, to indicate, for each i, L_i and $\mathcal{B}'_i$, being straightforward the blocks in $\mathcal{B}_i$.

Lemma 2.2.2. *For* $n = 25, 49, 73$ *there is an* $S(2,4,n)$ $(X, \mathcal{B})$, *having a* $\{C_4, K_3 + e\}$*-metamorphosis with empty leaves.*

Proof **n=25:** $X = \mathbb{Z}_{25}$, $\mathcal{B} = \{\{1,5,12,0\}, \{1,6,13,2\}, \{3,7,14,2\}, \{8,4,3,10\}, \{4,9,11,0\}, \{5,10,17,6\}, \{7,11,18,6\}, \{7,12,19,8\}, \{9,15,13,8\}, \{14,5,16,9\}, \{10,15,22,11\}, \{12,16,23,11\}, \{12,24,17,13\}, \{13,18,20,14\}, \{10,14,21,19\}, \{15,2,20,16\}, \{16,21,3,17\}, \{17,22,4,18\}, \{0,23,19,18\}, \{19,24,1,15\}, \{21,20,7,0\}, \{21,8,1,22\}, \{2,22,9,23\}, \{23,5,3,24\}, \{6,20,24,4\}, \{2,0,24,10\}, \{3,20,11,1\}, \{4,2,21,12\}, \{3,0,13,22\}, \{4,14,23,1\}, \{7,5,4,15\}, \{6,8,16,0\}, \{7,9,17,1\}, \{2,8,5,18\}, \{19,3,9,6\}, \{9,20,12,10\}, \{5,21,13,11\}, \{6,14,22,12\}, \{7,23,10,13\}, \{14,8,24,11\}, \{15,17,0,14\}, \{10,18,1,16\}, \{17,19,2,11\}, \{15,18,12,3\}, \{16,19,13,4\}, \{22,20,19,5\}, \{15,21,23,6\}, \{24,16,22,7\}, \{20,17,23,8\}, \{9,21,24,18\}\}$; $\mathcal{C} = \{(2,3,1,0), (7,5,3,0), (14,13,4,0), (23,7,16,0), (10,11,2,1), (8,6,4,1), (24,10,17,1), (18,15,4,2), (20,11,9,2), (21,22,4,3), (19,5,12,3), (9,7,6,5), (21,24,8,5), (22,20,9,6), (15,22,13,6), (14,10,8,7), (17,18,9,8), (13,11,12,10), (18,16,14,11), (19,15,13,12), (21,23,14,12), (17,19,16,15), (23,24,17,16), (20,21,19,18), (24,22,23,20)\}$
$\mathcal{K} = \{(4,0,1) - 20,\ (5,9,0) - 22,\ (1,3,2) - 12,\ (6,2,7) - 1,\ (10,8,9) - 1,\ (6,10,4) -$

$12, (11,5,6)-3,\ (15,8,7)-16, (11,8,12)-6,\ (18,9,14)-1, (11,16,15)-6,$ $(19,10,11)-14,\ (14,12,13)-7,\ (23,13,24)-7,\ (15,14,19)-16,\ (10,2,16)-4,$ $(11,21,17)-8,\ (18,16,17)-14,\ (21,18,22)-3,\ (23,2,18)-15,\ (8,0,18)-3,$ $(24,15,5)-22,\ (10,0,20)-4, (21,0,6)-19,\ (20,8,22)-23\}$

n=49: $X=\mathbb{Z}_{49}$. The starters blocks of $\mathcal{B}$ are $\{0,8,3,1\}$, $\{0,29,4,18\}$, $\{6,33,21,0\}$, $\{32,19,9,0\}$. The starters blocks of $\mathcal{C}$ are $(0,5,4,22)$ and $(0,9,34,13)$. The starters blocks of $\mathcal{K}$ are $(0,1,19)-12$, $(6,17,0)-16$.

n=73: $X=\mathbb{Z}_{73}$. The starters blocks of $\mathcal{B}$ are $\{1,4,6,0\}$, $\{7,28,0,20\}$, $\{9,33,44,0\}$, $\{0,25,47,15\}$, $\{46,12,30,0\}$, $\{0,31,14,50\}$. The starters blocks of $\mathcal{C}$ are $(0,1,3,13)$, $(0,26,54,24)$ and $(0,29,65,31)$. The starters blocks of $\mathcal{K}$ are $(10,1,0)-4$, $(40,27,0)-12$, $(0,23,8)-22$. □

Lemma 2.2.3. *For $n\equiv 1 \pmod{24}$, there exists an $S(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For $n=25,49,73$, the result follows from Lemma 2.2.2. Let Γ be the 4-GDD in Lemma 2.2.1 with $t=12$. Add an infinite point to each group $G_i=\mathbb{Z}_{24}\times\{i\}$, $i=0,1,2,3$, and place on it a copy of the $S(2,4,25)$ given in Lemma 2.2.2. The result is an $S(2,4,97)$ having a $\{C_4,K_3+e\}$-metamorphosis. Now let $n=24u+1$, with $u\geq 5$. Add an infinite point to the vertex set of a 4-GDD of type 6^u [11] and apply to it the weighting construction with $r=s=4$, $\alpha=1$ and $w=4$. This completes the proof. □

Lemma 2.2.4. *There exist an $S(2,4,16)$ and an $S(2,4,40)$ having a $\{C_4,K_3+e\}$-metamorphosis where $L_{\mathcal{C}}$ is an 1-factor and $L_{\mathcal{K}}$ is the empty graph.*

Proof **n=16:** $X=\mathbb{Z}_{16}$, $\mathcal{B}=\{\{1,2,0,3\}, \{4,6,0,5\}, \{0,7,8,9\}, \{11,13,0,12\}, \{15,0,10,14\}, \{4,1,7,11\}, \{1,12,14,5\}, \{1,8,15,6\}, \{9,13,10,1\}, \{2,13,15,4\}, \{2,10,5,7\}, \{2,9,12,6\}, \{8,11,14,2\}, \{3,9,14,4\}, \{3,5,8,13\}, \{3,11,10,6\}, \{3,7,12,15\}, \{8,10,4,12\}, \{9,15,5,11\}, \{7,14,6,13\}\}$;

$\mathcal{C}=\{(1,2,9,8), (11,13,9,3), (0,3,5,7), (11,7,14,5), (13,2,10,8), (4,15,12,1), (6,10,14,4), (0,15,6,12)\}$;

$L_C=\{(0,5),(1,10),(2,14),(3,7),(4,12),(6,13),(8,11),(9,15)\}$;

$\mathcal{K}=\{(4,1,0)-6, (10,0,7)-1, (13,14,12)-7, (2,6,8)-13, (6,3,1)-12, (3,9,13)-15, (14,7,8)-10, (11,12,15)-0, (13,2,10)-4, (11,6,9)-14\}$.

n=40: $X=\mathbb{Z}_{40}$. $\mathcal{B}=\{\{i,1+i,4+i,13+i\}, \{i,2+i,7+i,24+i\}, \{i,6+i,14+i,25+i\},\{j,10+j,20+j,30+j\} \mid 0\leq i\leq 39,\ 0\leq j\leq 9\}$;

$\mathcal{C}=\{(i,4+i,20+i,24+i), (i,5+i,20+i,25+i), (i,8+i,20+i,28+i) \mid 0\leq i\leq 19\}$;

$L_C=\{j,20+j),(10+j,30+j) \mid 0\leq j\leq 9\}$;

$\mathcal{K}=\{(6,21,15)-25, (7,22,16)-26, (7,22,16)-26, (8,23,17)-27, (9,24,18)-28,$

$(10,25,19)-29$, $(11,26,20)-30$, $(12,27,21)-31$, $(13,28,22)-32$, $(14,29,23)-33$, $(15,30,24)-34$, $(16,31,25)-30$, $(17,32,26)-31$, $(18,33,27)-32$, $(19,34,28)-33$, $(20,35,29)-34$, $(21,36,30)-35$, $(22,37,31)-36$, $(23,38,32)-37$, $(24,39,33)-38$, $(25,0,34)-39$, $(26,1,35)-0$, $(27,2,36)-1$, $(28,3,37)-2$, $(29,4,38)-3$, $(30,5,39)-4$, $(31,6,0)-17$, $(32,7,1)-18$, $(33,8,2)-19$, $(34,9,3)-20$, $(35,10,4)-21$, $(36,11,5)-22$, $(37,12,6)-23$, $(38,13,7)-24$, $(39,14,8)-25$, $(0,15,9)-26$, $(1,16,10)-27$, $(2,17,11)-28$, $(3,18,12)-29$, $(4,19,13)-30$, $(5,20,14)-31$, $(0,5,17)-29$, $(1,6,18)-30$, $(2,7,19)-31$, $(3,8,20)-32$, $(4,9,21)-33$, $(5,10,22)-34$, $(6,11,23)-35$, $(7,12,24)-36$, $(8,13,25)-37$, $(9,14,26)-38$, $(10,15,27)-39$, $(11,16,28)-0$, $(12,17,29)-1$, $(13,18,30)-2$, $(14,19,31)-3$, $(15,20,32)-2$, $(16,21,33)-3$, $(17,22,34)-4$, $(18,23,35)-5$, $(19,24,36)-6$, $(20,25,37)-7$, $(21,26,38)-8$, $(22,27,39)-9$, $(23,28,0)-10$, $(24,29,1)-11\}$. □

Remark 2.2.1. In the $S(2,4,16)$ given in Lemma 2.2.4, it is possible to choose a path of lenght 2 from each $B \in \mathcal{B} \setminus \{0,1,2,3\}$ so that the edges belonging to these paths can be reassembled into the set of (K_3+e)s $\{(13,14,2)-5$, $(12,8,7)-13$, $(2,8,6)-15$, $(6,3,5)-14$, $(3,13,9)-14$, $(11,12,15)-10$, $(13,10,12)-5$, $(9,6,11)-4$, $(4,5,7)-9\}$ and into the edges $\{0,15\}, \{2,4\}$.

Remark 2.2.2. In the $S(2,4,16)$ given in Lemma 2.2.4, it is possible to choose a path of lenght 2 from each $B \in \mathcal{B} \setminus \{0,1,2,3\}$ so that the edges belonging to these paths can be reassembled into the set of (K_3+e)s $\{(12,8,7)-11$, $(6,2,8)-15$, $(3,6,5)-12$, $(3,13,9)-14$, $(11,12,15)-13$, $(13,12,10)-15$, $(9,11,6)-14$, $(4,7,5)-14\}$ and into the triangles $(0,7,10), (2,13,14)$.

The $6t+4$ Construction[46]. Let $n = 6t+4$, where t is even and $t \geq 10$. Let $X = \{1,2,\ldots,t\}$ and let R be a skew room frame of type $2^{t/2}$ with holes $H = \{h_1, h_2, \ldots, h_{t/2}\}$ of size 2. For the definition of a skew room frame and for results on its existence see [25].

1. For the hole $h_1 \in H$, let $(X_{h_1}, \mathcal{B}_1)$ be a copy of the $S(2,4,16)$ in Lemma 2.2.4 on $X_{h_1} = \{a,b,c,d\} \cup (h_1 \times \mathbb{Z}_6)$.
2. For each hole $h_i \in H \setminus \{h_1\}$, let $(X_{h_i}, \mathcal{B}_i)$ be a copy of the $S(2,4,16)$ in Lemma 2.2.4 on $X_{h_i} = \{a,b,c,d\} \cup (h_i \times \mathbb{Z}_6)$ such that $\{a,b,c,d\} \in \mathcal{B}_i$.
3. If x and y belong to different holes in H, then there exists only one cell (r,c) in R containing the pair $\{x,y\}$. Let $\mathcal{D} = \{\{(x,i),(y,i),(r,i+1),(c,i+4)\} \mid i \in \mathbb{Z}_6\}$.

Let $X = \bigcup_{h_i \in H} X_{h_i}$ and $\mathcal{B} = (\bigcup_{h_i \in H\setminus\{h_1\}} \mathcal{B}_i \setminus \{\{a,b,c,d\}\}) \cup \mathcal{B}_1 \cup \mathcal{D}$. It is straightforward to see that $(X,\mathcal{B})$ is an $S(2,4,n)$. For $i,j \in \mathbb{Z}_6$, the vertices $(x,i) \in X$ will be called "of level i" and the edge $\{(x,i),(y,j)\}$ will be called "between levels i and j".

Lemma 2.2.5. *For $n \equiv 16 \pmod{24}$, there exists an $S(2,4,n)$ having $\{C_4, K_3+e\}$-metamorphosis.*

Proof Let $n = 16+24k$. By Lemma 2.2.4 we can assume $k \geq 2$. Let $(X,\mathcal{B})$ the $S(2,4,n)$ given by the $6t+4$ Construction with $t = 4k+2$. It is proved in [46](Lemma 2.5) that $(X,\mathcal{B})$ has a C_4-metamorphosis with leave a 1-factor. So we have only to prove the (K_3+e)-metamorphosis of $(X,\mathcal{B})$.

- Take a (K_3+e)-metamorphosis of $(X_{h_1},\mathcal{B}_1)$ as in Lemma 2.2.4.
- For each hole h_{2i}, $1 \leq i \leq k$, delete the edges from type 2 blocks and reassemble them as in Remark 2.2.1, where we put a,b,c,d instead of $0,1,2,3$.
- For each hole h_{2i+1}, $1 \leq i \leq k$, delete the edges from type 2 blocks and reassemble them as in Remark 2.2.2, where we put a,b,c,d instead of $0,1,2,3$. Note that the edges from Remark 2.2.1 and the triangles from 2.2.2 can be reassembled into (K_3+e)s.
- Delete the paths $[(x,2),(c,0),(y,2)]$, $[(x,3),(c,1),(y,3)]$ and $[(x,4), (r,5),(y,4)]$ from all blocks in $\mathcal{D}$ of the form $\{(x,2), (y,2), (c,0), (r,3)\}$, $\{(x,3), (y,3), (c,1), (r,4)\}$ and $\{(x,4), (y,4), (r,5), (c,2)\}$. Delete the paths $[(y,0), (x,0), (r,1)]$, $[(y,1), (x,1), (r,2)]$, $[(y,5), (x,5), (r,0)]$ from all blocks in $\mathcal{D}$ of the form $\{(x,0), (y,0), (r,1), (c,4)\}$, $\{(x,1), (y,1), (r,2), (c,5)\}$ and $\{(x,5), (y,5), (r,0), (c,3)\}$, respectively.

The deleted edges don't belong to the same hole and we can split them into the following classes:

1. edges between levels 0 and 2;
2. edges between levels 1 and 3;
3. edges between levels 4 and 5;
4. edges on level 0;

5. edges on level 1;

6. edges on level 5;

7. edges between levels 0 and 1;

8. edges between levels 1 and 2;

9. edges between levels 0 and 5.

Reassemble the edges of type 1, 4, 7 into the (K_3+e)s $((c,2),(y,0),(x,0))-(r,1)$, the edges of type 2, 5, 8 into the (K_3+e)s $((c,3),(y,1),(x,1))-(r,2)$, the edges of type 3, 6, 9 into the (K_3+e)s $((c,4),(y,5),(x,5))-(r,0)$. Note that, for example, $\{\{(x,2),(c,0)\},\{(y,2),(c,0)\}\} = \{\{(c,2),(y,0)\},\{(c,2),(x,0)\}\} = \{\{(a,2),(1,0)\},\{(a,2),(2,0)\},\{(b,2),(3,0)\},\{(b,2),(4,0)\},\ldots \mid a \neq 1,2, b \neq 3,4,\ldots\} = \{\{(1,2),(a,0)\},\{(2,2),(a,0)\},\{(3,2),(b,0)\},\{(4,2),(b,0)\},\ldots \mid a \neq 1,2, b \neq 3,4,\ldots\}$. Therefore we obtain a (K_3+e)-design of order n. □

Theorem 2.2.6. *For $n \equiv 1,4 \pmod{12}$, there exists an $S(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof The result follows from Theorem 2.1.1 and Lemmas 2.2.3 and 2.2.5. □

2.3 $\lambda = 3$

Lemma 2.3.1. *There exist $\{4,5\}$-GDDs of type $2^1 4^5$, $3^1 5^4$, $6^1(6u+4)^4$, $u \geq 2$.*

Proof Let $(S,\mathcal{G},\mathcal{B})$ be a 5-GDD of type 5^5 [11], where the groups are $G_i = \mathbb{Z}_5 \times \{i\}$, $i = 1,\ldots,5$. Let $B_1,\ldots,B_5$ be the blocks of $\mathcal{B}$ meeting $(0,1)$. Remove the vertices $(0,1),(1,1),(2,1)$ and form a new GDD of type $2^1 4^5$ having $G_1 \setminus \{(0,1),(1,1),(2,1)\}$ and $B_i \setminus \{(0,1)\}$, $i = 1,\ldots,5$ as groups and $G_i, i = 2,3,4,5$ and $B \setminus \{(1,1),(2,1)\}$, for every $B \in \mathcal{B} \setminus \{B_1,B_2,\ldots,B_5\}$, as blocks. Note that the blocks of size 5 of this new GDD are those meeting $(3,1)$ or $(4,1)$. The remaining blocks are of size 4.
Now delete $(0,1),(1,1)$ in $(S,\mathcal{G},\mathcal{B})$. We get a $\{4,5\}$-GDD of type $3^1 5^4$. The blocks of the new GDD have size 5 if they contain one of the points $(2,1)$, $(3,1)$, $(4,1)$, otherwise have size 4.

Let $(S, \mathcal{G}, \mathcal{B})$ be a 5-GDD of type $(6u+4)^5$ $u \geq 2$ [11], where the groups are $G_i = \mathbb{Z}_{6u+4} \times \{i\}$, for $1 \leq i \leq 5$. By deleting the points $(0,1), (1,1), \ldots, (6u-3, 1)$, we obtain a $\{4,5\}$-GDD of type $6^1(6u+4)^4$. The blocks of the new GDD have size 4 or 5. The blocks of size 5 are those containing $(x, 1)$, for some $6u-2 \leq x \leq 6u+3$. □

Lemma 2.3.2. *For $t \geq 2, t \neq 3$, there exist 4-GDDs of index* 3 *and type $(2t)^4$ or $(2t)^5$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof Take the 4-GDD of type $(2t)^4$ constructed in Lemma 2.2.1 and repeat three times its blocks. The result is a 4-GDD of type $(2t)^4$ and index $\lambda = 3$. Now let $(X, \mathcal{B})$ be an $S_3(2,4,5)$. Place in each block $\{x_1, x_2, x_3, x_4\} \in \mathcal{B}$ a 4-GDD of type $(2t)^4$ with groups $G_i = \{x_i\} \times \mathbb{Z}_{2t}$ having a $\{C_4, K_3+e\}$-metamorphosis. The result is the required 4-GDD of index 3 and type $(2t)^5$ having a $\{C_4, K_3+e\}$-metamorphosis. □

Lemma 2.3.3. *For $n \equiv 1 \pmod 8$, $n \geq 9$, there exists an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof
n = 9. $X = \mathbb{Z}_9$. The starters blocks of $\mathcal{B}$ are $\{2,0,4,1\}$, $\{1,6,0,4\}$. If we delete the edges $\{a,b\}$, $\{c,d\}$ from each block $\{a,b,c,d\}$, we can reassemble these edges into a set $\mathcal{C}$ = with starter block $(0,4,8,2)$. If we delete the paths with starters $[4,2,1]$, $[1,0,6]$, we can reassemble these edges into a set $\mathcal{K}$ with starter block $(0,1,3)-4$.
n = 17. $X = \mathbb{Z}_{17}$. The starters blocks of $\mathcal{B}$ are $\{6,4,1,0\}$, $\{2,12,8,0\}$, $\{16,7,4,0\}$, $\{15,8,14,0\}$. If we delete the edges $\{a,b\}$, $\{c,d\}$ from each block $\{a,b,c,d\}$, we can reassemble these edges into a set $\mathcal{C}$ with starter blocks $(0,8,16,3)$, $(0,1,3,10)$. If we delete the paths with starters $[1,4,0]$, $[8,0,12]$, $[16,4,7]$, $[0,15,14]$, we can reassemble these edges into a set $\mathcal{K}$ with starter blocks $(0,1,4)-9$, $(0,5,8)-10$.
n = 24u + 1, $u \geq 1$. Take 3 copies of the $S(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis given in Lemma 2.2.3.
n = 33. Take the 4-GDD of index 3 and type 8^4 constructed in Lemma 2.3.2. Add an infinite point to each group $G_i, i = 0,1,2,3$, and place on it a copy of the $S_3(2,4,9)$ above constructed. We obtain an $S_3(2,4,33)$ having a $\{C_4, K_3+e\}$-metamorphosis.
n = 24u + 9, $u \geq 2$ or $n = 48u+17$, $u \geq 1$. Add an infinite point to the vertex set of a 4-GDD of type 2^{3u+1} (4^{3u+1})[11] and apply Theorem 2.1.2

with $r = s = 4$ and $w = 4$. The result is an $S_3(2, 4, n)$ having a $\{C_4, K_3 + e\}$-metamorphosis.

n = 96u + 41, $u \geq 0$. Blow up by 8 an $S_3(2, 4, 12u + 5)$ $(\mathbb{Z}_{12u+5}, \mathcal{B})$ and place in each expanded block a 4-GDD of type 8^4 having a $\{C_4, K_3 + e\}$-metamorphosis (see Lemma 2.2.1). To complete the proof add an infinite point to each expanded vertex of $\mathbb{Z}_{12u+5}$ and place on it an $S_3(2, 4, 9)$ having a $\{C_4, K_3 + e\}$-metamorphosis.

n = 96u + 89, $u \geq 0$. Apply Theorem 2.1.2 with $\lambda = 3$, $\alpha = 1$, $r = 4$, $s = 5$ (Lemma 2.3.2) and the following ingredients given in Lemma 2.3.1:

- if $u = 0$: $w = 4$, a $\{4, 5\}$-GDD of type $2^1 4^5$;
- if $u = 1$: $w = 8$, a $\{4, 5\}$-GDD of type $3^1 5^4$;
- if $u \geq 2$: $w = 4$, a $\{4, 5\}$-GDD of type $6^1 (6u + 4)^4$.

□

Lemma 2.3.4. *For $n = 8, 24$ there exist an $S_3(2, 4, n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof

n=8: $X = \mathbb{Z}_8, \mathcal{B} = \{\{0, 1, 3, 7\}, \{1, 2, 4, 7\}, \{2, 3, 5, 7\}, \{3, 4, 6, 7\}, \{4, 5, 0, 7\}, \{5, 6, 1, 7\}, \{0, 6, 2, 7\}, \{2, 4, 5, 6\}, \{3, 5, 6, 0\}, \{4, 6, 0, 1\}, \{5, 1, 0, 2\}, \{6, 3, 1, 2\}, \{0, 3, 2, 4\}, \{1, 3, 4, 5\}\}$. Delete the edges (a, b),(c, d) from each block $\{a, b, c, d\} \in \mathcal{B}$ and reassemble them into $\mathcal{C} = \{(0, 1, 2, 7), (6, 5, 1, 7), (5, 4, 3, 7), (2, 3, 5, 4), (6, 0, 2, 4), (0, 6, 3, 1)\}$ and $L_C = \{(1, 2), (3, 0), (4, 7), (5, 6)\}$. Delete from the blocks in $\mathcal{B}$ the paths [1, 0, 3], [1, 4, 7], [4, 6, 7], [0, 5, 7], [5, 1, 7], [2, 0, 7], [5, 4, 6], [3, 6, 5], [6, 0, 4], [1, 0, 5], [2, 1, 3], [0, 3, 4], [1, 3, 5] and reassemble their edges into $\mathcal{K} = \{(2, 1, 0) - 4, (3, 5, 0) - 1, (3, 7, 1) - 4, (6, 7, 4) - 3, (0, 6, 3) - 1, (0, 7, 5) - 1, (4, 6, 5) - 3\}$.

n=24: $X = \mathbb{Z}_{12} \times \{1, 2\}$. $\mathcal{B} = \{\{(i, 1), (11 + i, 2), (1 + i, 1), (2 + i, 2)\}, \{(i, 1), (i, 2), (3+i, 1), (5+i, 1)\}, \{(i, 1), (9+i, 2), (4+i, 1), (6+i, 1)\}, \{(i, 1), (7+i, 2), (3+i, 1), (5+i, 1)\}, \{(i, 1), (6 + i, 2), (4 + i, 1), (5 + i, 1)\}, \{(i, 1), (8 + i, 2), (3 + i, 1), (4 + i, 1)\}, \{(i, 1), (6 + i, 2), (10 + i, 2), (11 + i, 2)\}, \{(i, 1), (4 + i, 2), (8 + i, 2), (9 + i, 2)\}, \{(i, 1), (11+i, 2), (8+i, 2), (10+i, 2)\}, \{(i, 1), (i, 2), (3+i, 2), (5+i, 2)\}, \{(i, 1), (7+i, 2), (1 + i, 2), (3 + i, 2)\}, \{(j, 1), (j, 2), (6 + j, 1), (6 + j, 2)\} \mid i \in \mathbb{Z}_{12}, j \in \mathbb{Z}_6\}$. Delete the edges $\{a, b\}$, $\{c, d\}$ from each block $\{a, b, c, d\}$ and reassemble them into $\mathcal{C} = \{((i, 1), (2 + i, 1), (1 + i, 2), (11 + i, 2)), ((i, 1), (2 + i, 1), (2 + i, 2), (1 + i, 2)), ((i, 1), (1+i, 1), (10+i, 2), (8+i, 2)), ((j, 1), (6+j, 1), (j, 2), (6+j, 2)) \mid i \in \mathbb{Z}_{12}, j \in$

$\mathbb{Z}_6\}$ and $L_{\mathcal{C}} = \{\{(j,1),(j,2)\},\{(6+j,1),(6+j,2)\} \mid j \in \mathbb{Z}_6\}$.
$\mathcal{K} = \{((i,2),(5+i,1),(i,1)) - (2+i,2),\ ((9+i,2),(6+i,1),(i,1)) - (5+i,1),$ $((3+i,2),(1+i,2),(i,1)) - (4+i,2),\ ((11+i,2),(8+i,2),(i,1)) - (i,2) \mid i \in \mathbb{Z}_{12}\} \cup$ $\{((10,2),(0,1),(6,2)) - (0,2),\ ((11,2),(1,1),(7,2)) - (1,2),\ ((12,2),(2,1),(8,2)) -$ $(2,2),\ ((13,2),(3,1),(9,2)) - (3,2),\ ((14,2),(4,1),(10,2)) - (4,2),$
$((3,2),(5,1),(11,2)) - (7,2),\ ((4,2),(6,1),(12,2)) - (8,2),\ ((4,1),(0,1),(3,1)) -$ $(3,2),\ ((5,1),(1,1),(4,1)) - (4,2),\ ((6,1),(2,1),(5,1)) - (5,2),\ ((7,1),(3,1),(6,1)) -$ $(6,2),\ ((8,1),(4,1),(7,1)) - (7,2),\ ((8,1),(5,1),(9,1)) - (3,2),\ ((9,1),(6,1),(10,1)) -$ $(4,2),\ ((10,1),(7,1),(11,1)) - (5,2),\ ((0,1),(1,1),(9,1)) - (7,2),$
$((1,1),(2,1),(10,1)) - (8,2),\ ((2,1),(3,1),(11,1)) - (9,2),\ ((1,2),(7,1),(5,2)) -$ $(11,2),\ ((2,2),(8,1),(6,2)) - (10,2),\ ((0,1),(11,1),(8,1)) - (8,2)\}$. □

The $4t$ Construction. [**46**] Let $n = 4t$, where $t \geq 4$ and $t \neq 6$. Let $S = \{1,2,\ldots,t\}$ and let $(S,\circ)$ be an idempotent self-orthogonal quasigroup of order t [2]. Set $X = S \times \mathbb{Z}_4$ and define a collection of blocks $\mathcal{B}$ as follows:

1. For each $x \in S$, place in $\mathcal{B}$ three copies of the block $\{(x,0),(x,1),(x,2),(x,3)\}$.

2. For each pair $x,y \in S, x < y$, place in $\mathcal{B}$ the blocks $\{(x,i),(y,i),(x \circ y, i+1),(y \circ x, i+1)\}$, where $i \in \mathbb{Z}_4$ and the second coordinates are reduced modulo 4.

3. For each pair $x,y \in S, x < y$, place in $\mathcal{B}$ the blocks $\{(x,i),(y,i),(x \circ y, i+2),(y \circ x, i+2)\}$, where $i = 0,1$ and the second coordinates are reduced modulo 4.

4. For each pair $x,y \in S, x \neq y$, place in $\mathcal{B}$ the block $\{(x,0),(y,1),(x \circ y, 2),(y \circ x, 3)\}$.

Then $(X,\mathcal{B})$ is an $S_3(2,4,n)$. For $i \in \mathbb{Z}_4$, the vertices $(x,i) \in X$ will be called "of level i" and the edge $\{(x,i),(x,j)\}$ will be called "belonging to the same column".

Lemma 2.3.5. *For $n \equiv 0 \pmod 8$, there exists an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof Let $n = 8k$. For $k = 1,3$, the result follows from Lemma 2.3.4. Now let $k \neq 1,3$ and let $(X,\mathcal{B})$ be the $S_3(2,4,8k)$ given in the $4t$ Construction with $t = 2k$. Lemma 4.4 in [46] proves that $(X,\mathcal{B})$ has a C_4-metamorphosis.

Now we prove that $(X,\mathcal{B})$ has a (K_3+e)-metamorphosis:

- For each odd $x \in S$, delete the paths $2[(x,1), (x,0), (x,2)]$ and $[(x,1),(x,2),(x,3)]$ from type 1 blocks; for each even $x \in S$, delete the paths $2[(x,0),(x,1),(x,2)]$ and $[(x,0),(x,2),(x,3)]$ from type 1 blocks. Reassemble these paths into $(K_3+e)s$ with leave $[(x,1),(x,0),(x,2)]$ for x odd and
$[(x,0),(x,1),(x,2)]$ for x even.

- From each type 2 block delete the path $[(x,i),(x\circ y,i+1),(y,i)]$. The deleted edges don't belong to the same column and we can split them into the following classes: (I) edges between levels 0 and 1, (II) edges between levels 1 and 2, (III) edges between levels 2 and 3, (IV) edges between levels 0 and 3.

- From each type 3 block delete the path $[(y,i),(x,i),(y\circ x,i+2)]$ if $x=2j-1$ and $y\circ x=2j$, $j=1,\dots,k$, otherwise delete the path $[(x,i),(y,i),(y\circ x,i+2)]$. The deleted edges don't belong to the same column and we can split them into the following classes: (V) edges on level 0, (VI) edges on level 1, (VII) edges between levels 0 and 2, (VIII) edges between levels 1 and 3.

- From each type 4 block delete the path $[(y,1),(x,0),(x\circ y,2)]$. The deleted edges don't belong to the same column and we can split them into the following classes: (IX) edges between levels 0 and 2, (X) edges between levels 0 and 1.

Reassemble the deleted edges (I), (V) and (VII) into the $(K_3{+}e)$s $((y,0),(x\circ y,1),(x,0))-(y\circ x,2)$ if $x=2j-1$ and $y\circ x=2j$, $j=1,\dots,k$; otherwise, into the (K_3+e)s $((x,0),(x\circ y,1),(y,0))-(y\circ x,2)$.

Reassemble the deleted edges (II), (VI), (VIII) into the $(K_3{+}e)$s $((y,1),(x\circ y,2),(x,1))-(y\circ x,3)$ if $x=2j-1$ and $y\circ x=2j$, $j=1,\dots,k$; otherwise, into the (K_3+e)s $((x,1),(x\circ y,2),(y,1))-(y\circ x,3)$.

Reassemble the deleted edges (III), (IV), (IX) and (X) into the (K_3+e)s $((y\circ x,3),(x\circ y,2),(x,0))-(y,1)$.

Next we need to rearrange these (K_3+e)s to use the paths obtained from type 1 blocks, $[(x,1),(x,0),(x,2)]$, for x odd, and $[(x,0),(x,1),(x,2)]$, for x even. For each $j=1,\dots,k$, replace the (K_3+e) $((y,0),(x\circ y,1),(2j-1,0))-(2j,2)$, obtained by rearranging the deleted edges (I), (V) and (VII), by $((y,0),(x\circ y,1),(2j-1,0))-(2j-1,2)$. Replace the (K_3+e) $((y,3),(x\circ y,2),(2j-1,0))-(2j,1)$, obtained by rearranging the deleted edges (III),

(IV), (IX) and (X), by $((y,3),(x \circ y,2),(2j-1,0)) - (2j-1,1)$.
Next arrange the remaining edges $\{(2j,0),(2j,1)\}$, $\{(2j,1),(2j,2)\}$, $\{(2j-1,0),(2j,1)\}$ and $\{(2j-1,0),(2j,2)\}$, $j=1,\ldots,k$, into the (K_3+e)s $((2j-1,0),(2j,2),(2j,1)) - (2j,0)$, $j=1,\ldots,k$.

We obtain a 3-fold (K_3+e)-design of order n and so an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis. □

Theorem 2.3.6. *For $n \equiv 0,1 \pmod 4$, there exists an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For $n \equiv 4,5 \pmod 8$, the result follows from Theorem 2.1.1. For $n \equiv 0 \pmod 8$ and for $n \equiv 1 \pmod 8$, the result follows from Lemmas 2.3.5 and 2.3.3, respectively. □

2.4 Summary

Lemma 2.4.1. *For $\lambda = 2$ with $n \equiv 1,4 \pmod{12}, n \geq 4$, $\lambda = 6$ with $n \equiv 0,1 \pmod 4, n \geq 4$, $\lambda = 4,8$ with $n \equiv 1 \pmod 3, n \geq 4$ and $\lambda = 12$, with $n \geq 4$, there exists an $S_\lambda(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For the values of λ and n as in hypothesis, there exists an $S_{\lambda/2}(2,4,n)$, $(X,\mathcal{B})$. By repeating two times each block of $(X,\mathcal{B})$, we obtain an $S_\lambda(2,4,n)$. For each $B_1, B_2 \in \mathcal{B}$ such that $B_1 = B_2 = \{x,y,z,t\}$, remove the edges $\{x,y\}$, $\{z,t\}$ ($\{x,y\}$ and $\{x,t\}$) from B_1 and the edges $\{x,t\}$, $\{y,z\}$ ($\{y,t\}$, $\{z,t\}$) from B_2. Rearrange the removed edges into the 4-cycle (x,y,z,t) (into the K_3+e $(x,y,t)-z$). This completes the proof. □

Theorem 2.4.2. *There exists an $S_\lambda(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod{12}$ and $\lambda(n-1) \equiv 0 \pmod 3$.*

Proof The necessity is trivial. For $\lambda = 1,3$ the result follows from Theorems 2.2.6, 2.3.6. For $\lambda = 2$ with $n \equiv 1,4 \pmod{12}$, $\lambda = 6$ with $n \equiv 0,1 \pmod 4$, $\lambda = 4,8,12$, the result follows from Lemma 2.4.1. For $\lambda = 2$, $n = 7,10,19$, the result follows from Theorem 2.1.1. For $\lambda = 2$, $n \equiv 7,10 \pmod{12}$, $n \geq 22$, take a $PBD(n)$ with one block of size 7 and others of size 4 [71] and place an $S_2(2,4,4)$ or an $S_2(2,4,7)$ having a $\{C_4, K_3+e\}$-metamorphosis on each block. For $\lambda = 6$ and $n \equiv 2,3 \pmod 4$, the result follows from Theorem 2.1.1. For $\lambda = 5,7,9,10,11$ combine a $S_\nu(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis with a $S_\mu(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis, with $(\lambda,\nu,\mu) = (5,4,1)$, $(7,6,1)$, $(9,6,3)$, $(10,8,2)$, $(11,6,5)$, respectively. For $\lambda = 12k + h$, with $0 \leq h \leq 11$, combine k $S_{12}(2, 4, n)$ having a $\{C_4, K_3 + e\}$-metamorphosis with an $S_h(2, 4, n)$ having a $\{C_4, K_3 + e\}$-metamorphosis. $\square$

Chapter 3

Complete simultaneous metamorphoses of kite designs

3.1 Preliminaries

We say that a $\{G_1, G_2, \ldots, G_\mu\}$-metamorphosis is *complete* if $\{G_i \mid i = 1, 2, \ldots, \mu\}$ coincides with the family of all nonisomorphic proper subgraphs of G without isolated vertices (see Definition 10).

Theorem 3.1.1. *[49] Table 1 shows the leaves of maximum packings of λK_n with triangles, where $\emptyset$ denotes the empty graph, G is a graph on n vertices of odd degrees and $(n+4)/2$ edges, D is a graph with 4 edges and even vertex degrees and a tripole is a graph consisting of $(n-4)/2$ disjoint edges and a 3-star:*

It is not difficult to settle the maximum packings of λK_n with S_3s and P_4s.

Theorem 3.1.2. *The leaves of maximum packings of λK_n with S_3s (or with P_4s) are collections of $m = 0, 1, 2$ edges, with $m \equiv \lambda n(n-1)/2 \pmod 3$.*

C.C. Lindner, G. Lo Faro and A. Tripodi [51] gave a complete answer to the existence problem of metamorphoses of a λ-fold kite system into a maximum packing of λK_n with triangles.

G. Lo Faro and A. Tripodi [54] gave also a complete answer to the existence problem of metamorphoses of a λ-fold kite system into a maximum packing of λK_n with P_4s.

λ	$n \pmod 6$					
	0	1	2	3	4	5
=1	1-factor	$\emptyset$	1-factor	$\emptyset$	tripole	C_4
$\equiv 0 \pmod 6$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
≥ 7 and $\equiv 1 \pmod 6$	1-factor	$\emptyset$	1-factor	$\emptyset$	tripole	D
$\equiv 2 \pmod 6$	$\emptyset$	$\emptyset$	$2P_2$	$\emptyset$	$\emptyset$	$2P_2$
$\equiv 3 \pmod 6$	1-factor	$\emptyset$	G	$\emptyset$	tripole	$\emptyset$
$\equiv 4 \pmod 6$	$\emptyset$	$\emptyset$	D	$\emptyset$	$\emptyset$	D
$\equiv 5 \pmod 6$	1-factor	$\emptyset$	tripole	$\emptyset$	tripole	$2P_2$

Table 3.1: Leaves of maximum packings of λK_n with triangles

In this chapter, we give a complete answer to the existence problem of a λ-fold kite system having a complete simultaneous metamorphosis. More precisely we prove the following

Main Theorem. *There exists a λ-fold kite system of order n having a complete simultaneous metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod 8$ and $(\lambda, n) \neq (1, 8)$. There is not a kite system of order 8 having an S_3-metamorphosis, but there is a kite system of order 8 having a $\{K_3, P_4, P_3, P_2, E_2\}$-metamorphosis. We will make use of this*

STANDARD WEIGHTING CONSTRUCTION. Suppose there exist:

1. *an r-GDD of type $g_1^{u_1} g_2^{u_2} \ldots g_h^{u_h}$;*
2. *a λ-fold kite system of order wg_i (or $1 + wg_i$), $i = 1, \ldots, h$;*
3. *a λ-fold kite design of the complete r-partite graph K_w^r.*

Then there is a λ-fold kite system of order $w(g_1u_1 + \ldots + g_hu_h)$ (or $1 + w(g_1u_1 + \ldots + g_hu_h)$).

Note that if the kite designs given as ingredients in the above construction have a Γ-metamorphosis with empty leaves and $\Gamma \subseteq \{K_3, S_3, P_4, P_3, P_2, E_2\}$ then the resulting kite system has a Γ-metamorphosis. Of course the same result is not always true if the leave of some metamorphosis of some ingredient is nonempty.

3.2 $\{K_3, S_3, P_4\}$-metamorphosis

Let $(X, \mathcal{B})$ be a λ-fold G-design having a $\{K_3, S_3, P_4\}$-metamorphosis. Then, basing on the definition of simultaneous metamorphosis (Definition 10), we have $\mu = 3$, $G_1 = K_3$, $G_2 = S_3$, and $G_3 = P_4$. In the following we put $\mathcal{T}, \mathcal{S}, \mathcal{P}$ instead of B'_1, B'_2, B'_3 and L_T, L_S, L_P instead of L_1, L_2, L_3. If B is the kite $(a, b, c) - d$, we put $B_1 = (a, b, c)$, $B_2 = [c; a, b, d]$, $B_3 = [b, a, c, d]$. Moreover we omit to explicitely mention the empty leave(s), we write (a, b) instead of $\{a, b\}$ and $m(a, b)$ to denote the edge $\{a, b\}$ m times repeated. We use the subscript notation x_i to denote the ordered pair (x, i).

3.2.1 Kite systems

Lemma 3.2.1. *There is not an S_3-metamorphosis of a kite system of order 8.*

Proof Let $(\mathbb{Z}_8, \mathcal{B}_1 \cup \mathcal{B}'_1, L)$ be an S_3-metamorphosis of a kite system $(\mathbb{Z}_8, \mathcal{B})$. Then $|\mathcal{B}| = 7$, so the two S_3s in $\mathcal{B}'_1$ cover 6 bases. Denote by i and j the centers of these stars. It is $i \neq j$, otherwise the vertex $i = j$ should appear as vertex of degree 2 in at least 6 kites. This is impossible. Let $\mathcal{I}$ and $\mathcal{J}$ be the sets of kites from which we picked up the bases of the stars with centers i and j, respectively. Being $|\mathcal{I}| = |\mathcal{J}| = 3$, there exists only one kite $B(i) \in \mathcal{B} \setminus \mathcal{I}$ meeting i and only one kite $B(j) \in \mathcal{B} \setminus \mathcal{J}$ meeting j. Moreover the degree of i in $B(i)$ and of j in $B(j)$ is 1. Let B be the kite of $\mathcal{B}$ covering the edge (i, j). If $B \in \mathcal{I}$ then there are at least 4 kites in $\mathcal{B}$ having j as a vertex of degree $d(j) \geq 2$. This is impossible. Analogously we obtain that $B \notin \mathcal{J}$. Then $\mathcal{B} \setminus (\mathcal{I} \cup \mathcal{J}) = \{B\}$ and $B = B(i) = B(j)$, a contradiction, because the degree of i in $B(i)$ and of j in $B(j)$ is 1. □

The following example gives a simultaneous metamorphosis of a kite system of order 8 into a maximum packing of K_8 with K_3s and P_4s and into a (not maximum) packing with S_3s having two 2-stars as leave.

Example 3.2.1. Let $X = \cup_{i=0}^{3} \{\alpha_i, \beta_i\}$ and let $\mathcal{B} = \{(\alpha_0, \alpha_1, \alpha_3) - \beta_1$, $(\beta_0, \beta_1, \alpha_2) - \alpha_1$, $(\beta_2, \alpha_2, \alpha_0) - \beta_0$, $(\beta_0, \alpha_1, \beta_3) - \beta_1$, $(\alpha_1, \beta_2, \beta_1) - \alpha_0$, $(\alpha_3, \alpha_2, \beta_3) - \alpha_0$, $(\beta_0, \alpha_3, \beta_2) - \beta_3\}$. Then $(X, \mathcal{B})$ is a kite system of order 8 having

- a K_3-metamorphosis with $\mathcal{T} = \{(\alpha_0, \beta_1, \beta_3)\}$ and $L_T = \{(\beta_2, \beta_3)$, (α_3, β_1), (α_0, β_0), $(\alpha_1, \alpha_2)\}$;

- a P_4-metamorphosis with $\mathcal{P} = \{[\beta_3, \alpha_1, \alpha_3, \beta_2],\ [\beta_3, \alpha_2, \beta_1, \beta_2]\}$ and $L_P = (\alpha_0, \alpha_2)$.
- a metamorphosis into a packing of K_8 with S_3s such that $\mathcal{S} = \{[\beta_0; \alpha_3, \alpha_1, \beta_1]\}$ and $L_S = \{[\alpha_1; \alpha_0, \beta_2],\ [\alpha_2; \alpha_3, \beta_2]\}$.

In order to handle the remaining cases we need the following example:

Example 3.2.2. Let $K_{4,4,4}$ be the complete tripartite graph with partition classes $V_1 = \{\alpha_0, \ldots, \alpha_3\}$, $V_2 = \{\beta_0, \ldots, \beta_3\}$ and $V_3 = \{\gamma_0, \ldots, \gamma_3\}$. Let $\mathcal{B} = \{(\gamma_3, \alpha_1, \beta_3) - \alpha_3, (\beta_1, \alpha_1, \gamma_1) - \alpha_3, (\alpha_1, \beta_2, \gamma_2) - \alpha_0,\ (\alpha_2, \beta_3, \gamma_2) - \beta_1,\ (\gamma_0, \alpha_2, \beta_1) - \alpha_0,\ (\alpha_2, \beta_2, \gamma_1) - \beta_3,\ (\gamma_2, \alpha_3, \beta_0) - \alpha_2,\ (\alpha_3, \beta_1, \gamma_3) - \alpha_2,\ (\beta_2, \alpha_3, \gamma_0) - \alpha_1,\ (\alpha_0, \gamma_1, \beta_0) - \alpha_1,\ (\alpha_0, \beta_2, \gamma_3) - \beta_0,\ (\alpha_0, \beta_3, \gamma_0) - \beta_0\}$. Then $(V_1 \cup V_2 \cup V_3, \mathcal{B})$ is a kite-decomposition of $K_{4,4,4}$ having a $\{K_3, S_3, P_4\}$-metamorphosis with

- $\mathcal{T} = \{(\alpha_3, \beta_3, \gamma_1),\ (\alpha_0, \beta_1, \gamma_2),\ (\alpha_1, \beta_0, \gamma_0),\ (\alpha_2, \beta_0, \gamma_3)\}$;
- $\mathcal{S} = \{[\alpha_1; \gamma_3, \beta_1, \beta_2],\ [\alpha_2; \gamma_0, \beta_3, \beta_2],\ [\alpha_3; \beta_1, \gamma_2, \beta_2],\ [\alpha_0; \gamma_1, \beta_3, \beta_2]\}$;
- $\mathcal{P} = \{[\alpha_2, \beta_1, \gamma_3, \beta_2],\ [\beta_3, \alpha_1, \gamma_1, \beta_2],\ [\beta_2, \gamma_2, \beta_3, \gamma_0],\ [\gamma_0, \alpha_3, \beta_0, \gamma_1]\}$.

Lemma 3.2.2. *For n=32,40,48,56,64,80 there exist kite systems of order n having a $\{K_3, S_3, P_4\}$-metamorphosis.*

Proof Suppose at first $n = 32, 40, 48, 56, 64$. The existence of a 3-GDD $(S, \mathcal{G}, \mathcal{U})$ of type 2^4, $2^3 4^1$, 2^6, $2^4 6^1$ and $2^3 4^1 6^1$ is well-known [11]. Apply the standard weighting construction by giving weight $w = 4$ and placing in each expanded block a copy of the kite-decomposition in Example 3.2.2 and in each expanded group a copy of the kite-designs in Examples 3.2.1, 3.4.1, 3.4.2, 3.4.3, 3.4.4. Starting from any 3-GDD, the result is a kite system of order n having a K_3-metamorphosis but not a $\{K_3, S_3, P_4\}$-metamorphosis: the kite systems induced by the expanded groups of size 8 cannot have an S_3-metamorphosis (see Lemma 3.2.1). Moreover the leaves produced in their P_4-metamorphoses don't share any vertex. Now we present a procedure that, starting from a suitable 3-GDD $(S, \mathcal{G}, \mathcal{U})$, shows how to rearrange the leaves and some blocks of $\mathcal{S} \cup \mathcal{P}$ in order to construct new S_3s and P_4s. We will write $\{\underline{a}, b\}$ if, inflating by 4 the group $\{a, b\} \in \mathcal{G}$, we apply Example 3.2.1 in order to produce an S_3-metamorphosis with leave $\{[a_1; a_0, b_2], [a_2; a_3, b_2]\}$ and a P_4-metamorphosis with leave $\{(a_0, a_2)\}$.

Step 1 (Building S_3s). Suppose that $(S, \mathcal{G}, \mathcal{U})$ contains 3 groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and 3 blocks (x, a, w), (x, c, y), (x, e, z) such that $x \notin \{a, b, c, d, e, f\}$.

Using the standard weighting construction we produce (from above groups and blocks) the following S_2s and S_3s: $[a_1; a_0, b_2]$, $[a_2; a_3, b_2]$, $[c_1; c_0, d_2]$, $[c_2; c_3, d_2]$, $[e_1; e_0, f_2]$, $[e_2; e_3, f_2]$, $[x_1; w_3, a_1, a_2]$, $[x_1; y_3, c_1, c_2]$, $[x_1; z_3, e_1, e_2]$. It is easy to rearrange the edges of these stars to construct the following S_3s: $[a_1; a_0, b_2, x_1]$, $[a_2; a_3, b_2, x_1]$, $[c_1; c_0, d_2, x_1]$, $[c_2; c_3, d_2, x_1]$, $[e_1; e_0, f_2, x_1]$, $[e_2; e_3, f_2, x_1]$, $[x_1; y_3, z_3, w_3]$.

Step 2 (Building P_4s). Suppose that $(S, \mathcal{G}, \mathcal{U})$ contains 3 groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and 2 blocks (a, c, t), (e, u, t). Using the standard weighting construction we produce (from above groups and blocks) the following P_2s and P_4s: $(a_0, a_2), (c_0, c_2), (e_0, e_2)$, $[a_2, c_1, t_3, c_2]$ and $[e_2, u_1, t_3, u_2]$. It is easy to rearrange the edges of these paths to construct the following P_4s: $[a_0, a_2, c_1, t_3], [e_0, e_2, u_1, t_3]$, $[c_0, c_2, t_3, u_2]$.

Case $n = 32$. Take the 3-GDD of type 2^4 with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{c}, d\}, \{\underline{e}, f\}, \{\underline{h}, g\}\}$ and blocks $\mathcal{U} = \{(g, a, d), (g, c, f), (g, e, b), (b, h, c), (e, d, h), (b, d, f), (c, e, a), (h, f, a)\}$.

Apply Step 1 to groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks (g, a, d), (g, c, f), (g, e, b). To complete the S_3-metamorphosis take $[h_1; h_0, g_2]$, $[h_2; h_3, g_2]$, $[b_1; c_3, h_1, h_2]$ and form the stars $[h_1; h_0, g_2, b_1]$, $[h_2; h_3, g_2, b_1]$ and leave $\{(b_1, c_3)\}$.

Apply Step 2 to groups $\{\underline{c}, d\}$, $\{\underline{e}, f\}$, $\{\underline{h}, g\}$ and blocks (c, e, a), (h, f, a). The result is the required P_4-metamorphosis having leave (a_0, a_2).

Case $n = 40$. Take the 3-GDD of type $2^4 4^1$ with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{c}, d\}, \{\underline{e}, f\}, \{x, y, z, t\}\}$ and $\mathcal{U} = \{(x, a, d), (x, c, f), (x, e, b), (a, c, z), (e, d, z), (f, b, z), (y, a, f), (y, b, d), (y, e, c), (e, d, z), (t, a, e), (t, b, c), (t, f, d)\}$.

Apply Step 1 to groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks (x, a, d), (x, c, f), (x, e, b).

Apply Step 2 to groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks (a, c, z), (e, d, z).

Case $n = 48$. Take the 3-GDD of type 2^6 with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{e}, f\}, \{\underline{c}, d\}, \{\underline{g}, h\}, \{\underline{m}, p\}, \{\underline{q}, r\}\}$ and $\mathcal{U} = \{(p, a, f), (p, e, h), (p, c, g), (a, e, r), (c, f, r), (a, d, h), (a, c, m), (a, g, q), (b, d, f), (b, c, h), (b, m, q), (b, r, p), (d, p, q), (d, r, g), (h, m, r), (e, g, b), (e, m, d), (e, q, c), (g, m, f), (q, h, f)\}$.

Apply Step 1 to the following sets of groups and blocks:

- $\{\underline{a}, b\}$, $\{\underline{e}, f\}$, $\{\underline{c}, d\}$, (p, a, f), (p, e, h), (p, c, g);
- $\{\underline{g}, h\}$, $\{\underline{m}, p\}$, $\{\underline{q}, r\}$, (e, g, b), (e, m, d), (e, q, c).

Apply Step 2 to the following sets of groups and blocks:

- $\{\underline{a}, b\}$, $\{\underline{e}, f\}$, $\{\underline{c}, d\}$, (a, e, r), (c, f, r);
- $\{\underline{g}, h\}$, $\{\underline{m}, p\}$, $\{\underline{q}, r\}$, (g, m, f), (q, h, f).

Case $n = 56$. Take the 3-GDD of type $2^4 6^1$ with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{c}, d\}, \{\underline{e}, f\}, \{\underline{g}, h\}, \{1, 2, 3, 4, 5, 6\}\}$ and $\mathcal{U} = \{(1, a, d)$, $(1, c, h)$, $(1, e, b)$, $(a, c, 3)$, $(e, h, 3)$, $(1, g, f)$ $(2, a, e)$, $(2, b, d)$, $(2, c, g)$, $(2, f, h)$, $(3, d, g)$, $(3, b, f)$, $(4, a, f)$, $(4, c, e)$, $(4, b, g)$, $(4, d, h)$ $(5, a, g)$, $(5, c, f)$, $(5, d, e)$, $(5, b, h)$, $(6, a, h)$, $(6, b, c)$, $(6, e, g)$, $(6, d, f)\}$.

Apply Step 1 to the groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks $(1, a, d)$, $(1, c, h)$, $(1, e, b)$. To complete the S_3-metamorphosis take $[g_1; g_0, h_2]$, $[g_2; g_3, h_2]$, $[1_1; f_3, g_1, g_2]$ and form the stars $[g_1; g_0, h_2, 1_1]$, $[g_2; g_3, h_2, 1_1]$ and the leave $\{(1_1, f_3)\}$.

Apply Step 2 to the groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks $(a, c, 3)$, $(e, h, 3)$. The result is a P_4-metamorphosis having leave $\{(g_0, g_2)\}$.

Case $n = 64$. Take the 3-GDD of type $2^3 4^1 6^1$ with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{d}, c\}, \{\underline{f}, e\}, \{x, y, z, t\}, \{1, 2, 3, 4, 5, 6\}\}$ and $\mathcal{U} = \{(1, a, x)$, $(1, d, y)$, $(1, f, z)$, $(a, d, 6)$, $(f, t, 6)$, $(1, c, b)$, $(1, e, t)$, $(2, x, e)$, $(2, y, a)$, $(2, z, b)$, $(2, t, d)$, $(2, c, f)$ $(3, x, d)$, $(3, y, e)$, $(3, z, c)$, $(3, t, b)$, $(3, a, f)$ $(4, x, f)$, $(4, y, b)$, $(4, z, a)$, $(4, t, c)$, $(4, d, e)$, $(5, x, c)$, $(5, y, f)$, $(5, z, d)$, $(5, t, a)$, $(5, b, e)$, $(6, x, b)$, $(6, y, c)$, $(6, z, e)$, (a, c, e), $(b, d, f)\}$.

Apply Step 1 to the groups $\{\underline{a}, b\}$, $\{\underline{d}, c\}$, $\{\underline{f}, e\}$, and blocks $(1, a, x)$, $(1, d, y)$, $(1, f, z)$.

Apply Step 2 to the groups $\{\underline{a}, b\}$, $\{\underline{d}, c\}$, $\{\underline{f}, e\}$, and blocks $(a, d, 6)$, $(f, t, 6)$.

To complete the proof we prove the case $n = 80$. We can proceed as above by applying the standard weighting construction to the 3-GDD of type $2^6 8^1$ with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{c}, d\}, \{\underline{e}, f\}, \{\underline{g}, h\}, \{\underline{n}, m\}, \{\underline{p}, q\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}$ and $\mathcal{U} = \{(1, a, d)$, $(1, c, h)$, $(1, e, q)$, $(a, c, 3)$, $(e, b, 3)$, $(2, e, n)$, $(2, f, p)$, $(2, m, q)$, $(2, d, h)$, $(2, b, c)$, $(2, a, g)$, $(3, d, f)$, $(3, g, m)$, $(3, n, q)$, $(3, h, p)$, $(4, e, m)$, $(4, n, p)$, $(4, a, h)$, $(4, b, d)$, $(4, c, g)$, $(4, f, q)$, $(5, d, m)$, $(5, c, e)$, $(5, a, q)$, $(5, g, p)$, $(5, b, f)$, $(5, h, n)$, $(6, a, f)$, $(6, c, n)$, $(6, d, e)$, $(6, g, q)$, $(6, b, p)$, $(6, h, m)$, $(7, a, m)$, $(7, c, p)$, $(7, d, q)$, $(7, e, g)$, $(7, b, n)$, $(7, f, h)$, $(8, a, n)$, $(8, c, q)$, $(8, e, h)$, $(8, f, g)$, $(8, b, m)$, (c, f, m), (a, e, p), (b, h, q), $(1, g, b)$, $(1, n, f)$, $(1, p, m)$, (g, n, d), $(p, 8, d)\}$.

Apply Step 1 to the following sets of groups and blocks:

- $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$, $(1, a, d)$, $(1, c, h)$, $(1, e, q)$;
- $\{\underline{g}, h\}$, $\{\underline{n}, m\}$, $\{\underline{p}, q\}$, $(1, g, b)$, $(1, n, f)$, $(1, p, m)$.

Apply Step 2 to the following sets of groups and blocks:

- $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$, $(a, c, 3)$, $(e, b, 3)$;
- $\{\underline{g}, h\}$, $\{\underline{n}, m\}$, $\{\underline{p}, q\}$, (g, n, d), $(p, 8, d)$.

□

Theorem 3.2.3. *There exists a kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \equiv 0, 1 \pmod 8$, $n \geq 9$. There exists a kite system of order 8 having a $\{K_3, P_4\}$-metamorphosis.*

Proof The necessary part is trivial, so we prove only the sufficient part. The proof for $n = 8, 9, 16, 17, 24, 32, 40, 48, 56, 64, 80$ follows from Examples 3.2.1, 3.4.1, 3.4.2, 3.4.3 and Lemma 3.2.2. For the remaining $n \geq 25$, apply the standard weighting construction by giving weight $w = 4$ to a 3-GDD as shown in Table 3.2. L_T, L_S, L_P are obtained by joining the leaves from the metamorphoses on each expanded group. □

n	k	3-GDD of type	L_T	L_S, L_P
24k	≥ 3	6^k	1-factor	$\emptyset$
24k+1	≥ 1	2^{3k}	$\emptyset$	$\emptyset$
24k+8	≥ 4	$6^{k-1}8$	1-factor	P_2
24k+9	≥ 1	2^{3k+1}	$\emptyset$	$\emptyset$
24k+16	≥ 3	6^k4	tripole	$\emptyset$
24k+17	≥ 1	$2^{3k}4$	C_4	P_2

Table 3.2: $\lambda = 1$ ($\emptyset$ denotes the empty graph)

Remark 3.2.1. Note that in Theorem 3.2.3 we have:

- for $n \equiv 8 \pmod{24}$, $n \geq 32$, L_T is an 1-factor which contains the edge (a_0, a_1), $L_S = (b_1, c_3)$ and $L_P = (a_0, a_2)$;
- for $n \equiv 17 \pmod{24}$, $L_T = (1, 2, 3, 16)$, $L_S = (2, 16)$, $L_P = (8, 16)$.

3.2.2 2-fold kite systems

Example 3.2.3. Let $2K_{2,2,2}$ be two copies of the complete tripartite graph with partition classes $V_1 = \{a_0, a_1\}$, $V_2 = \{b_0, b_1\}$ and $V_3 = \{c_0, c_1\}$. Let $\mathcal{B} = \{(c_0, a_0, b_1)-c_1, (b_0, a_0, c_1)-a_1, (a_0, c_1, b_1)-a_1, (b_0, a_0, c_0)-a_1, (b_0, a_1, c_0)-b_1, (b_0, c_1, a_1) - b_1\}$. Then $(V_1 \cup V_2 \cup V_3, \mathcal{B})$ is a 2-fold kite-decomposition of $2K_{2,2,2}$ having a $\{K_3, S_3, P_4\}$-metamorphosis with:

- $\mathcal{T} = \{(a_1, b_1, c_1), (a_1, b_1, c_0)\}$;
- $\mathcal{S} = \{[a_0; b_0, c_1, c_0], [b_0; a_0, a_1, c_1]\}$;
- $\mathcal{P} = \{[c_0, a_0, b_1, c_1], [a_0, c_1, a_1, c_0]\}$.

Theorem 3.2.4. *There exists a 2-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \equiv 0, 1 \pmod 4$, $n \geq 4$.*

Proof The proof of the necessary part is trivial, so we prove only the sufficient part. The proof for $n = 4, 5, 8, 9$ follows from Examples 3.4.5, 3.4.6, 3.4.7, 3.4.8.

Let $n \equiv 0 \pmod 4$, $n \geq 12$. Put $n = 4k$. Let $(S, \mathcal{G}, \mathcal{U})$ be a 3-GDD of type 2^k if $k \equiv 0, 1 \pmod 3$, $2^{k-2}4^1$ if $k \equiv 2 \pmod 3$. Apply the standard weighting construction by giving weight $w = 2$. By Examples 3.2.3, 3.4.5 and 3.4.7 we obtain the proof.

Let $n \equiv 1 \pmod 4$, $n \geq 13$. Put $n = 1 + 4k$. Let $(S, \mathcal{G}, \mathcal{U})$ be a 3-GDD of type 2^k if $k \equiv 0, 1 \pmod 3$, $2^{k-2}4$ if $k \equiv 2 \pmod 3$ having groups $G_1 = \{1, 2\}, G_2 = \{3, 4\}, \ldots, G_k = \{2k-1, 2k\}$ or $G_1 = \{1, 2\}, G_2 = \{3, 4\}, \ldots, G_{k-1} = \{2k-3, 2k-2, 2k-1, 2k\}$, respectively. Let $X = \{\infty\} \cup (S \times \mathbb{Z}_2)$, then $|X| = 4k + 1 = n$. We define a 2-fold kite system as follows:

1. For each G_i, let $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ be a copy of the 2-fold kite-system in Example 3.4.6 obtained by renaming its vertices as follows: $0 \to \infty, 1 \to (2i-1)_0, 2 \to (2i)_0, 3 \to (2i-1)_1, 4 \to (2i)_1$, with $1 \leq i \leq k$ if $k \equiv 0, 1 \pmod 3$ and $1 \leq i \leq k-2$ if $k \equiv 2 \pmod 3$; in the latter case, for $i = k-1$, take a copy of the system in Example 3.4.8 by renaming its vertices as follows: $j \to (2k-4+j)_0$, if $1 \leq j \leq 4$, $j \to (2k-8+j)_1$, if $5 \leq j \leq 8$, $0 \to \infty$.

2. For each $U = (a, b, c) \in \mathcal{U}$, let $((a \times \mathbb{Z}_2) \cup (b \times \mathbb{Z}_2) \cup (c \times \mathbb{Z}_2), \mathcal{B}_U)$ be the 2-fold kite-system, given in Example 3.2.3.

Let $\mathcal{B} = (\bigcup_{G\in\mathcal{G}} \mathcal{B}_G) \cup (\bigcup_{U\in\mathcal{U}} \mathcal{B}_U)$. Then $(X, \mathcal{B})$ is a 2-fold kite system of order n. Now we show that

- $(X, \mathcal{B})$ has a K_3-metamorphosis. To prove it note that $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ has a K_3-metamorphosis whose leave L_T^i is $\{2((2i-1)_1, (2i)_1)\}$ (the empty set) if the size of the starting group G_i is 2 (4 respectively). The set of tails of the blocks of the 2-fold kite system placed on $(a, b, c) \in \mathcal{U}$ is $\{2(a_1, b_1), (a_1, c_1), (b_1, c_1), (a_1, c_0), (b_1, c_0)\}$. Using three of them construct the triangle (a_1, b_1, c_0). The edges (a_1, b_1), (a_1, c_1), (b_1, c_1) can be assembled with $\bigcup L_T^i$ as follows:

  1. if $k \equiv 0 \pmod 3$, $(\bigcup_{i=1}^{k} L_T^i) \cup (\bigcup_{U\in\mathcal{U}} U \times \{1\}) = \{(1_1, 2_1), (3_1, 4_1), \ldots, ((2k-1)_1, (2k)_1)\} \cup K_{2k}$, where K_{2k} is the complete graph on vertex set $S \times \{1\}$. Since an 1-factor is the padding of a minimum covering with triangles of order $2k \equiv 0 \pmod 6$ (see [49]), there exists a K_3-decomposition of $(\bigcup_{i=1}^{k} L_T^i) \cup (\bigcup_{U\in\mathcal{U}} U \times \{1\})$ with empty leave;
  2. if $k \equiv 1 \pmod 3$, $(\bigcup_{i=1}^{k} L_T^i) \cup (\bigcup_{U\in\mathcal{U}} U \times \{1\})=\{((2i-1)_1, (2i)_1, (2k)_1), ((2i-1)_1, (2i)_1, (2k-1)_1) \mid 1 \le i \le k-1\} \cup \{2((2k-1)_1, (2k)_1)\} \cup K_2^{k-1}$, where K_2^{k-1} is the complete $(k-1)$-partite graph with partition classes $G_1, G_2, \ldots, G_{k-1}$. Since there exists a 3-GDD of type 2^{k-1} (see [11]), with $k \equiv 1 \pmod 3$, $(\bigcup_{i=1}^{k} L_T^i) \cup (\bigcup_{U\in\mathcal{U}} U \times \{1\})$ is decomposable into triangles with leave $\{2((2k-1)_1, (2k)_1)\}$;
  3. if $k \equiv 2 \pmod 3$, $(\bigcup_{i=1}^{k-2} L_T^i) \cup (\bigcup_{U\in\mathcal{U}} U \times \{1\})= \{((2i-1)_1, (2i)_1, (2k)_1), ((2i-1)_1, (2i)_1, (2k-1)_1) \mid 1 \le i \le k-2\} \cup K_2^{k-1}$, where K_2^{k-1} is the complete $(k-1)$-partite graph with partition classes $G_1, G_2, \ldots, G_{k-2}, \{(2k-3)_1, (2k-2)_1\}$. Since there exist a 3-GDD of type 2^{k-1} (see [11]), with $k \equiv 2 \pmod 3$, $(\bigcup_{i=1}^{k-1} L_T^i) \cup (\bigcup_{U\in\mathcal{U}} U \times \{1\})$ is decomposable into triangles with empty leave.

- $(X, \mathcal{B})$ has an S_3-metamorphosis. To prove it note that $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ has a S_3-metamorphosis whose leave L_S^i is $\{(\infty, (2i-1)_0), (\infty, (2i)_0)\}$, for $1 \le i \le k$ or $1 \le i \le k-2$ (if $k \equiv 2 \pmod 3$). These edges can be assembled into 3-stars $[\infty; i_0, (i+1)_0, (i+2)_0]$ with $i \equiv 1 \pmod 3$. The leave is empty if $k \equiv 0, 2 \pmod 3$ and $\{(\infty, (2k-1)_0), (\infty, (2k)_0)\}$ if $k \equiv 1 \pmod 3$.

- $(X, \mathcal{B})$ has a P_4-metamorphosis. To prove it note that $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ has a P_4-metamorphosis whose leave L_P^i is $\{(\infty, (2i)_1), ((2i)_0, (2i-1)_1)\}$, for $1 \leq i \leq k$ or $1 \leq i \leq k-2$ (if $k \equiv 2 \pmod 3$). For each i, with $i \neq k$ if $k \equiv 1 \pmod 3$, remove the path $[(2i)_0, (2i)_1, \infty, (2i-1)_1]$ and let Γ the set of edges covered by these paths and by $\bigcup L_P^i$. Construct the following paths, for $i \equiv 1 \pmod 3$: $[(2i)_0, (2i)_1, \infty, (2i+4)_1]$, $[(2i+2)_0, (2i+2)_1, \infty, (2i)_1]$, $[(2i+3)_1, (2i+4)_0, (2i+4)_1, \infty]$, $[(2i)_0, (2i-1)_1, \infty, (2i+3)_1]$, $[(2i+2)_0, (2i+1)_1, \infty, (2i+2)_1]$. The above paths cover all edges in Γ if $k \equiv 0, 2 \pmod 3$ and all edges in $\Gamma \setminus \{(\infty, (2k)_1), ((2k)_0, (2k-1)_1)\}$ if $k \equiv 1 \pmod 3$. It follows that $L_P = \emptyset$ if $k \equiv 0, 2 \pmod 3$ and $L_P = \{(\infty, (2k)_1), ((2k)_0, (2k-1)_1)\}$ if $k \equiv 1 \pmod 3$.

□

Remark 3.2.2. Note that the nonemty leaves of the 2-fold kite systems having a $\{K_3, S_3, P_4\}$-metamorphosis constructed in this section are as follows:

- if $n \equiv 5 \pmod{12}$, $L_T = 2(a, b)$, $L_S = [c; e, d]$, $L_P = \{(c, b), (a, e)\}$;
- if $n \equiv 8 \pmod{12}$, $L_T = 2(a, b)$, $L_S = [c; e, d]$, $L_P = \{(c, b), (f, e)\}$.

3.2.3 3-fold kite systems

Theorem 3.2.5. *There exists a 3-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \equiv 0, 1 \pmod 8$, $n \geq 8$.*

Proof The necessary part is trivial. The sufficiency for $n = 8$ is given in Example 3.4.9. Now construct on the same set X of size $n \geq 9$ a copy $(X, \mathcal{B}_1)$ of the kite system given in Section 2.1 and a copy $(X, \mathcal{B}_2)$ of the 2-fold kite system given in Section 2.2. It is clear that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ have a $\{K_3, S_3, P_4\}$-metamorphosis. Denote by $\mathcal{T}^i, \mathcal{S}^i, \mathcal{P}^i, L_T^i, L_S^i, L_P^i$ the sets $\mathcal{T}, \mathcal{S}, \mathcal{P}, L_T, L_S, L_P$ corresponding to $(X, \mathcal{B}_i)$, $i = 1, 2$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 3-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis. To prove this it is sufficient to put $\mathcal{T} = \mathcal{T}^1 \cup \mathcal{T}^2$, $\mathcal{S} = \mathcal{S}^1 \cup \mathcal{S}^2$, $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2$ and:

- for $n \equiv 1, 9 \pmod{24}$, $n \geq 9$, $L_T^1 = L_T^2 = L_S^1 = L_S^2 = L_P^1 = L_P^2 = \emptyset$. Then $L_T = L_S = L_P = \emptyset$;
- for $n \equiv 0 \pmod{24}$, $n \geq 24$, $L_S^1 = L_P^1 = L_S^2 = L_P^2 = \emptyset$, $L_T^1 =$ 1-factor, $L_T^2 = \emptyset$. Then $L_S = L_P = \emptyset$, $L_T = L_T^1$ is an 1-factor;

- for $n \equiv 16 \pmod{24}$, $L_S^1 = L_P^1 = L_S^2 = L_P^2 = \emptyset$, L_T^1 is a tripole and $L_T^2 = \emptyset$. Then $L_S = L_P = \emptyset$ and L_T is a tripole;

- for $n \equiv 17 \pmod{24}$, by Remarks 3.2.1 and 3.2.2 the leaves are of the type: $L_T^1 = \{(1,2,3,16)\}$, $L_S^1 = \{(2,16)\}$, $L_P^1 = \{(16,8)\}$, $L_T^2 = \{2(a,b)\}$, $L_S^2 = \{[d,c,e]\}$, $L_P^2 = \{(c,b),(a,e)\}$. Construct the required 2-fold kite system by renaming c,a,b,d,e as follows: $c \to 16, a \to 1, b \to 3, d \to 0, e \to 8$. The leaves can be reassembled into the triangles $(1,2,3),(1,3,16)$, the star $[16;2,0,8]$, the path $[3,16,8,1]$. Then $L_T = L_S = L_P = \emptyset$;

- for $n \equiv 8 \pmod{24}$, $n \geq 32$, by Remarks 3.2.1 and 3.2.2 the leaves are of the type: L_T^1 = 1-factor containing the edge (a_0,a_1), $L_S^1 = \{(b_1,c_3)\}$, $L_P^1 = \{(a_0,a_2)\}$, $L_T^2 = \{2(a,b)\}$, $L_S^2 = \{(c,d),(c,e)\}$, $L_P^2 = \{(c,b),(f,e)\}$. Construct the required kite system by renaming b_1, c_3, a_1, a_0, a_2 as follows: $b_1 \to c$, $c_3 \to f$, $a_1 \to a$, $a_0 \to b$, $a_2 \to e$. The leaves can be assembled into the star $[c;d,e,f]$ and the path $[c,b,e,f]$. Then $L_S = L_P = \emptyset$, L_T contains the 3-times repeated edge $3(a,b)$ and an 1-factor on the vertices $X \setminus \{a,b\}$.

□

3.2.4 4-fold kite systems

Example 3.2.4 ($4(K_6 \setminus K_2)$)**.** Let $X = \{\infty_1,\infty_2,0,1,2,3\}$, $\mathcal{B} = \{(1,2,\infty_1) - 3,\ (2,3,\infty_1) - 1,\ (0,3,\infty_2) - 2,\ (1,\infty_1,0) - \infty_2,\ (\infty_2,1,2) - 3,\ (\infty_1,0,3) - \infty_2,\ (\infty_2,0,1) - 3,\ (3,1,\infty_1) - 2,\ (3,\infty_2,2) - 0,\ (2,\infty_2,0) - \infty_1,\ (3,\infty_2,1) - 2,\ (2,\infty_1,0) - 3,\ (1,0,2) - 3,\ (0,3,1) - \infty_2\}$. Then $(X,\mathcal{B})$ is a 4-fold kite-system of order 6 with hole $\{\infty_1,\infty_2\}$ having:

- a K_3-metamorphosis with $\mathcal{T} = \{(\infty_1,1,2),\ (\infty_1,0,3),\ (\infty_2,0,2),\ (\infty_2,1,3)\}$ and leave $\{(2,3),(2,3)\}$;

- an S_3-metamorphosis with $\mathcal{S} = \{[1;\infty_2,2,3],\ [\infty_2;0,2,3],\ [3;0,2,\infty_2],\ [0;1,\infty_1,3]\}$ and leave $\{(\infty_1,1),\ (\infty_1,2)\}$;

- a P_4-metamorphosis with $\mathcal{P} = \{[0,\infty_1,2,1],\ [\infty_1,0,3,1],\ [\infty_2,1,\infty_1,3],\ [3,\infty_2,2,0]\}$ and leave $\{(0,\infty_2),(0,1)\}$.

Example 3.2.5 ($4(K_7 \setminus K_3)$)**.** Let $X = \{\infty_1,\infty_2,\infty_3,4,5,6,7\}$, $\mathcal{B} = \{(\infty_1,7,4) - \infty_2,\ (\infty_1,5,6) - \infty_2,\ (5,\infty_2,7) - \infty_1,\ (6,\infty_3,7) - 5,\ (5,\infty_3,4) - 6,\ (7,\infty_2,4) -$

∞_1, $(5,\infty_2,6)-\infty_1$, $(5,7,\infty_1)-4$, $(6,7,\infty_3)-4$, $(\infty_3,5,4)-7$, $(7,\infty_1,4)-6$, $(\infty_1,5,6)-\infty_3$, $(5,\infty_3,7)-\infty_2$, $(6,7,\infty_2)-5$, $(5,\infty_2,4)-6$, $(5,\infty_3,4)-\infty_2$, $(5,\infty_1,6)-\infty_2$, $(\infty_3,7,6)-4\}$. Then $(X,\mathcal{B})$ is a 4-fold kite-system of order 7 with hole $\{\infty_1,\infty_2,\infty_3\}$ having:

- a K_3-metamorphosis with $\mathcal{T}=\{(\infty_2,4,6), (\infty_2,4,6), (\infty_1,4,6), (\infty_3,4,6), (\infty_1,4,7), (\infty_2,5,7)\}$;
- an S_3-metamorphosis with $\mathcal{S}=\{[7;\infty_1,5,6], [7;\infty_1,\infty_2,6], [5;\infty_1,\infty_2,\infty_3], [5;\infty_1,\infty_2,\infty_3], [5;\infty_1,\infty_2,\infty_3], [\infty_3;5,6,7]\}$;
- a P_4-metamorphosis with $\mathcal{P}=\{[4,7,\infty_2,6], [\infty_2,4,\infty_3,7], [\infty_2,4,\infty_3,7], [6,5,4,\infty_1], [6,\infty_1,7,\infty_3], [\infty_2,7,6,5]\}$.

Theorem 3.2.6. *For every $n\geq 4$ there exists a 4-fold kite system having a $\{K_3,S_3,P_4\}$-metamorphosis. Moreover for $n\equiv 2,5 \pmod 6$ L_T is either a 4-cicle or a $2P_3$.*

Proof If $n\equiv 0,1 \pmod 4$, let $(X,\mathcal{B})$ be the 2-fold kite system of order n, having a $\{K_3,S_3,P_4\}$-metamorphosis, constructed in Section 2.2. By Remark 3.2.2, for $n\equiv 5,8 \pmod{12}$ it is $L_T^1=\{2(a,b)\}$, $L_S^1=\{(c,d),(c,e)\}$, $L_P^1=\{(c,b),(e,f)\}$ with $|\{a,b,c,d,e\}|=5$ and $|\{b,c,e,f\}|=4$ (note that for $n\equiv 5 \pmod{12}$ in L_P^1 it is $f=a$). Let $\mathcal{B}'$ be the block set obtained by changing b with e in each block of $\mathcal{B}$. Then $(X,\mathcal{B}')$ is a 2-fold kite system having a $\{K_3,S_3,P_4\}$-metamorphosis with empty leaves or, for $n\equiv 5,8 \pmod{12}$, $L_T^2=\{2(a,e)\}$, $L_S^2=\{(c,d),(c,b)\}$, $L_P^2=\{(c,e),(b,f)\}$. Then $(X,\mathcal{B}\cup\mathcal{B}')$ is a 4-fold kite system of order n having a $\{K_3,S_3,P_4\}$-metamorphosis with empty leaves or, for $n\equiv 5,8 \pmod{12}$, $L_T=\{2[b,a,e]\}$, $L_S=\{(c,d)\}$, $L_P=\{(b,f)\}$ and $[c;b,e,d]\in\mathcal{S}$, $[f,e,c,b]\in\mathcal{P}$.

If $n\equiv 2,3 \pmod 4$, $n=4k+s$, $k\geq 3$ and $s\in\{2,3\}$, let $S=\{1,2,\ldots,2k\}$, $R_s=\{\infty_1,\ldots,\infty_s\}$ and $(\mathcal{S},\mathcal{G},\mathcal{U})$ a 3-GDD of type 2^k(if $k\equiv 0,1 \pmod 3$) or $2^{k-2}4$ (if $k\equiv 2 \pmod 3$), with groups $G_1=\{1,2\}, G_2=\{3,4\},\ldots,G_k=\{2k-1,2k\}$ or $G_1=\{1,2,3,4\}, G_2=\{5,6\},\ldots,G_{k-1}=\{2k-1,2k\}$, respectively. Set $X=R_s\cup(S\times\mathbb{Z}_2)$ and define a collection $\mathcal{B}$ of kites as follows:

1. Let $(R_s\cup(G_1\times\mathbb{Z}_2),\mathcal{B}_{G_1})$ be a copy of the 4-fold kite system of order $2|G_1|+s$ given in Examples 3.4.11,3.4.12,3.4.14,3.4.15 having a $\{K_3,S_3,P_4\}$-metamorphosis with leaves L_T^1, L_S^1, L_P^1; put $\mathcal{B}_{G_1}\subseteq\mathcal{B}$.
2. For every $U=(x,y,z)\in\mathcal{U}$, let $(S_U,\mathcal{B}_U)$ be a copy of the $2K_{2,2,2}$ kite-decomposition of Example 3.2.3; put $2\mathcal{B}_U\subseteq\mathcal{B}$.

3. For every $G_i \in \mathcal{G}$, with $i > 1$, construct a 4-fold kite system $(R_s \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ of order $2|G_i| + s$ with a hole of size s by taking a copy of the designs in Examples 3.2.4, 3.2.5 and renaming the vertices $0, 1, 2, 3$ of Example 3.2.4 and $4, 5, 6, 7$ of Example 3.2.5 as follows:

   - if $s = 2, k \equiv 0, 1 \pmod 3$: $0 \to (2i-1)_0$, $1 \to (2i)_0$, $2 \to (2i-1)_1$, $3 \to (2i)_1$;
   - if $s = 2, k \equiv 2 \pmod 3$: $0 \to (2i+1)_0$, $1 \to (2i+2)_0$, $2 \to (2i+1)_1$, $3 \to (2i+2)_1$;
   - if $s = 3, k \equiv 0, 1 \pmod 3$: $4 \to (2i-1)_1$, $5 \to (2i)_1$, $6 \to (2i-1)_0$, $7 \to (2i)_0$;
   - if $s = 3, k \equiv 2 \pmod 3$: $4 \to (2i+1)_1$, $5 \to (2i+2)_1$, $6 \to (2i+1)_0$, $7 \to (2i+2)_0$.

Denote the leaves by L^i_T, L^i_S, L^i_P. Put $\mathcal{B}_{G_i} \subseteq \mathcal{B}$. Then $(X, \mathcal{B})$ is a 4-fold kite system of order $4k + s$. The metamorphoses are obtained as follows: apply the metamorphoses showed in steps 1. 2. 3. to the blocks of $\mathcal{B}_{G_1}, 2\mathcal{B}_U$ and $\mathcal{B}_{G_i}, i > 1$, respectively. In order to complete our metamorphoses and so to obtain the leaves L_T, L_S, L_P, proceed as follows:

- For $s = 3$ and $k \equiv 0, 1 \pmod 3$, we have $L^i_T = L^i_S = L^i_P = \emptyset$, for all $G_i \in \mathcal{G}$. Then $L_T = L_S = L_P = \emptyset$.

- For $s = 3$ and $k \equiv 2 \pmod 3$, we have $L^i_T = L^i_S = L^i_P = \emptyset$, for $i > 1$, then $L_T = L^1_T, L_S = L^1_S, L_P = L^1_P$.

- For $s = 2$ and $k \equiv 0, 1 \pmod 3$, we have $L^1_T = L^1_S = L^1_P = \emptyset$. Moreover

  - in the K_3-metamorphosis, it is $\bigcup_{i=2}^k L^i_T = \{2(3_1, 4_1), 2(5_1, 6_1), \dots, 2((2k-1)_1, (2k)_1)\}$. Remove from $2\mathcal{B}_U$ the blocks $2(x_1, y_1, z_1)$ for each $(x, y, z) \in \mathcal{U}$. These blocks and the edges in $\bigcup_{i=2}^k L^i_T$ cover the graph $2(K_{2k} \setminus K_2)$ on vertex set $S \times \{1\}$ with the hole $\{1_1, 2_1\}$. For $k \equiv 0 \pmod 3$, take a decomposition $(S \times \{1\}, \mathcal{T}')$ of $2K_{2k}$ into triangles (see Section 2.2) such that $\{(1_1, 2_1, y_1), (1_1, 2_1, z_1)\} \subseteq \mathcal{T}'$, with $y_1 \neq z_1$. Delete the edges $2(1_1, 2_1)$. The result is a maximum packing of $2(K_{2k} \setminus K_2)$ with triangles having the 4-cycle $(1_1, y_1, 2_1, z_1)$ as leave; we have $L_T = \{(1_1, y_1, 2_1, z_1)\}$. For $k \equiv 1 \pmod 3$, take a decomposition of $2K_{2k}$ on vertex set $S \times \{1\}$ with leave $\{2(1_1, 2_1)\}$. The result is a decomposition of $2(K_{2k} \setminus K_2)$ into triangles. Then we have $L_T = \ emptyset.$

- in the S_3-metamorphosis, it is $\bigcup_{i=2}^{k} L_S^i = \{(\infty_1, 4_0), (\infty_1, 3_1), (\infty_1, 6_0), (\infty_1, 5_1), \ldots, (\infty_1, (2k)_0), (\infty_1, (2k-1)_1)\}$. The edges of $\bigcup_{i=2}^{k} L_S^i$ can be assembled into stars $[\infty_1; i_0, (i+2)_0, (i+4)_0]$, $i = 4 + 6h, h \geq 0$ and $[\infty_1; i_1, (i+2)_1, (i+4)_1]$, $i = 3 + 6h, h \geq 0$. It is easy to verify that $L_S = \emptyset$, if $k \equiv 1 \pmod 3$, or $L_S = \{(\infty_1, (2k)_0)\}$, if $k \equiv 0 \pmod 3$.
- in the P_4-metamorphosis, it is $L_P^i = \{(\infty_2, (2i-1)_0), ((2i-1)_0, (2i)_0)\}$, $i \geq 2$. For every $i = 2, 3, \ldots, k$ remove the path $[(2i-1)_0, \infty_1, (2i-1)_1, (2i)_0]$. Let Γ be the set of edges covered by these paths and by $\bigcup_{i=2}^{k} L_P^i$. Construct the following paths with $i \equiv 2 \pmod 3$: $[\infty_1, (2i-1)_1, (2i)_0, (2i-1)_0]$, $[\infty_1, (2i+1)_1, (2i+2)_0, (2i+1)_0]$, $[\infty_1, (2i+3)_1, (2i+4)_0, (2i+3)_0]$, $[\infty_1, (2i+1)_0, \infty_2, (2i+1)_0]$, $[(2i+1)_0, \infty_1, (2i+3)_0, \infty_2]$. The above paths cover all edges in Γ if $k \equiv 1 \pmod 3$ or all edges in $\Gamma \setminus \{((2k-1)_0, (2k)_0)\}$ if $k \equiv 0 \pmod 3$. It follows that $L_P = \emptyset$ for $k \equiv 1 \pmod 3$ and $L_P = \{((2k-1)_0, (2k)_0)\}$ for $k \equiv 0 \pmod 3$.

- $s = 2, k \equiv 2 \pmod 3$. $L_T^1 = L_S^1 = L_P^1 = \emptyset$; leaves of the other groups are:

  - in the K_3-metamorphosis, it is $\bigcup_{i=2}^{k-1} L_T^i = \{2(5_1, 6_1), 2(7_1, 8_1), \ldots, 2((2k-1)_1, (2k)_1)\}$. Remove from $2\mathcal{B}_U$ the blocks $2(x_1, y_1, z_1)$ for each $(x, y, z) \in \mathcal{U}$. These blocks and the edges in $\bigcup_{i=2}^{k-1} L_T^i$ cover the graph $2K_{2k}$ on vertex set $S \times \{1\}$ with the hole $\{1_1, 2_1, 3_1, 4_1\}$. Then a maximum packing of $2(K_{2k} \setminus K_4)$ with triangles with leave empty (see [20]) completes the K_3-metamorphosis.
  - in the S_3-metamorphosis, $\bigcup_{i=2}^{k-1} L_S^i = \{(\infty_1, 6_0), (\infty_1, 5_1), (\infty_1, 8_0), (\infty_1, 7_1), \ldots, (\infty_1, (2k)_0), (\infty_1, (2k-1)_1)\}$. The edges of $\bigcup_{i=2}^{k-1} L_S^i$ can be assembled into the 3-stars $[\infty_1; i_0, (i+2)_0, (i+4)_0]$, $i = 6h$, $h \geq 1$ and $[\infty_1; i_1, (i+2)_1, (i+4)_1]$ with $i = 5 + 6h, h \geq 0$. Therefore the leave is empty.
  - in the P_4-metamorphosis, it is $L_P^i = \{(\infty_2, (2i+1)_0), ((2i+1)_0, (2i+2)_0)\}, 2 \leq i \leq k-1$. For every $i = 2, 3, \ldots, k-1$ remove the path $[(2i+1)_0, \infty_1, (2i+1)_1, (2i+2)_0]$. Let Γ be the set of

edges covered by these paths and by $\bigcup_{i=2}^{k-1} L_P^i$. Construct the following paths with $i \equiv 2 \pmod 3$: $[\infty_1, (2i+1)_1, (2i+2)_0, (2i+1)_0]$, $[\infty_1, (2i+3)_1, (2i+4)_0, (2i+3)_0]$, $[\infty_1, (2i+5)_1, (2i+6)_0, (2i+5)_0]$, $[\infty_1, (2i+1)_0, \infty_2, (2i+3)_0]$, $[(2i+3)_0, \infty_1, (2i+5)_0, \infty_2]$. The above paths cover all edges in Γ, because $k \equiv 2 \pmod 3$ and so $2k-4 \equiv 0 \pmod 3$. Therefore the leave is empty.

□

Remark 3.2.3. The nonempty leaves of $\{K_3, S_3, P_4\}$-metamorphoses constructed in this section are

- if $n \equiv 5 \pmod 6$ or $n \equiv 8 \pmod{12}$, $L_T = 2[a, e, b]$, $L_S = (f, c)$, $L_P = (e, c)$;
- if $n \equiv 2 \pmod{12}$ and $n \geq 14$, $L_T = (a, b, c, d)$, $L_S = (e, f)$, $L_P = (f, g)$.

3.2.5 λ-fold kite systems

Lemma 3.2.7. *For every $n \geq 4$ there exists a 12-fold kite system of order n having $\{K_3, S_3, P_4\}$-metamorphosis with empty leaves.*

Proof If $n \equiv 0, 1, 3, 4 \pmod 6$, combine 3 copies of the 4-fold kite system constructed in Section 2.4. If $n \equiv 5 \pmod 6$ or $n \equiv 8 \pmod{12}$, let $(X, \mathcal{B}_1)$ be the 4-fold kite system of order n constructed in Section 2.4. By Remark 3.2.3, we can suppose $L_T^1 = \{2[d, a, e]\}$, $L_S^1 = \{(b, c)\}$, $L_P^1 = \{(a, c)\}$. Applying the permutation $\varphi = (a, d, e, b)$ ($\psi = (e, d, b)$) to the vertices of X, we obtain the 4-fold kite system $(X, \mathcal{B}_2)$ ($(X, \mathcal{B}_3)$ respectively) having a $\{K_3, S_3, P_4\}$-metamorphosis with $L_T^2 = \{2[e, d, b]\}$, $L_S^2 = \{(a, c)\}$, $L_P^2 = \{(d, c)\}$ and $L_T^3 = \{2[b, a, d]\}$, $L_S^3 = \{(e, c)\}$, $L_P^3 = \{(a, c)\}$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a 12-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the edges of $L_T^1 \cup L_T^2 \cup L_T^3$ into the triangles $2(a, d, e), 2(a, b, d)$, the edges of $L_S^1 \cup L_S^2 \cup L_S^3$ into the star $[c; a, b, e]$, the edges of $L_P^1 \cup L_P^2 \cup L_P^3$ into the path $[d, c, a, e]$.

If $n \equiv 2 \pmod{12}$, $n \geq 14$, let $(X, \mathcal{B}_1)$ be the 4-fold kite system of order n given in Section 2.4. By Remark 3.2.3, we can suppose $L_T^1 = \{(a, b, c, d)\}$, $L_S^1 = \{(e, f)\}$, $L_P^1 = \{(f, g)\}$. Let h, m be two vertices distinct from a, b, c, d, e, f, g. Applying the permutation $\varphi = (f, g, h)$ and changing b with c, we obtain a 4-fold kite system $(X, \mathcal{B}_2)$ of order n with $L_T^2 = \{(a, c, b, d)\}$, $L_S^2 = \{(e, g)\}$, $L_P^2 = \{(g, h)\}$. Applying to $(X, \mathcal{B}_1)$ the permutation $\varphi = (b, c, d)$ and changing g with m and f with h, we obtain a 4-fold kite system $(X, \mathcal{B}_3)$ of order n with $L_T^3 = \{(a, c, d, b)\}$, $L_S^3 = \{(e, h)\}$, $L_P^3 = \{(h, m)\}$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a 12-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the edges of $L_T^1 \cup L_T^2 \cup L_T^3$ into the triangles (a, c, b), (a, c, d), (c, d, b), (a, b, d), the edges of $L_S^1 \cup L_S^2 \cup L_S^3$ into the star $[e; f, g, h]$, the edges of $L_P^1 \cup L_P^2 \cup L_P^3$ into the path $[m, h, g, f]$. □

Theorem 3.2.8. *There exists a λ-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod 8$, $(\lambda, n) \neq (1, 8)$. There exists a kite system of order 8 having a $\{K_3, P_4\}$-metamorphosis*

Proof The necessity is trivial. For $1 \leq \lambda \leq 4$ the proof follows from Sections 2.1, 2.2, 2.3, 2.4. Let $\lambda \geq 5$ and $n \geq 4$, with $\lambda n(n-1) \equiv 0 \pmod 8$. If $n = 8$ and $\lambda = 5, 7$ the proof follows from Examples 3.4.16 and 3.4.18. If $n = 5$ and $\lambda = 6$, the proof follows from Example 3.4.17.

Let F_n be a 1-factor of K_n containing the edges $(a, d), (b, c)$. Define the following set of edges: $T_n = [a; b, c, d] \cup (F_n \setminus \{(a, d), (b, c)\})$, $2P_3 = 2[b, a, c]$, $C_4 = (a, b, d, c)$ and $2P_2 = 2(b, c)$. Put $A = 2P_3 \cup F_n = (a, b, c) \cup T_n$, $C = 2P_3 \cup 2P_2 = 2(a, b, c)$; $F = 2[a, c, b] \cup 2[a, b, c] = 2(a, b, c) \cup 2P_2$, $H = (a, b, c, d) \cup (a, d, b, c) = \{(a, b, d), (a, c, d)\} \cup 2P_2$.
Let $5 \leq \lambda \leq 11$. Combine a suitable λ_1-fold kite system having $\{K_3, S_3, P_4\}$-metamorphosis (with leaves L_T^1, L_S^1, L_P^1) and a suitable λ_2-fold kite-system having $\{K_3, S_3, P_4\}$-metamorphosis (with leaves L_T^2, L_S^2, L_P^2), for suitable values of λ_1 and λ_2, and replace the leaves where it is necessary (see Table 3). For example, for $\lambda = 6$ and $n \equiv 5, 8 \pmod{12}, n \geq 8$, let $(X, \mathcal{B}_1)$ be a copy of the 2-fold kite-system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis given in Section 2.2 and let $(X, \mathcal{B}_2)$ be a copy of the 4-fold kite-system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis given in Section 2.4. Therefore, by Remarks 3.2.2, 3.2.3, we can suppose $L_T^1 = 2(a, b)$, $L_S^1 = [c; d, e]$, $L_P^1 = \{(c, b), (e, f)\}$ and $L_T^2 = 2[a, e, b]$, $L_S^2 = (f, c)$, $L_P^2 = (e, c)$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 6-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the edges of $L_T^1 \cup L_T^2 \cup L_T^3$ into the triangles $2(a, e, b)$, the edges of $L_S^1 \cup L_S^2 \cup L_S^3$ into the star $[c; d, e, f]$, the edges of $L_P^1 \cup L_P^2 \cup L_P^3$ into the path $[f, e, c, b]$ ($f = a$ if $n \equiv 5 \pmod{12}$).

For $\lambda = 12$ the proof follows from Lemma 3.2.7. Let $\lambda \equiv 1 \pmod 6), \lambda \geq 13$. Write $\lambda = 6k + 7$ and combine k copies of a 6-fold kite-system having a $\{K_3, S_3, P_4\}$-metamorphosis with a 7-fold kite-system having a $\{K_3, S_3, P_4\}$-metamorphosis. For each $\lambda = 12k+h$, with $0 \leq h \leq 11$ and $h \neq 1, 7$, combine k copies of a 12-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis with an h-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. $\square$

λ	$n \geq 4$	λ_1	λ_2	L_T^1	L_T^2	$L_T^1 \cup L_T^2$	L_T	L_S^1	L_S^2	L_S	L_P^1	L_P^2	L_P
5	$0 \pmod{24}$	1	4	F_n	$\emptyset$	F_n	F_n	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	$1, 9 \pmod{24}$	1	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	$16 \pmod{24}$	1	4	T_n	$\emptyset$	T_n	T_n	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	$8 \pmod{24}, n \neq 8$	1	4	F_n	$2P_3$	A	T_n	P_2	P_2	S_2	P_2	P_2	E_2
5	$17 \pmod{24}$	2	3	$2P_2$	$\emptyset$	$2P_2$	$2P_2$	S_2	$\emptyset$	S_2	E_2	$\emptyset$	E_2
6	$0, 1, 4, 9 \pmod{12}$	2	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	$5, 8 \pmod{12}, n \neq 5$	2	4	$2P_2$	$2P_3$	C	$\emptyset$	S_2	P_2	$\emptyset$	E_2	P_2	$\emptyset$
7	$17 \pmod{24}$	3	4	$\emptyset$	$2P_3$	$2P_3$	$2P_3$	$\emptyset$	P_2	P_2	$\emptyset$	P_2	P_2
7	$0, 1, 9, 16 \pmod{24}$	1	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	$8 \pmod{24}, n \neq 8$	1	6	L_T^1	$\emptyset$	L_T^1	L_T^1	P_2	$\emptyset$	P_2	P_2	$\emptyset$	P_2
8	$0, 1 \pmod{3}$	4	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	$2 \pmod{3}$	4	4	$2P_3$ or C_4	$2P_3$ or C_4	F or H	$2P_2$	P_2	P_2	S_2	P_2	P_2	E_2
9	$0, 1 \pmod{8}$	3	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
10	$0, 1, 4, 9 \pmod{12}$	4	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
10	$5, 8 \pmod{12}$	4	6	L_T^1	$\emptyset$	L_T^1	L_T^1	P_2	$\emptyset$	P_2	P_2	$\emptyset$	P_2
11	$0, 1, 9, 16 \pmod{24}$	5	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
11	$8, 17 \pmod{24}$	5	6	L_T^1	$\emptyset$	L_T^1	L_T^1	S_2	$\emptyset$	S_2	E_2	$\emptyset$	E_2

Table 3.3: $\lambda = 5, 6, 7, 8, 9, 10, 11$ ($\emptyset$ denotes the empty graph)

3.3 Proof of Main Theorem

Theorem 3.3.1. *Every λ-fold kite system of order 4 has an E_2-metamorphosis.*

Proof Let $(\mathbb{Z}_4, \mathcal{B})$ be a λ-fold kite system of order 4. Then $|\mathcal{B}| = \frac{3\lambda}{2}$. For each $B = (a, b, c) - d \in \mathcal{B}$, let $B_1 = \{(a, b), (c, d)\}$ and $B_1' = \{(a, c), (b, c)\}$. Let L be the graph $(\mathbb{Z}_4, \bigcup_{B \in \mathcal{B}} B_1')$. We denote by $d_G(x)$ the degree of the vertex x in the graph G. It is easy to check that for every $x \in \mathbb{Z}_4$ and for every $B \in \mathcal{B}$, $d_{B_1'}(x) = d_B(x) - 1$. Then $d_L(x) = d_{\lambda K_4}(x) - |\mathcal{B}| = 3\lambda - \frac{3\lambda}{2} = \frac{3\lambda}{2}$, for every $x \in \mathbb{Z}_4$. Therefore L is a regular graph. Suppose that the edge (x, y) appears α times in L. Let $\{z, t\} = \mathbb{Z}_4 \setminus \{x, y\}$. Then (z, t) appears α times in L, otherwise L couldn't be regular. Using the edges $(x, y), (z, t)$ costruct α E_2s. Since each B_1 is an E_2, the E_2-metamorphosis is trivially completed. $\square$

Theorem 3.3.2. *Every λ-fold kite system of order n, with $n \geq 10$ if $\lambda \geq 2$ has an E_2-metamorphosis.*

Proof Let $(X, \mathcal{B})$ be a λ-fold kite system of order n. Then $|\mathcal{B}| = \frac{\lambda n(n-1)}{8}$. For each $B = (a, b, c) - d \in \mathcal{B}$, let $B_1 = \{(a, b), (c, d)\}$ and $B_1' = \{(a, c), (b, c)\}$. Let $L = \bigcup_{B \in \mathcal{B}} B_1'$. The degree of each vertex of L is at most $\left\lfloor \lambda \frac{2(n-1)}{3} \right\rfloor$. Combine at random the edges of L into E_2s. The result is a set $\mathcal{E}$ of E_2s and a graph G having $2h \geq 0$ edges. For $h = 0$ the theorem is proved. Let $h > 0$. Then every two edges of G share a common vertex. Let $\mathcal{E}'_{vw} = \{E \in \mathcal{E} \mid E$ is not incident in $v, w\}$, $\mathcal{E}_{vw} = \{E \in \mathcal{E} \mid E$ is incident in v and $w\}$, $\mathcal{E}_v = \{E \in \mathcal{E} \mid E$ incident in $v\}$, $\mathcal{E}'_v = \{E \in \mathcal{E} \mid E$ is not incident in $v\}$. The following two cases arise:

Case 1. G is a star, possibly with repeated edges. Let $G = S_{2h} = [0; v_1, v_2, \ldots, v_{2h}]$.

Case 1a. Let $v_1 = v_2 = \ldots = v_{2h} = 1$. Then $|\mathcal{E}'_{01}| = |\mathcal{E}| - |\mathcal{E}_0| - |\mathcal{E}_1| + |\mathcal{E}_{01}| \geq |\mathcal{E}| - 2(\left\lfloor \lambda \frac{2(n-1)}{3} \right\rfloor - 2h) + (\lambda - 2h) \geq \lambda \frac{n(n-1)}{8} - h - \frac{4}{3}\lambda(n-1) + 4h + \lambda - 2h = \lambda(n-1)(\frac{3n-32}{24}) + h + \lambda$. It follows $|\mathcal{E}'_{01}| > h$ for $n \geq 10$. Choose h blocks $E \in \mathcal{E}'_{01}$. Combining each of this block with two edges $(0, 1)$ we complete the E_2-metamorphosis.

Case 1b. Let $|\{v_1, v_2, \ldots, v_{2h}\}| \geq 2$. Take v_i, v_j with $v_i \neq v_j$. Note that $|\mathcal{E}'_0| = |\mathcal{E}| - |\mathcal{E}_0| \geq \lambda \frac{n(n-1)}{8} - h - (\left\lfloor \lambda \frac{2(n-1)}{3} \right\rfloor - 2h) \geq \lambda \frac{n(n-1)}{8} - \lambda \frac{2(n-1)}{3} + h = \lambda \frac{(n-1)(3n-16)}{24} + h$. Then $|\mathcal{E}'_0| > \lambda + h$, for $n \geq 7$. Choose a block $\overline{E} \in \mathcal{E}'_0$ not containing the edge (v_i, v_j). It is possible to rearrange the edges $(0, v_i), (0, v_j)$

of S_{2h} with the edges of $\overline{E}$ in order to form two new E_2s. Remove $(0, v_i), (0, v_j)$ from S_{2h}, substitue $\overline{E}$ with the new E_2s in $\mathcal{E}$ and reapply the procedure, that will stop when S_{2h} is empty.

Case 2. G is a triangle with repeated edges. Suppose G contains the edges $m(0,1)$, $p(1,2)$, $q(2,0)$. Since $m + p + q = 2h$, at least one of m, p, q must be even. Suppose $m = 2k$. Then $|\mathcal{E}'_{01}| = |\mathcal{E}| - |\mathcal{E}_0| - |\mathcal{E}_1| + |\mathcal{E}_{01}| \geq |\mathcal{E}| - (\left\lfloor \lambda \frac{2(n-1)}{3} \right\rfloor - (m+q)) - (\left\lfloor \lambda \frac{2(n-1)}{3} \right\rfloor - (m+p)) + (\lambda - m) \geq \lambda \frac{n(n-1)}{8} - h - \frac{4}{3}\lambda(n-1)\lambda + 2h = \lambda(n-1)(\frac{3n-32}{24}) + h + \lambda$. Then $|\mathcal{E}'_{01}| > h > k$, for $n \geq 10$. Choose k blocks $E \in \mathcal{E}'_{01}$. Combine each of this block with two edges $(0,1)$. The left edges make a star $S_{2(h-k)} = [2; 0, 0, \ldots, 1, 1, \ldots]$ that we can assemble as in Case 1.

For $\lambda = 1$, only subcase 1b holds and $n \geq 8$, so every kite system has an E_2-metamorphosis. □

Main Theorem. *There exists a λ-fold kite system of order n having a complete simultaneous metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod 8$ and $(\lambda, n) \neq (1, 8)$. There is not a kite system of order 8 having an S_3-metamorphosis, but there is a kite system of order 8 having a $\{K_3, P_4, P_3, P_2, E_2\}$-metamorphosis.*

Proof Every λ-fold G-design has P_2-metamorphoses. Let $B = (a, b, c) - d$ be a block of a λ-fold kite system $(X, \mathcal{B})$. Decompose B into the two paths $[a, b, c]$ and $[a, c, d]$. Then every λ-fold kite system $(X, \mathcal{B})$ has a P_3-metamorphosis. Let $n \geq 4$, λ such that $\lambda n(n-1) \equiv 0 \pmod 8$. Let $(X, \mathcal{B})$ be the λ-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis (see Theorem 3.2.8) or, if $(\lambda, n) = (1, 8)$, having a $\{K_3, P_4\}$-metamorphosis. Then $(X, \mathcal{B})$ has a $\{K_3, S_3, P_4, P_3, P_2\}$-metamorphosis or, if $(\lambda, n) = (1, 8)$, a $\{K_3, P_4, P_3, P_2\}$-metamorphosis. $(X, \mathcal{B})$ has also an E_2-metamorphosis. This follows from Theorems 3.3.1 and 3.3.2 for $n = 4$, $n \geq 10$ and for $\lambda = 1, \forall n \equiv 0, 1 \pmod 8$. For the remaining values of n and λ, the E_2-metamorphosis of $(X, \mathcal{B})$ follows easily from the proof of Theorems 3.2.4, 3.2.5, 3.2.6, 3.2.8 and from the observation that the starting designs (see Examples 3.4.6, 3.4.7, 3.4.8, 3.4.9, 3.4.16, 3.4.18, 3.4.17) have also an E_2-metamorphosis. □

3.4 Appendix to Chapter 3

The following are λ-fold kite-systems of order n having a $\{K_3, S_3, P_4\}$-metamorphosis. Except otherwise specified, the vertex set is $\mathbb{Z}_n$.

Example 3.4.1 ($\lambda = 1, n = 9$). $\mathcal{B} = \{(1,3,4)-8,\ (1,0,6)-5,\ (1,8,7)-5,\ (1,2,5)-0,\ (2,7,4)-6,\ (0,2,8)-6,\ (3,2,6)-7,\ (8,3,5)-4,\ (3,7,0)-4\}$; $\mathcal{T} = \{(5,7,6),\ (8,4,6),\ (4,5,0)\}$; $\mathcal{S} = \{[1;8,3,0],[2;0,1,7],[3;2,8,7]\}$; $\mathcal{P} = \{[7,8,2,6],[4,3,5,2],[6,0,7,4]\}$.

Example 3.4.2 ($\lambda = 1, n = 16$). $\mathcal{B} = \{(4,0,9)-2,\ (11,0,5)-13,\ (3,0,1)-15,\ (1,5,10)-3,\ (12,1,6)-14,\ (4,1,2)-15,\ (6,2,11)-4,\ (13,2,7)-0,\ (0,2,14)-15,\ (7,3,12)-5,\ (3,14,8)-1,\ (6,3,4)-15,\ (8,4,13)-6,\ (12,8,2)-10,\ (7,8,10)-15,\ (14,9,5)-15,\ (7,9,6)-15,\ (8,9,11)-15,\ (6,10,0)-8,\ (14,10,4)-12,\ (10,12,9)-15,\ (11,7,1)-9,\ (11,10,13)-15,\ (11,14,12)-15,\ (9,13,3)-11,\ (13,12,0)-15,\ (1,13,14)-7,\ (4,5,7)-15,\ (5,6,8)-15,\ (2,5,3)-15\}$;
$\mathcal{T} = \{(2,10,15),\ (3,11,15),\ (7,14,15),\ (1,9,15),\ (0,8,15),\ (5,12,15),\ (6,13,15)\}$;
$\mathcal{S} = \{[0;4,3,11],\ [1;5,12,4],\ [2;6,13,0],\ [3;14,7,6],\ [8;4,12,7],\ [9;14,7,8],\ [10;6,14,12],\ [11;7,10,14],\ [13;9,12,1],\ [5;4,6,2]\}$;
$\mathcal{P} = \{[9,0,5,10],\ [1,0,10,13],\ [6,1,2,11],\ [7,2,14,8],\ [12,3,4,13],\ [2,8,10,4],\ [5,9,6,8],\ [11,9,12,0],\ [1,7,5,3],\ [12,14,13,3]\}$;
$L_T = \{[4;11,12,15],\ (1,8),\ (2,9),\ (3,10),\ (6,14),\ (0,7),\ (5,13)\}$.

Example 3.4.3 ($\lambda = 1, n = 17$). $\mathcal{B} = \{(1,4,14)-3,\ (1,5,7)-15,\ (1,10,8)-14,\ (15,2,5)-10,\ (2,6,8)-3,\ (2,11,9)-1,\ (3,4,6)-14,\ (3,7,9)-15,\ (3,10,12)-15,\ (11,4,7)-12,\ (10,4,2)-3,\ (8,4,13)-10,\ (8,5,12)-14,\ (5,11,3)-1,\ (5,9,14)-2,\ (6,13,9)-12,\ (6,12,1)-16,\ (6,10,15)-1,\ (7,13,2)-1,\ (13,11,1)-0,\ (15,13,3)-16,\ (16,5,4)-9,\ (16,10,11)-14,\ (16,6,7)-10,\ (16,13,12)-2,\ (9,16,8)-11,\ (16,14,15)-11,\ (0,5,6)-11,\ (0,11,12)-4,\ (0,8,7)-14,\ (0,14,13)-5,\ (4,0,15)-8,\ (0,9,10)-14,\ (16,2,0)-3\}$;
$\mathcal{T} = \{(1,3,0),\ (7,10,14),\ (8,11,15),\ (9,12,4),\ (8,14,3),\ (9,15,1),\ (10,13,5),\ (11,14,6),\ (12,15,7),\ (2,14,12)\}$;
$\mathcal{S} = \{[4;8,10,11],\ [2;6,11,15],\ [6;10,12,13],\ [5;8,9,11],\ [0;4,9,14],\ [0;5,8,11],\ [1;4,5,10],\ [3;4,7,10],\ [13;7,11,15],\ [16;5,6,10],\ [16;9,13,14]\}$;
$\mathcal{P} = \{[14,4,6,8],\ [7,5,2,0],\ [8,10,12,5],\ [7,4,2,13],\ [4,13,9,14],\ [9,11,1,12],\ [15,10,11,12],\ [0,15,14,13],\ [4,5,6,7],\ [10,9,7,8],\ [12,13,3,11]\}$;
$L_T = (1,2,3,16)$; $L_S = (8,16)$; $L_P = (8,16)$.

Example 3.4.4 ($\lambda = 1, n = 24$). $\mathcal{B} = \{(3,1,10)-11,\ (4,1,9)-10,\ (5,1,11)-12,\ (4,2,11)-23,\ (5,2,10)-22,\ (2,12,6)-7,\ (5,3,12)-13,\ (6,3,11)-22,\ (3,7,13)-0,\ (6,4,13)-14,\ (7,4,12)-0,\ (8,4,14)-0,\ (5,14,7)-8,\ (5,8,13)-1,\ (5,9,15)-1,$

$(8,6,15)-19$, $(9,6,14)-2$, $(10,6,16)-23$, $(9,7,16)-14$, $(10,7,15)-14$, $(11,7,17)-14$, $(10,8,17)-20$, $(11,8,16)-4$, $(12,8,18)-16$, $(11,9,18)-19$, $(12,9,17)-5$, $(13,9,19)-5$, $(12,10,19)-20$, $(13,10,18)-5$, $(14,10,20)-6$, $(13,11,20)-21$, $(14,11,19)-6$, $(15,11,21)-7$, $(21,12,14)-18$, $(15,12,20)-7$, $(12,16,22)-5$, $(15,13,22)-17$, $(16,13,21)-6$, $(13,17,23)-7$, $(15,16,2)-9$, $(17,15,0)-11$, $(15,18,23)-14$, $(17,16,3)-8$, $(19,16,0)-18$, $(16,20,1)-12$, $(18,17,4)-10$, $(19,17,1)-2$, $(21,17,2)-13$, $(20,18,2)-3$, $(21,18,1)-14$, $(22,18,3)-14$, $(21,19,3)-4$, $(2,19,22)-14$, $(4,19,23)-12$, $(20,22,4)-5$, $(20,23,3)-15$, $(0,20,5)-6$, $(22,21,8)-19$, $(21,23,5)-16$, $(21,0,4)-15$, $(22,23,9)-21$, $(22,0,6)-18$, $(22,1,7)-18$, $(0,23,10)-21$, $(1,23,6)-17$, $(2,23,8)-20$, $(0,1,8)-9$, $(0,2,7)-19$, $(3,0,9)-20\}$;
$\mathcal{T}=\{(1,2,13), (2,3,14), (3,4,15), (4,5,16), (5,6,17), (6,7,18), (7,8,19), (8,9,20), (9,10,21), (10,11,22), (11,12,23), (12,13,0), (14,15,1), (14,16,23), (14,17,22), (14,18,0), (18,19,5), (19,20,6), (20,21,7)\}$;
$\mathcal{S}=\{[1;3,4,5], [2;4,5,12], [3;5,6,7], [4;6,7,8], [5;14,8,9], [6;8,9,10], [7;9,10,11], [8;10,11,12], [9;11,12,13], [10;12,13,14], [11;13,14,15], [12;21,15,16], [13;15,16,17], [15;16,17,18], [16;17,19,20], [17;18,19,21], [18;20,21,22], [19;21,2,4], [20;22,23,0], [21;22,23,0], [22;23,0,1], [23;0,1,2], [0;1,2,3]\}$;
$\mathcal{P}=\{[1,10,2,11], [1,9,15,6], [1,11,3,12], [6,12,4,13], [4,14,7,16], [13,8,17,9], [14,6,16,8], [15,7,17,23], [8,18,9,19], [19,10,18,23], [10,20,11,19], [11,21,13,22], [14,12,20,1], [22,16,2,7], [15,0,16,3], [4,17,1,18], [17,2,18,3], [3,19,22,4], [19,23,5,20], [3,23,9,0], [13,7,1,8], [21,8,23,10], [4,0,6,23]\}$;
$L_T=\{(1,12), (2,9), (3,8), (4,10), (13,14), (15,19), (16,18), (17,20), (21,6), (22,5), (23,7), (0,11)\}$.

Example 3.4.5 ($\lambda=2, n=4$)**.** $\mathcal{B}=\{(3,0,1)-2, (0,1,2)-3, (0,2,3)-1\}$, $\mathcal{T}=(1,2,3)$, $\mathcal{S}=[0;1,2,3]$, $\mathcal{P}=[0,1,2,3]$.

Example 3.4.6 ($\lambda=2, n=5$)**.** $\mathcal{B}=\{(1,0,4)-3$, $(2,0,4)-3$, $(1,2,3)-0$, $(1,3,0)-2$, $(1,4,2)-3\}$, $\mathcal{T}=\{(0,2,3)\}$, $\mathcal{S}=\{[1;2,3,4]\}$, $\mathcal{P}=\{[2,4,0,3]\}$, $L_T=2(4,3)$, $L_S=[0;1,2]$, $L_P=\{(2,3),(0,4)\}$.

Example 3.4.7 ($\lambda=2, n=8$)**.** $\mathcal{B}=\{(6,7,1)-5, (4,7,5)-3, (7,2,3)-1, (4,6,2)-1, (3,0,6)-5, (0,1,4)-3, (2,5,0)-7, (0,5,1)-2, (6,5,3)-1, (3,2,4)-1, (7,0,2)-5, (1,7,6)-2, (4,5,7)-3, (4,6,0)-3\}$
$\mathcal{T}=\{(1,3,4), (2,5,6), (3,7,0), (1,3,5)\}$,
$\mathcal{S}=\{[7;1,4,0], [6;7,5,4], [0;1,3,5], [2;7,3,5]\}$,
$\mathcal{P}=\{[1,7,5,0], [0,6,2,3], [3,5,1,4], [2,0,6,7]\}$,
$L_T=2(1,2)$, $L_S=[4;5,6]$, $L_P=\{(4,2),(7,5)\}$.

Example 3.4.8 ($\lambda = 2, n = 9$)**.** It is sufficient doubling the blocks of the above kite-system of order 9.

Example 3.4.9 ($\lambda = 3, n = 8$)**.** $\mathcal{B} = \{(7,4,2)-3,\ (6,7,3)-1,\ (7,5,1)-2,\ (4,6,5)-2,\ (1,0,4)-3,\ (2,0,6)-1,\ (5,0,3)-7,\ (1,0,2)-5,\ (6,1,3)-2,\ (5,3,4)-2,\ (7,0,5)-1,\ (2,7,6)-5,\ (4,1,7)-0,\ (4,6,0)-3,\ (4,6,2)-1,\ (7,4,1)-3,\ (4,5,3)-2,\ (6,7,5)-2,\ (3,0,6)-1,\ (2,0,7)-3,\ (1,5,0)-4\}$.
$\mathcal{T} = \{(2,3,1),\ (2,1,3),\ (2,3,4),\ (3,7,0),\ (1,6,5)\}$*;*
$\mathcal{S} = \{[4;6,7,5],\ [7;6,2,0],\ [0;2,1,3],\ [5;7,3,1],\ [6;1,7,4],\ [0;1,5,2],\ [4;6,1,7]\}$*;*
$\mathcal{P} = \{[7,3,1,4],\ [5,6,0,3],\ [4,2,6,0],\ [3,5,1,7],\ [2,0,5,7],\ [3,4,0,6],\ [6,7,0,5]\}$*;*
$L_T = \{3(2,5),\ (1,6),\ (3,7),\ (4,0)\}$.

Example 3.4.10 ($\lambda = 4, n = 5$)**.** $\mathcal{B} = \{(1,2,0)-4, (1,3,0)-4, (2,3,4)-1, (2,4,1)-3, (2,0,3)-4, (1,2,0)-3, (1,4,0)-3, (2,4,3)-1, (2,3,1)-4, (2,0,4)-3\}$*;* $\mathcal{T} = \{2(1,3,4)\}$*;* $\mathcal{S} = \{[1;2,4,3],\ [2;0,3,4],\ [2;0,4,3]\}$*;* $\mathcal{P} = \{[2,0,3,4],\ [1,4,0,3],\ [1,3,4,0]\}$*;* $L_T = 2[3,0,4]$*;* $L_S = (1,2)$*;* $L_P = (0,2)$.

Example 3.4.11 ($\lambda = 4, n = 6$)**.** $X = \mathbb{Z}_5 \cup \{\infty\}$*,* $\mathcal{B} = \{(i, 2+i, \infty) - (i+1),\ (i+1, 2+i, i) - \infty,\ (2+i, 4+i, i) - (i+1) | i \in \mathbb{Z}_5\}$;
$\mathcal{T} = \{(i, 1+i, \infty) | i \in \mathbb{Z}_5)\}$; $\mathcal{S} = \{[1;2,4,3], [0;2,3,1], [2;0,4,3], [3;0,1,4], [4;2,0,1]\}$;
$\mathcal{P} = \{[\infty, 2+i, i, 4+i]\}$.

Example 3.4.12 ($\lambda = 4, n = 7$)**.** $\mathcal{B} = \{(1,0,4)-2,\ (1,5,6)-2,\ (5,0,2)-1,\ (0,6,3)-2,\ (3,5,4)-6,\ (0,2,4)-1,\ (5,2,6)-1,\ (5,0,1)-2,\ (0,6,3)-1,\ (5,3,4)-6,\ (1,0,4)-3,\ (1,5,6)-3,\ (5,0,3)-1,\ (0,6,2)-3,\ (2,5,4)-6,\ (5,0,4)-2,\ (5,1,6)-2,\ (0,1,2)-5,\ (0,6,3)-2,\ (1,3,4)-6,\ (2,1,3)-5\}$*;*
$\mathcal{T} = \{(2,4,6), (1,4,6), (3,4,6), (2,4,6), (1,2,3), (1,2,3), (2,3,5)\}$*;*
$\mathcal{S} = \{[0;1,5,6], [5;1,3,2], [0;2,5,6],\ [5;1,2,3], [0;1,5,6],\ [1;2,3,5], [0;1,5,6]\}$*;*
$\mathcal{P} = \{[4,0,2,6], [0,4,5,6], [4,3,6,1],\ [4,2,6,3], [1,0,3,4],\ [6,3,1,2], [6,5,4,0]\}$.

Example 3.4.13 ($\lambda = 4, n = 8$)**.** Take two copies of the 2-fold kite-system of order 8. In one of them change 5 with 2. The result is a 4-fold kite-system of order 8 having a $\{K_3, S_3, P_4\}$-metamorphosis, with
$\mathcal{T} = \{(1,3,4),\ (2,5,6),\ (3,7,0),\ (1,3,5),\ (1,3,4),\ (5,2,6),\ (3,7,0),\ (1,3,2)\}$;
$\mathcal{S} = \{[7;1,4,0],\ [6;7,5,4],\ [0;5,1,3],\ [2;7,3,5],\ [7;1,4,0],\ [6;7,2,4],\ [0;2,1,3],\ [5;7,3,2],\ [4;2,6,5]\}$;
$\mathcal{P} = \{[1,7,5,0],\ [0,6,2,3],\ [3,5,1,4],\ [2,0,6,7],\ [6,7,2,0],\ [0,6,5,3],\ [3,2,1,7],\ [6,0,5,4],\ [4,2,7,5]\}$;
$L_T = 2[2,1,5]$; $L_S = (6,4)$; $L_P = (1,4)$.

Example 3.4.14 ($\lambda = 4, n = 10$)**.** $X = \mathbb{Z}_9 \cup \{\infty\}$, $\mathcal{B} = \{(i, 4+i, \infty) - (i+1),\ (i+6, 4+i, i) - \infty,\ (3+i, 5+i, 1+i) - i,\ (4+i, 1+i, i) - (3+i),\ (2+i, 1+i, i) - (3+i) | i \in$

$\mathbb{Z}_9\}$; $\mathcal{T} = \{(i, 1+i, \infty) | i \in \mathbb{Z}_9\} \cup (2\{(0,3,6), (1,4,7), (2,5,8)\})$;
$\mathcal{S} = \{[0;2,1,7], [8;0,1,7], [2;1,4,3], [3;4,1,5], [5;4,6,7], [6;4,7,8]\} \cup \{[4+i; i, 6+i, 1+i] | i \in \mathbb{Z}_9\}$;
$\mathcal{P} = \{[\infty, 4+i, i, 1+i] | i \in \mathbb{Z}_9\} \cup \{[0,5,6,2], [1,5,4,8], [1,6,7,3], [1,2,7,8], [3,8,0,1], [2,3,4,0]\}$.

Example 3.4.15 ($\lambda = 4, n = 11$)**.** Let $S = \{x_j | j \in \mathbb{Z}_5\}$ and $X = S \cup \mathbb{Z}_5 \cup \{\infty\}$.

1. Let (S,$\mathcal{B}'$) be a copy of the 4-fold kite-system of order 5 above constructed with the leaves L_T, L_S, L_P.
2. For each $i \in \mathbb{Z}_5$, let $\mathcal{B}_i = \{(i-1, i+1, x_i) - \infty$, $(i-2, i+2, x_i) - (i+1)$, $(x_i, i, \infty) - (i+1)$, $(i, \infty, x_i) - (i+2)$, $(i-1, i+1, x_i) - i$, $(x_i, i-2, i+2) - i$, $(\infty, i, x_i) - (i-2)$, $(i-2, i+2, x_i) - (i-1)$, $(x_i, i+1, i-1) - (i-2)\}$. It is easy to check that $(X, \cup\mathcal{B}_i)$ is a 4-fold kite-system of order 11 with a hole of size 5 on S having:
   - a K_3-metamorphosis with $\mathcal{T} = \{(x_i, 1+i, \infty), (i, i+2, x_i), (i-1, i-2, x_i)\}$ and empty leave;
   - an S_3-metamorphosis with $\mathcal{S} = \{[x_i; i-2, i, i+1], 0 \le i \le 4\} \cup \{[i-1; i, i+1, i+3], 0 \le i \le 3\} \cup \{[1;3,4,\infty], [2;0,4,\infty], [3;0,4,\infty], [3;0,2,\infty], [\infty;0,2,4], [\infty;0,1,4]\}$ and empty leave;
   - a P_4-metamorphosis with $\mathcal{P} = \{[i+1, x_i, i+2, i-2], [i, \infty, x_i, i+2], [i, x_i, i+1, i-1]\}$ and empty leave.

Then $(X, \mathcal{B}' \cup (\bigcup \mathcal{B}_i))$ is a 4-fold kite system of order 11 having $\{K_3, S_3, P_4\}$-metamorphosis with leaves L_T, L_S, L_P.

Example 3.4.16 ($\lambda = 5, n = 8$)**.** Let $\mathcal{B}_1 = \{(1,2,5) - 6$, $(1,7,4) - 6$, $(1,3,6) - 5$, $(2,7,3) - 5$, $(6,0,2) - 4$, $(5,0,7) - 6$, $(4,3,0) - 1$, $(2,5,4) - 6$, $(2,7,6) - 3$, $(2,1,3) - 4$, $(5,7,1) - 4$, $(3,0,5) - 4$, $(4,0,7) - 3$, $(6,1,0) - 2$, $(7,2,5) - 1$, $(7,6,1) - 3$, $(7,4,3) - 5$, $(2,6,4) - 5$, $(3,0,2) - 1$, $(5,0,6) - 3$, $(1,4,0) - 7\}$. Then $(\mathbb{Z}_8, \mathcal{B}_1)$ is a 3-fold kite system having

- a K_3-metamorphosis with $\mathcal{T} = \{(5,6,4), (3,4,6), (5,3,1), (5,6,3), (4,1,2)\}$ and leave $L_T = \{(7,6), (7,3), (7,0), (0,1), (0,2), (5,4)\}$;
- an S_3-metamorphosis with $\mathcal{S} = \{[2;6,1,7], [1;6,3,7], [2;5,1,7], [4;3,1,7], [0;5,6,3], [0;5,3,4], [7;5,6,2]\}$;
- a P_4-metamorphosis with $\mathcal{P} = \{[5,2,0,7], [4,7,3,6], [3,0,1,7], [7,0,5,4], [7,6,1,3], [5,2,0,4], [3,4,6,0]\}$.

Let $(\mathbb{Z}_8, \mathcal{B}_2)$ be the above 2-fold kite system. Then $(\mathbb{Z}_8, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 5-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. Note that we can rearrange the leaves of the K_3-metamorphosis of $(\mathbb{Z}_8, \mathcal{B}_1)$ and $(\mathbb{Z}_8, \mathcal{B}_2)$ into the triangle $(1, 2, 0)$ and the leave $\{[7; 6, 3, 0], (1, 2), (5, 4)\}$.

Example 3.4.17 $(\lambda = 6, n = 5)$**.** Let $(\mathbb{Z}_5, \mathcal{B}_1)$ be the above 4-fold kite system of order 5 and $(\mathbb{Z}_5, \mathcal{B}_2)$ be the above 2-fold kite system when we change 0 with 2. Then $(\mathbb{Z}_5, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 6-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis, with:
$\mathcal{T} = \{(2, 0, 3), 2(1, 3, 4), 2(0, 3, 4)\}$;
$\mathcal{S} = \{[1; 0, 3, 4], [1; 2, 4, 3], [2; 0, 1, 4], [2; 0, 4, 3], [2; 1, 0, 3]\}$;
$\mathcal{P} = \{[0, 4, 2, 3], [2, 0, 3, 4], [1, 4, 0, 3], [1, 3, 4, 0], [3, 0, 2, 4]\}$.

Example 3.4.18 $(\lambda = 7, n = 8)$**.** Let $(\mathbb{Z}_8, \mathcal{B}_1)$ be the above 4-fold kite-system of order 8 and $(\mathbb{Z}_8, \mathcal{B}_2)$ be the above 3-fold kite system. Then $(\mathbb{Z}_8, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 7-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the leaves of the K_3-metamorphosis of $(\mathbb{Z}_8, \mathcal{B}_1)$ and $(\mathbb{Z}_8, \mathcal{B}_2)$ into the triangles $2(1, 2, 5)$ and the leave $\{(2, 5), (1, 6), (3, 7), (4, 0)\}$.

Chapter 4

Block Colourings of C_4−designs

4.1 Preliminaries

It is well-known that a C_4-system of order v, briefly $4CS(v)$, exists if and only if $v = 1 + 8k,\ k \geq 1$. Every vertex of a $4CS(v)$ is contained exactly in $r = \frac{v-1}{2} = 4k$ blocks. The integer r is called, using the graph theoretic terminology, *degree* or also *replication number*.

A colouring of a $4CS(v)$ $\Sigma = (V, \mathcal{B})$ is a mapping $\phi : \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours. An h-colouring is a colouring in which exactly h colours must be used. For each $i = 1, \ldots, h$, the subset $\mathcal{B}_i$ of $\mathcal{B}$, containing all the blocks coloured with colour i, is a colour class. A $4CS(v)$ Σ is said to be h-uncolourable if there is not any h-colouring of Σ.

For a partition of degree r into s parts, an h-colouring of type s is a colouring of blocks such that, for each element $x \in V$, the blocks containing x are coloured with s colours. For a $4CS(v)$ $\Sigma = (V, \mathcal{B})$, we define the colour spectrum $\Omega_s(\Sigma) = \{h :$ there exists an h-block-colouring of type s of $\Sigma\}$, and also define $\Omega_s(v) = \cup\Omega_s(\Sigma)$, where the union is taken over the set of all $4CS(v)s$.

The lower s-chromatic index $\chi'_s(\Sigma)$ and the upper s-chromatic index $\overline{\chi}'_s(\Sigma)$ of Σ are defined as $\chi'_s(\Sigma) = \min \Omega_s(\Sigma)$, $\overline{\chi}'_s(\Sigma) = \max \Omega_s(\Sigma)$, and similarly, $\chi'_s(v) = \min \Omega_s(v)$, $\overline{\chi}'_s(v) = \max \Omega_s(v)$. If $\Omega_s(\Sigma) = \emptyset$ ($\Omega_s(v) = \emptyset$), then we say that Σ (any $4CS(v)$) is uncolorable.

For a vertex x and for every $i = 1, 2, \ldots, s$, $\mathcal{B}_{x,i}$ is the set of all the blocks incident with x and coloured by the ith colour. A colouring of type s is *equitable* if for every vertex x and for $i, j = 1, \ldots, s$, $|\mathcal{B}_{x,i} - \mathcal{B}_{x,j}| \leq 1$. A bicolouring, tricolouring or quadricolouring is an equitable colouring with $s = 2$, $s = 3$ and $s = 4$, respectively.

If x is a vertex of V, then we will say that x is of type $A^i B^j \ldots C^u$, if i blocks containing x are coloured by A, j blocks containing x are coloured by B,... and so on until u blocks containing x are coloured by C.

4.2 Bicolourings

In this section we will consider bicolourings.

Lemma 4.2.1. *Let Σ be a $4CS(v)$, with $v = 1 + 8k$. If k is odd, then Σ is not 3-bicolourable.*

Proof Let $\Sigma = (V, \mathcal{B})$ be a $4CS(v)$ and suppose that $\phi : \mathcal{B} \to \{1, 2, 3\}$ is a 3-bicolouring of Σ. Partition V into three sets X, Y, Z of size x, y, z, respectively, such that:

each element of X is incident with blocks of colour 1 and 2,
each element of Y is incident with blocks of colour 1 and 3,
each element of Z is incident with blocks of colour 2 and 3.

Observe that there is not any block incident with all three types of elements. Then the blocks either contain all elements of the same type or contain elements of two types.
Further, no block contains an odd number of edges having the extremes of different type.
This implies that it is impossible that two among x, y, z are odd numbers. Further, since $x+y+z = 8k+1$, it follows that exactly one among x, y, z is an odd number and exactly two among $x+y, x+z, y+z$ are odd numbers.
Finally, since:

$$\mathcal{B}_1 = \frac{2k(x+y)}{4} = \frac{k(x+y)}{2},$$
$$\mathcal{B}_2 = \frac{k(x+z)}{2},$$
$$\mathcal{B}_3 = \frac{k(z+y)}{2},$$

it follows that k is an even number, necessarily. □

In [39] the author proved the following:

Theorem 4.2.2. *[39] The complete bipartite graph $K_{X,Y}$ can be decomposed into edge disjoint cycles of length $2k$ if and only if (1) $|X| = x$ and $|Y| = y$ are even, (2) $x \geq k$ and $y \geq k$, and (3) $2k$ divides xy.*

This permits to prove the following:

Lemma 4.2.3. *For all even k, there is a 2-bicolorable $4CS(1+8k)$ and a 3-bicolorable $4CS(1+8k)$.*

Proof Let $k = 2h$. It is not difficult to prove that $\Sigma = (\mathbb{Z}_{8k+1}, \mathcal{B})$, with starter blocks $\{(0, i, 4k+1, k+i) \mid 1 \leq i \leq k\}$, is a $4CS(8k+1)$.
If we assign the colour A to all the blocks obtained for $i = 1, 2, \ldots, h$ and to all their translated, and assign the colour B to all the blocks obtained for $i = h+1, h+2, \ldots, 2h$ and to all their translated, we define a 2-bicolouring of Σ.

Now, let $A = \{a_1, a_2, \ldots, a_{8h}\}$, $B = \{b_1, b_2, \ldots, b_{8h}\}$, $C = \{\infty\}$ and let $\Sigma_1 = (A \cup C, \mathcal{B}_1)$, $\Sigma_2 = (B \cup C, \mathcal{B}_2)$ be two C_4-systems of order $8h+1$. By Theorem 4.2.2, there exists a C_4-decomposition of the bipartite graph $K_{A,B}$ $(K_{A,B}, \mathcal{B}_3)$. Observe that $\Sigma = (A \cup B \cup C, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a $4CS(1+8k)$. By colouring with a colour i the blocks of $\mathcal{B}_i$, we obtain a 3-bicolouring, because each vertex of A (of B) has degree $4h$ in $(K_{A,B}, \mathcal{B}_3)$ and degree $4h$ in $(A \cup C, \mathcal{B}_1)$ (in $(B \cup C, \mathcal{B}_2)$). □

Theorem 4.2.4. *For 4-cycle systems it is $\Omega_2(1+8k) = \emptyset$, if k is odd, $\Omega_2(1+8k) = \{2, 3\}$, if k is even.*

Proof Let $\Sigma = (V, \mathcal{B})$ be an $4CS(v)$ and $\phi : \mathcal{B} \to \mathcal{C}$ an h-bicolouring of Σ. Let $c \in \mathcal{C}$ and let $x \in V$ an element incident with blocks of colour c. There are $2k$ blocks of colour c incident with x. Thus there are at least $1 + 4k$ elements in V incident with blocks of colour c. Then $h(1+4k) \leq 2v = 2 + 16k$. Therefore

$$h \leq \left\lfloor \frac{16k+2}{4k+1} \right\rfloor = 3$$

and so $\overline{\chi}'_2(v) \leq 3$.
Now, let $h = 2$. It is

$$|\mathcal{B}_c| = \frac{v \cdot 2k}{4} = \frac{8k^2 + k}{2}.$$

Then, if k is odd Σ is 2-uncolourable and, by Lemma 4.2.1, uncolourable. If k is even, by Lemma 4.2.3 this is sufficient to prove that $\Omega_2(1+8k) = \{2, 3\}$. □

4.3 Tricolourings

In this section we will consider tricolourings.

Lemma 4.3.1. *There exist 3-tricolourable $4CS(9)s$ and $4CS(17)s$.*

Proof Let $v = 9$ (in these systems each vertex has degree 4).
Consider the following $4CS(9)$ $\Sigma = (\mathbb{Z}_9, \mathcal{B})$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and:

- $\mathcal{B}_1 = \{(1,2,7,4),\ (2,0,8,5),\ (3,1,0,6)\}$,
- $\mathcal{B}_2 = \{(4,5,1,8),\ (5,6,2,3),\ (6,4,0,7)\}$,
- $\mathcal{B}_3 = \{(7,8,6,1),\ (8,3,4,2),\ (3,7,5,0)\}$.

If we assign the colour A to all the blocks belonging to $\mathcal{B}_1$, the colour B to all the blocks belonging to $\mathcal{B}_2$ and the colour C to all the blocks belonging to $\mathcal{B}_3$, we define a 3-tricolouring in Σ, with the vertices $0,1,2$ of type A^2BC, the vertices $4,5,6$ of type AB^2C and the vertices $7,8,3$ of type ABC^2.
Let $v = 17$. In the systems of order 17, each vertex has degree 8.
Consider the $4CS(9)$ $\Sigma_1 = (V_1, \mathcal{C}_1)$, where $V_1 = \{0\} \cup \{a_i : 1 \leq i \leq 8\}$, isomorphic to the previous system $\Sigma = (\mathbb{Z}_9, \mathcal{B})$, by the isomorphism $\varphi : V_1 \to Z_9$ such that:

$\varphi(0) = 0$,

$\varphi(a_i) = i,\ for\ i = 1, 2, ..., 8.$

Consider the $4CS(9)$ $\Sigma_2 = (V_2, \mathcal{C}_2)$, where $V_2 = \{0\} \cup \{b_i : 1 \leq i \leq 8\}$, isomorphic to the previous system $\Sigma = (\mathbb{Z}_9, \mathcal{B})$, by the isomorphism $\psi : V_2 \to Z_9$ such that:

$\psi(0) = 0$,

$\psi(b_i) = i,\ for\ i = 1, 2, ..., 8.$

Let $\Delta = (V, \mathcal{C})$ be the $4CS(17)$ such that:

$V = V_1 \cup V_2$,

$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$,

and:

- $\mathcal{C}_3 = \{(a_3, b_6, a_4, b_1),\ (a_7, b_2, a_8, b_1),\ (a_3, b_4, a_4, b_3),\ (a_1, b_4, a_2, b_5),$ $(a_5, b_6, a_6, b_5),\ (a_5, b_8, a_6, b_7)\}$,
- $\mathcal{C}_4 = \{(a_5, b_2, a_6, b_1),\ (a_1, b_6, a_2, b_3),\ (a_7, b_4, a_8, b_3),\ (a_3, b_2, a_4, b_5),$ $(a_1, b_8, a_2, b_7),\ (a_7, b_8, a_8, b_7)\}$,
- $\mathcal{C}_5 = \{(a_1, b_2, a_2, b_1),\ (a_5, b_4, a_6, b_3),\ (a_7, b_6, a_8, b_5),\ (a_3, b_8, a_4, b_7)\}$.

If we colour:

- the blocks of $\mathcal{C}_1$ with the same colour of the correspondent isomorphic blocks of Σ;
- the blocks of $\mathcal{C}_2$, assigning the colour C to the blocks of $\psi^{-1}(\mathcal{B}_1)$, the colour B to the blocks of $\psi^{-1}(\mathcal{B}_2)$ and the colour A to the blocks of $\psi^{-1}(\mathcal{B}_3)$;
- the blocks of $\mathcal{C}_3$ with A;
- the blocks of $\mathcal{C}_4$ with B;
- the blocks of $\mathcal{C}_5$ with C;

then a 3-tricolouring is defined in the system Δ, with the property that all the vertices are of type $A^3B^3C^2$, except the vertices $a_3, b_1, 0$ of type $A^3B^2C^3$ and the vertices a_7, a_8, b_2 of type $A^2B^3C^3$. □

Theorem 4.3.2. *For all* $k \equiv 0 \pmod 3$, $k > 0$, *there exist* 3*-tricolourable* $4CS(1+8k)s$ *.*

Proof Let $k = 3h$. Consider the $4CS(8k+1)$ $\Phi = (\mathbb{Z}_{8k+1}, \mathcal{B})$ defined in Lemma 4.2.3, having starter blocks $\{(0, i, 4k+1, k+i) \mid 1 \le i \le k\}$. If we assign the colour A to the blocks in which $i = 1, 2, \ldots, h$ and to all their translated, the colour B to the blocks in which $i = h+1, h+2, \ldots, 2h$ and to all their translated, the colour C to the blocks in which $i = 2h+1, 2h+2, \ldots, 3h$ and to all their translated, then we obtain a 3-tricolouring of Φ having all the vertices of type $A^{4h}B^{4h}C^{4h}$. □

Theorem 4.3.3. *For all* $k \equiv 1 \pmod 3$, *there exist* 3*-tricolourable* $4CS(1+8k)s$*.*

Proof For $k = 1$, the result is proved in Lemma 4.3.1.
Let $k = 3h + 1$, $h > 0$.
Let $V_1 = \{0\} \cup \{x_i : 1 \le i \le 8\}$, $V_2 = \{0\} \cup \{y_i : 1 \le i \le 24h\}$.
Construct the $4CS(9)$ $(V_1, \mathcal{D}_1)$, isomorphic to the system Σ defined in Lemma 4.3.1, by the isomorphism $0 \to 0$ and $x_i \to i$, for every i, $1 \le i \le 8$.
Construct a $4CS(24h+1)$ $(V_2, \mathcal{D}_2)$ isomorphic to the system Φ defined in Theorem 4.3.2, by the isomorphism $0 \to 0$ and $y_i \to i$, for every i, $1 \le i \le 24h$.
Let $\Gamma = (V, \mathcal{D})$ be the $4CS(24h+9)$ where:

$$V = V_1 \cup V_2,$$

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5,$$

and:

- $\mathcal{D}_3 = \{(x_1, y_i, x_2, y_{i+1}), (x_3, y_{16h+i}, x_4, y_{16h+i+1}), (x_5, y_{8h+i}, x_6, y_{8h+i+1}), (x_7, y_i, x_8, y_{i+1}) : 1 \leq i \leq 8h-1, i \equiv 1 \pmod 2\}$,
- $\mathcal{D}_4 = \{(x_1, y_{8h+i}, x_2, y_{8h+i+1}), (x_3, y_i, x_4, y_{i+1}), (x_5, y_{16h+i}, x_6, y_{16h+i+1}), (x_7, y_{8h+i}, x_8, y_{8h+i+1}) : 1 \leq i \leq 8h-1, i \equiv 1 \pmod 2\}$,
- $\mathcal{D}_5 = \{(x_1, y_{16h+i}, x_2, y_{16h+i+1}), (x_3, y_{8h+i}, x_4, y_{8h+i+1}), (x_5, y_i, x_6, y_{i+1}), (x_7, y_{16h+i}, x_8, y_{16h+i+1}) : 1 \leq i \leq 8h-1, i \equiv 1 \pmod 2\}$.

If we colour

- the blocks of $\mathcal{D}_1$ as the correspondent isomorphic blocks of Σ defined in Lemma 4.3.1;
- the blocks of $\mathcal{D}_2$ as the correspondent isomorphic blocks of Φ defined in Theorem 4.3.2;
- the blocks of $\mathcal{D}_3$ with A;
- the blocks of $\mathcal{D}_4$ with B;
- the blocks of $\mathcal{D}_5$ with C;

then we obtain a 3-tricolouring of Γ such that:

- the vertices $0, x_1, x_2$ and y_i, for every i, $1 \leq i \leq 8h$, are of type $A^{4h+2}B^{4h+1}C^{4h+1}$,
- the vertices x_6, x_4, x_5 and y_i, for every i, $8h+1 \leq i \leq 16h$, are of type $A^{4h+1}B^{4h+2}C^{4h+1}$,
- the vertices x_3, x_7, x_8 and y_i, for every i, $16h+1 \leq i \leq 24h$, are of type $A^{4h+1}B^{4h+1}C^{4h+2}$.

□

Theorem 4.3.4. *For all* $k \equiv 2 \pmod 3$*, there exist* 3*-tricolourable* $4CS(1+8k)$.

Proof For $k = 2$, the result is proved in Lemma 4.3.1.
Let $k = 3h+2$, $h > 0$. Let $V_1 = \{0\} \cup \{\alpha_i : 1 \leq i \leq 16\}$, $V_2 = \{0\} \cup \{\beta_i : 1 \leq i \leq 24h\}$.
Construct a $4CS(17)$ $(V_1, \mathcal{F}_1)$ isomorphic to system Δ defined in Lemma 4.3.1, by the isomorphism $0 \to 0$, $\alpha_i \to a_i$, for every i, $1 \leq i \leq 8$, $\alpha_{8+i} \to b_i$, for every i, $1 \leq i \leq 8$.

Construct a $4CS(24h+1)$ $(V_2, \mathcal{F}_2)$ isomorphic to system Φ defined in Theorem 4.3.2, by the isomorphism $0 \to 0$ and $\beta_i \to i$, for every i, $1 \le i \le 24h$.
Let $\Omega = (V, \mathcal{F})$ be the $4CS(24h+17)$ where:

$V = V_1 \cup V_2$,

$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$,
and:

- $\mathcal{F}_3 = \{(\alpha_1, \beta_i, \alpha_2, \beta_{i+1}),\ (\alpha_3, \beta_{16h+i}, \alpha_4, \beta_{16h+i+1}),\ (\alpha_5, \beta_{8h+i}, \alpha_6, \beta_{8h+i+1}),$ $(\alpha_7, \beta_i, \alpha_8, \beta_{i+1}),\ (\alpha_9, \beta_{16h+i}, \alpha_{10}, \beta_{16h+i+1}),\ (\alpha_{11}, \beta_{8h+i}, \alpha_{12}, \beta_{8h+i+1}),$ $(\alpha_{13}, \beta_i, \alpha_{14}, \beta_{i+1}),\ (\alpha_{15}, \beta_{16h+i}, \alpha_{16}, \beta_{16h+i+1}) : 1 \le i \le 8h-1, i \equiv 1 \pmod 2\}$,

- $\mathcal{F}_4 = \{(\alpha_1, \beta_{8h+i}, \alpha_2, \beta_{8h+i+1}),\ (\alpha_3, \beta_i, \alpha_4, \beta_{i+1}),\ (\alpha_5, \beta_{16h+i}, \alpha_6, \beta_{16h+i+1}),$ $(\alpha_7, \beta_{8h+i}, \alpha_8, \beta_{8h+i+1}),\ (\alpha_9, \beta_i, \alpha_{10}, \beta_{i+1}),\ (\alpha_{11}, \beta_{16h+i}, \alpha_{12}, \beta_{16h+i+1}),$ $(\alpha_{13}, \beta_{8h+i}, \alpha_{14}, \beta_{8h+i+1}),\ (\alpha_{15}, \beta_i, \alpha_{16}, \beta_{i+1}) : 1 \le i \le 8h-1, i \equiv 1 \pmod 2\}$,

- $\mathcal{F}_5 = \{(\alpha_1, \beta_{16h+i}, \alpha_2, \beta_{16h+i+1}),\ (\alpha_3, \beta_{8h+i}, \alpha_4, \beta_{8h+i+1}),\ (\alpha_5, \beta_i, \alpha_6, \beta_{i+1}),$ $(\alpha_7, \beta_{16h+i}, \alpha_8, \beta_{16h+i+1}),\ (\alpha_9, \beta_{8h+i}, \alpha_{10}, \beta_{8h+i+1}),\ (\alpha_{11}, \beta_i, \alpha_{12}, \beta_{i+1}),$ $(\alpha_{13}, \beta_{16h+i}, \alpha_{14}, \beta_{16h+i+1}),\ (\alpha_{15}, \beta_{8h+i}, \alpha_{16}, \beta_{8h+i+1}) : 1 \le i \le 8h-1, i \equiv 1 \pmod 2\}$.

If we colour

- the blocks of $\mathcal{F}_1$ as the correspondent isomorphic blocks of Σ defined in Lemma 4.3.1;
- the blocks of $\mathcal{F}_2$ as the correspondent isomorphic blocks of Φ defined in Theorem 4.3.2;
- the blocks of $\mathcal{F}_3$ with A;
- the blocks of $\mathcal{F}_4$ with B;
- the blocks of $\mathcal{F}_5$ with C;

then we obtain a 3-tricolouring of Ω such that:

- the vertices $0, \alpha_3, \alpha_9$ and β_i, for every i, $16h+1 \le i \le 24h$, are of type $A^{4h+3}B^{4h+2}C^{4h+3}$,

- the vertices $\alpha_7, \alpha_8, \alpha_{10}$ and β_i, for every i, $8h+1 \leq i \leq 16h$, are of type $A^{4h+2}B^{4h+3}C^{4h+3}$,
- all the other vertices are of type $A^{4h+3}B^{4h+3}C^{4h+2}$. $\square$

Theorem 4.3.5. *For every $v \equiv 1 \pmod 8$, the lower 3-chromatic index $\chi_3'(v)$ for 4-cycle systems is 3.*

Proof The result follows from Theorems 4.3.2, 4.3.3, 4.3.4. $\square$

Theorem 4.3.6. *For the upper 3-chromatic index $\overline{\chi}_3'(v)$ of $4CS(v)$ the following inequalities hold:*

- $\overline{\chi}_3'(v) \leq 8$, *if* $v \equiv 1 \pmod{24}$*;*
- $\overline{\chi}_3'(v) \leq 9$, *if* $v \equiv 9, 17 \pmod{24}$, $v \neq 9, 17$*;*
- $\overline{\chi}_3'(v) \leq 8$, *if* $v = 9$*;*
- $\overline{\chi}_3'(v) \leq 10$, *if* $v = 17$.

Proof Let $\Sigma = (V, \mathcal{B})$ be a $4CS(v)$ and let $\phi : \mathcal{B} \to \mathcal{C}$ be a p-tricolouring of Σ. Let $c \in \mathcal{C}$ and let $x \in V$ be an element incident with the blocks of colour c.
For $v = 24h+1$, the degree partition is $(4h, 4h, 4h)$ and so there are $4h$ blocks of colour c incident with x. It follows that there are at least $1+8h$ elements of V incident with blocks of colour c.

Then: $p(1+8h) \leq 3v = 3+72h$.

Hence: $p \leq \left\lfloor \frac{72h+3}{8h+1} \right\rfloor = 8$,

and so: $\overline{\chi}_3'(v) \leq 8$.

For $v = 24h+9$, the degree partition is $(4h+1, 4h+1, 4h+2)$ and so there are at least $4h+1$ blocks of colour c incident with x. Thus, there are at least $3+8h$ elements of V incident with blocks of colour c.

Then: $p(3+8h) \leq 3v = 27+72h$.

Hence: $p \leq \left\lfloor \frac{72h+27}{8h+3} \right\rfloor = 9$,

and so: $\overline{\chi}_3'(v) \leq 9$.

If $v = 9$, since also the size of the block set is 9 and the degree of each vertex is 4, it follows that a 9-tricolouring is impossible.
So: $\overline{\chi}'_3(9) \leq 8$.
For $v = 24h + 17$, the degree partition is $(4h + 2, 4h + 3, 4h + 3)$ and so there are at least $4h + 2$ blocks of colour c incident with x. Thus, there are at least $5 + 8h$ elements of V incident with blocks of colour c.

Then: $p \cdot (5 + 8h) \leq 3v = 51 + 72h$.

Hence: $p \leq \left\lfloor \frac{72h+51}{8h+5} \right\rfloor$.

Therefore: $\overline{\chi}'_3(v) \leq 9$, for $v > 17$, and $\overline{\chi}'_3(v) \leq 10$, for $v = 17$. □

4.4 All possible tricolourings for $v = 9$

In this section we will determine completely the spectrum $\Omega_3(9)$ about tricolourings for 4-cycle systems of order 9. From Theorem 4.3.5, 4-cycle systems tricolourable with 3 colours there exist.

Lemma 4.4.1. *There exist* 4*-tricolourable* 4*-cycle systems of order* 9*.*

Proof Let $\Sigma = (\mathbb{Z}_9, \mathcal{D})$ be the C_4-system defined on $\mathbb{Z}_9$ as follows:

$$\mathcal{D} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4,$$

where

$\mathcal{B}_1 = \{(1, 2, 8, 4), (1, 3, 5, 0), (2, 3, 6, 7)\}$,

$\mathcal{B}_2 = \{(1, 5, 2, 6), (6, 4, 7, 5)\}$,

$\mathcal{B}_3 = \{(1, 7, 3, 8), (7, 0, 6, 8)\}$,

$\mathcal{B}_4 = \{(2, 4, 3, 0), (0, 4, 5, 8)\}$.

If we assign the colour A_i, for each $i = 1, 2, 3, 4$, to the blocks belonging to $\mathcal{B}_i$, we obtain a tricolouring of Σ with 4 colours. □

Theorem 4.4.2. *For 4-cycle systems, we have*

$$\Omega_3(9) = \{3, 4, 5\}.$$

Proof At first, we observe that, for tricolourings with 3 or 4 colours, the statement follows from Theorem 4.3.5 and Lemma 4.4.1, respectively. Now, let Σ=$(\mathbb{Z}_9, \mathcal{B})$ be the system defined on $\mathbb{Z}_9$ such that:

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5,$$

where

$\mathcal{B}_1 = \{(2, 8, 5, 0), (7, 3, 8, 6), (3, 5, 2, 6)\};$

$\mathcal{B}_2 = \{(1, 3, 2, 4), (1, 5, 4, 6)\};$

$\mathcal{B}_3 = \{(1, 0, 4, 7), (5, 7, 0, 6)\};$

$\mathcal{B}_4 = \{(1, 2, 7, 8)\};$

$\mathcal{B}_5 = \{(3, 4, 8, 0)\}.$

We can verify that Σ is a C_4-system of order 9 and if we assign the colour A_i, for each $i = 1, 2, .., 5$, to the blocks belonging to $\mathcal{B}_i$, we obtain a tricolouring of Σ with 5 colours.
Finally, we prove that no C_4-system of order 9 is tricolourable with 6 or more colours.
Let $\Sigma = (V, \mathcal{D})$ be a C_4-system of order 9 tricolourable with 6 colours. If f is the colouring, A_i the colours and $C_i = \{B \in \mathcal{B} : f(B) = A_i\}$ (*colouring classes*), we observe that necessarily:

i) *there are at least* 3 *colouring classes containing exactly one block;*

ii) *every vertex is of type* X^2YZ*;*

iii) *for every pair of blocks* B', B'', *we have* $|B' \cap B''| \leq 2$.

Let c be the number of colouring classes containing exactly one block, then $3 \leq c \leq 5$. Let $|C_1| \geq |C_2| ... \geq |C_6|$.

- It is not possible that $c = 5$, because C_1 contains 4 blocks and, therefore, there exist vertices belonging to 3 blocks of C_1.

- It is not possible that $c = 4$, because there are at most 2 vertices belonging to 2 blocks of C_2 and, therefore, at least 7 vertices belonging to 2 blocks of C_1.
- It is not possible that $c = 3$, because there are at most 2 vertices belonging to 2 blocks of C_1, at most 2 vertices belonging to 2 blocks of C_2 and at most 2 vertices belonging to 2 blocks of C_3. No other vertex belongs to 2 blocks of the same colour and this is a contradiction.

For tricolorings of 4-cycle systems of order 9 with 7 or more colours, it suffices to observe that in any case there are necessarily at least 3 vertices belonging to 4 blocks of 4 distinct colours. □

4.5 Constructions

In this and in the other sections we will use the following terminology and symbolism.

Let $A = \{a_1, a_2, \ldots, a_{2p}\}$, $B = \{b_1, b_2, \ldots, b_{2q}\}$ be two sets, such that $A \cap B = \emptyset$.

We will denote by $[A, B]$ the following family of cycles C_4:

$$[A, B] = \{(a_i, b_j, a_{i+p}, b_{j+q}) : 1 \leq i \leq p, 1 \leq j \leq q\}.$$

Observe that: $|[A, B]| = p \cdot q$.

Further, for $p = 4k$ and $\infty \notin A$, let $[A, \infty]$ be any 4-cycle system of order $v = 1 + 8k$ constructed on $A \cup \{\infty\}$.

CONSTRUCTION 1: $v \to k(v - 1) + 1$.

Theorem 4.5.1. *For every $v = 8h + 1$ and for every positive integer k, it is possible to construct a 4-cycle system Σ of order $k(v - 1) + 1$ containing k 4-cycle systems of order v.*

Proof Let

$$A_1 = \{a_{11}, a_{12}, \ldots, a_{1,8h}\},$$

$$A_2 = \{a_{21}, a_{22}, \ldots, b_{2,8h}\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_k = \{a_{k1}, a_{k2}, \ldots, a_{k,8h}\},$$

be any k sets of cardinality $8h$ and such that $A_i \cap A_j = \emptyset$, for every pair $i, j = 1, 2, ..., k$, $i \neq j$.
For each $i = 1, 2, ..., k$, let $\Sigma_i = (A_i \cup \{\infty\}, B_i)$ be any C_4-systems of order $1 + 8h$, constructed on $A_i \cup \{\infty\}$, where $\infty \notin \bigcup_{i=1,..,k} A_i$.
Further, consider the families of C_4-cycles:

$$\bigcup_{i,j=1,...,k}^{i<j} [A_i, A_j].$$

If

$$X = \bigcup_{i=1,...,k} A_i,$$

$$\mathcal{B} = (\bigcup_{i=1,...,k} \mathcal{B}_i) \cup (\bigcup_{i,j=1,...,k}^{i<j} [A_i, A_j]).$$

then it is possible to verify that $\Sigma = (X, B)$ is a C_4-system of order $k(v-1)+1$. It is immediate that, for every pair of distinct elements x, y of X, there exists at least a cycle C_4 of Σ containing the edge $\{x, y\}$. Further,

$$|\mathcal{B}| = k \cdot |\mathcal{B}_i| + \binom{k}{2} \cdot |[A_i, A_j]|,$$

where the indices i, j are fixed. It follows:

$$|\mathcal{B}| = k \cdot h \cdot (8h+1) + \binom{k}{2} \cdot 16h^2 =$$

$$= ..... = 8h^2k^2 + kh,$$

which is the number of blocks contained in a 4-cycle systems of order $1 + 8hk = k(v-1)+1$, exactly:

$$(1 + 8hk)hk.$$

This prove that Σ is a 4-cycle system of order $k(v-1)+1$, verifying the statement.□

CONSTRUCTION 2: $v \to v + 8kh$, for k odd, $k < v$

Theorem 4.5.2. *Let Σ' and Σ'' be any two 4-cycle systems of order $v = 8u+1$ and $w = 8h+1$ respectively. It is possible to construct a 4-cycle system Σ of order $v + 8kh$, for k odd and $k \leq v$, containing Σ' and k systems isomorphic to Σ''.*

Proof Let $\Sigma' = (\mathbb{Z}_v, D)$ be any C_4-system of order $v = 8u + 1$.
Let $k = 2p + 1$ be any odd positive integer with $k \leq v$.
Further, let

$$A_1 = \{a_{11}, a_{12}, \ldots, a_{1,8h}\},$$
$$A_2 = \{a_{21}, a_{22}, \ldots, b_{2,8h}\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_k = \{a_{k1}, a_{k2}, \ldots, a_{k,8h}\},$$

be any k sets of cardinality $8h$ and such that $A_i \cap A_j = \emptyset$, for every pair $i, j = 1, 2, ..., k$, $i \neq j$.
For each $i = 1, 2, ..., k$, let $\Sigma_i = (A_i \cup \{i\}, \mathcal{B}_i)$ be any C_4-system of order $1 + 8h$, constructed on $A_i \cup \{i\}$, where $i \in Z_v$.
Further, for each $i = 1, 2, .., k$, consider any 1-factor F_i of the complete graph K_{8u} defined on $\mathbb{Z}_v \setminus \{i\}$. Observe that two factors F_i, F_j can have pairs in common.
Finally, consider the following families of C_4-cycles:

$$\Gamma_{A_i} = \bigcup_{\{x,y\} \in F_i} [A_i, \{x, y\}],$$

for each $i = 1, 2, ..., k$.
If

$$X = (\bigcup_{i=1,\ldots,k} A_i) \cup \mathbb{Z}_v,$$

$$\mathcal{B} = (\bigcup_{i=1,\ldots,k} \mathcal{B}_i) \cup (\bigcup_{i=1,\ldots,k} \Gamma_{A_i}) \cup (\bigcup_{i,j=1,\ldots,k}^{i<j} [A_i, A_j]) \cup D.$$

then it is possible to verify that $\Sigma = (X, \mathcal{B})$ is a C_4-system of order $v + 8hk$.
It is immediate that, for every pair of distinct elements x, y of X, there exists at least a cycle C_4 of Σ containing the edge $\{x, y\}$. Further,

$$|\mathcal{B}| = k \cdot |\mathcal{B}_i| + k \cdot |\Gamma_{A_i})| + \binom{k}{2} |[A_i, A_j]| + |D|,$$

where the indices i, j are fixed. It follows:

$$|\mathcal{B}| = k \cdot h \cdot (8h + 1) + k \cdot 4h \cdot \frac{v - 1}{2} + \binom{k}{2} \cdot 16h^2 + u \cdot (8u + 1) =$$
$$= ..... = 8h^2k^2 + 16uhk + kh + 8u^2 + u,$$

which is exactly equal to the number of blocks contained in a 4-cycle systems of order $8u + 1 + 8hk$:

$$(8u + 1 + 8hk)(8u + 8hk)/8.$$

This prove that Σ is a 4-cycle system of order $v + 8hk$ verifying the statement. □

4.6 All possible tricolourings for $v = 1 + 24h$

Let $A = \{a_1, a_2, \ldots, a_{2p}\}$, $B = \{b_1, b_2, \ldots, b_{2q}\}$ be any two disjoint sets.

Theorem 4.6.1. *For every $v = 1 + 24h$ and for every $\sigma = 4, 5, 6, 7$ there exists a σ-tricolourable $4CS(v)$.*

Proof Let $A = \{a_1, a_2, \ldots, a_{8h}\}$, $B = \{b_1, b_2, \ldots, b_{8h}\}$, $C = \{c_1, c_2, \ldots, c_{8h}\}$, three sets such that $A \cap B = \emptyset$, $A \cap C = \emptyset$, $B \cap C = \emptyset$. Fixed $\infty \notin A \cup B \cup C$, let

$$\Sigma_A = [A, \infty] = (A \cup \{\infty\}, B_A),$$

$$\Sigma_B = [B, \infty] = (B \cup \{\infty\}, B_B),$$

$$\Sigma_C = [C, \infty] = (C \cup \{\infty\}, B_C),$$

be 4-cycle systems of order $1 + 8h$. By *Construction 1*, we can define a 4-cycle system $\Sigma = (X, \mathcal{B})$ of order $3(v-1) + 1 = 24h + 1$, where:

$$X = A \cup B \cup C \cup \{\infty\}$$

and

$$\mathcal{B} = B_A \cup B_B \cup B_C \cup [A, B] \cup [A, C] \cup [B, C].$$

The system Σ is tricolourable with 4 colours, with 5 colours and with 6 colours. In fact, if we define a block-colouring $f : \mathcal{B} \to \Delta$, where

$$\Delta = \{\alpha, \beta, \gamma, \delta, \mu, \varrho, \ldots\}$$

is a set of *colours*, as follows:

$$f(\square) = \alpha, \quad \forall \ \square \in B_A \cup [B, C],$$

$$f(\square) = \beta, \quad \forall \ \square \in B_B \cup [A, C],$$

and

$$f(\square) = \gamma, \qquad \forall \ \square \in B_C,$$

$$f(\square) = \delta, \quad \forall \ \square \in [A, B],$$

then we can verify that every vertex $x \in X$ is of type $X^{4h} Y^{4h} Z^{4h}$ and $\alpha, \beta, \gamma, \delta$ are used colours: therefore f is a tricolouring of Σ with 4 colours.

If we define the block-colouring $g' : \mathcal{B} \to \Delta$ as follows:

$$g'(\square) = \mu, \quad \forall \ \square \in [A, C],$$

$$g'(\square) = f(\square), \quad \forall \ \square \in B \setminus [A, C],$$

then we can verify that g' is a tricolouring of Σ which uses the colours $\alpha, \beta, \gamma, \delta, \mu$.

If we define a block-colouring $g'' : \mathcal{B} \to \Delta$ (set of colours), as follows:

$$g''(\square) = \varrho, \quad \forall \ \square \in [B, C],$$

$$g''(\square) = g'(\square), \quad \forall \ \square \in B \setminus [B, C],$$

then we can verify that g'' is a tricolouring of Σ which uses 6 colours: $\alpha, \beta, \gamma, \delta, \mu, \varrho$.

To prove the existence of 4-cycle systems tricolourable with 7 colours, at first consider the following partitions of A, B, C, respectively:

$$A_1 = \{a_1, a_2, \ldots, a_{4h}\}, \quad A_2 = \{a_{4h+1}, a_{4h+2}, \ldots, a_{8h}\},$$

$$B_1 = \{b_1, b_2, \ldots, b_{4h}\}, \quad B_2 = \{b_{4h+1}, b_{4h+2}, \ldots, b_{8h}\},$$

$$C_1 = \{c_1, c_2, \ldots, c_{4h}\}, \quad C_2 = \{c_{4h+1}, c_{4h+2}, \ldots, c_{8h}\},$$

Consider now the following set of 4-cycle systems:

$$\Gamma_1 = [A_1, B_1] \cup [A_1, C_1] \cup [B_1, C_1],$$

$$\Gamma_2 = [A_2, B_2] \cup [A_2, C_1] \cup [B_2, C_1],$$

$$\Gamma_3 = [A_1, B_2] \cup [A_1, C_2] \cup [B_2, C_2].$$

$$\Gamma_4 = [A_2, B_1] \cup [A_2, C_2] \cup [B_1, C_2].$$

It is easy to check that $(X, \mathcal{B})$ with $\mathcal{B} = (\bigcup_{i=1}^{4} \Gamma_i) \cup B_A \cup B_B \cup B_C$ is a $4CS(1 + 24h)$. Define a block-colouring $\varphi : \mathcal{B} \to \Delta$ of Σ as follows:

$$\varphi(\square) = \alpha, \quad \forall \ \square \in B_A,$$

$$\varphi(\square) = \beta, \quad \forall \ \square \in B_B,$$

$$\varphi(\square) = \gamma, \quad \forall \ \square \in B_C,$$

and for each $i = 1, 2, 3, 4$

$$\varphi(\square) = i, \quad \forall \ \square \in \Gamma_i.$$

We can verify that φ is a tricolouring of Σ which uses seven colours: $\alpha, \beta, \gamma, 1, 2, 3, 4$.
$\square$

In the next section we will prove that $m = 7$ is the maximum possible value for m-tricolourable 4-cycle systems of order $v = 1 + 24h$.

4.7 The exact value of $\overline{\chi'_3}(1+24h)$

We have already proved that $\overline{\chi}'_3(1+24h) \leq 8$. Here we prove that $\overline{\chi}'_3(1+24h) = 7$. This result follows from others, which we prove separately.

In what follows, in this section, we suppose always that

$\Sigma = (X, B)$ **is any** 4**-cycle system of order** $v = 24h + 1$ **for which there exists a tricolouring** $f : B \to \Omega$**,** $\Omega = \{A_1, A_2, \ldots, A_8, \ldots\}$ **set of colours and** $\Sigma_i = (X_i, B_i)$ **is a** 4**-cycle family whose blocks are coloured with the colour** A_i**, for every** $i = 1, 2, \ldots, 8, \ldots$.

$\Sigma_i = (X_i, B_i)$ is said to be a *colouring class* of Σ.

Theorem 4.7.1. *The following properties are verified in* Σ*:*

(1) *For each* $x \in X_i$*,* x *is contained in exactly* $4h$ *blocks of* Σ_i*;*

(2) *For each* $x \in X$*,* x *is contained in exactly* 3 *sets* X_{i1}, X_{i2}, X_{i3}*;*

(3) *For each* $i = 1, 2, \ldots, 8, \ldots,$ $|B_i| = |X_i| \cdot h$*,* $[|B_i|$ *is a multiple of* $h]$*;*

(4) *For each* $i = 1, 2, \ldots, 8, \ldots,$ $|X_i| \geq 8h + 1$*.*

Proof Properties (1),...,(4) follow from definition of Σ directly.
To prove (5), observe that if for some $|X_{i^*}|$ was $|X_{i^*}| \geq 16h - 3$, then:

$$\sum_{i=1,2,\ldots,8} |X_i| \geq 7(8h + 1) + 16h - 3 = 72h + 4 > 3v,$$

and this is not true. □

Considering (4) and (5) of Theorem 4.7.1, we can put:

$$|X_i| = 8h + 1 + k_i,$$

for each $i = 1, 2, ..., 8$ and $0 \leq k_i \leq 8h - 5$.

Theorem 4.7.2. *Let $\Sigma_i = (X_i, B_i)$ be any colouring class, for $i = 1, \ldots, 8, \ldots$ For every $x \in X_i$ there are in Σ_i exactly $8h$ vertices which form an edge with x in the blocks of B_i and exactly k_i vertices of X_i which do not form an edge with x in the blocks of B_i.*

Proof Easily, in every colouring-class, every vertex is contained in exactly $4h$ blocks. □

Theorem 4.7.3. *If $\Sigma_i = (X_i, B_i)$, $\Sigma_j = (X_j, B_j)$ are two any distinct colouring-classes, then:*

$$|X_i \cap X_j| \leq k_i + k_j + 1.$$

Proof Suppose that there are two coloring-classes, let $\Sigma' = (X', B')$, $\Sigma'' = (X'', B'')$, such that:

$$|X' \cap X''| \geq k' + k'' + 2.$$

Let $x \in X' \cap X''$. Since there are exactly k' vertices of X' which *does not* form an edge with x in Σ', it follows that there are at least $k'' + 1$ vertices of $X' \cap X''$ which form an edge with x in Σ'.
For the same reason, there are exactly k'' vertices of X'' which *does not* form an edge with x in Σ'' and therefore there are at least $k' + 1$ vertices in $X' \cap X''$ which form an edge with x in Σ''.
It follows that there exists an edge $\{x, y\}$ contained in a block of B' and in another block of B'' and this is not possible. □

Theorem 4.7.4. *If $\Sigma_{i1} = (X_{i1}, B_{i1})$, $\Sigma_{i2} = (X_{i2}, B_{i2})$, $\Sigma_{i3} = (X_{i3}, B_{i3})$ are any three distinct colouring-classes, then:*

$$|X_{i1} \cup X_{i2} \cup X_{i3}| \geq 24h - (k_{i1} + k_{i2} + k_{i3}).$$

Proof Let

$$|X_{i1} \cap X_{i2}| = \alpha_{12},$$
$$|X_{i2} \cap X_{i3}| = \alpha_{23},$$
$$|X_{i1} \cap X_{i3}| = \alpha_{13}.$$

From previous Theorem it follows:

$$|X_{i1} \cup X_{i2} \cup X_{i3}| \geq 24h + 3 + (k_{i1} + k_{i2} + k_{i3}) - (\alpha_{12} + \alpha_{23} + \alpha_{13}) \geq$$
$$24h + 3 + (k_{i1} + k_{i2} + k_{i3}) - [(k_{i1} + k_{i2} + 1) + (k_{i2} + k_{i3} + 1) +$$
$$+ (k_{i1} + k_{i3} + 1)] = 24h - (k_{i1} + k_{i2} + k_{i3}).$$

□

Theorem 4.7.5. *Let* $\Sigma_{i1} = (X_{i1}, B_{i1})$, $\Sigma_{i2} = (X_{i2}, B_{i2})$, $\Sigma_{i3} = (X_{i3}, B_{i3})$, $\Sigma_{i4} = (X_{i4}, B_{i4})$ *be four distinct colouring-classes. Then, for every* $j = 1, 2, 3, 4$, *there are at least* $8h-(k_{i1}+k_{i2}+k_{i3}+2k_{i4}+2)$ *vertices belonging to* $\Sigma_{ij} = (X_{ij}, B_{ij})$, *but not belonging to the other three classes.*

Proof Without loss of generality, we prove that there are at least $8h+1-(k_{i1} + k_{i2} + k_{i3} + 2k_{i4} + 2)$ vertices of Σ_{i4}, which do not belong to $X_{i1} \cup X_{i2} \cup X_{i3}$.
Let

$$|X_{i1} \cap X_{i4}| = \beta_{14},$$

$$|X_{i2} \cap X_{i4}| = \beta_{24},$$

$$|X_{i3} \cap X_{i4}| = \beta_{34}.$$

From previous Theorems, it follows:

$$|X_{i4} - (X_{i1} \cup X_{i2} \cup X_{i3})| \geq 8h + 1 + k_{i4} - (\beta_{14} + \beta_{24} + \beta_{34}) \geq$$

$$8h + 1 + k_{i4} - [(k_{i1} + k_{i4} + 1) + (k_{i2} + k_{i4} + 1) + (k_{i3} + k_{i4} + 1) =$$

$$= 8h - (k_{i1} + k_{i2} + k_{i3} + 2 \cdot k_{i4} + 2).$$

□

Theorem 4.7.6. *If* $\Sigma_{i1} = (X_{i1}, B_{i1})$, $\Sigma_{i2} = (X_{i2}, B_{i2})$, $\Sigma_{i3} = (X_{i3}, B_{i3})$, $\Sigma_{i4} = (X_{i4}, B_{i4})$ *are four distinct colouring-classes, then:*

$$|X_{i1} \cup X_{i2} \cup X_{i3} \cup X_{i4}| \geq 32h - 2 \cdot (k_{i1} + k_{i2} + k_{i3} + k_{i4} + 1).$$

Proof From previous Theorems:

$$|X_{i1} \cup X_{i2} \cup X_{i3} \cup X_{i4}| \geq 24h - (k_{i1} + k_{i2} + k_{i3}) + 8h - (k_{i1} + k_{i2} + k_{i3} + 2 \cdot k_{i4} + 2) =$$

$$32h - 2 \cdot (k_{i1} + k_{i2} + k_{i3} + k_{i4} + 1).$$

□

Theorem 4.7.7. *If* Σ *is tricolouable with 8 colours, then there are at least four colouring-classes, let* $\Sigma_{i1} = (X_{i1}, B_{i1})$, $\Sigma_{i2} = (X_{i2}, B_{i2})$, $\Sigma_{i3} = (X_{i3}, B_{i3})$, $\Sigma_{i4} = (X_{i4}, B_{i4})$, *such that:*

$$|X_{i1}| + |X_{i2}| + |X_{i3}| + |X_{i4}| \geq 36h + 2$$

Proof Otherwise, it should be:

$$(|X_1| + |X_2| + |X_3| + |X_4|) + (|X_5| + |X_6| + |X_7| + |X_8|) \leq$$

$$\leq (36h + 1) + (36h + 1) = 72h + 2,$$

while it should be:

$$\sum_{i=1,...8} |X_i| = 3v = 72h + 3.$$

□

Theorem 4.7.8. *If Σ is tricolourable with 8 colours, then there are at least four colouring-classes, let $\Sigma_{i1} = (X_{i1}, B_{i1})$, $\Sigma_{i2} = (X_{i2}, B_{i2})$, $\Sigma_{i3} = (X_{i3}, B_{i3})$, $\Sigma_{i4} = (X_{i4}, B_{i4})$, such that:*

$$|X_{i1}| + |X_{i2}| + |X_{i3}| + |X_{i4}| \leq 36h + 1$$

Proof From previous Theorem, if

$$|X_1| + |X_2| + |X_3| + |X_4| \geq 36h + 2,$$

then:

$$|X_5| + |X_6| + |X_7| + |X_8| =$$

$$= 72h + 3 - \sum_{i=1,..,4} |X_i| \leq 72h + 3 - (36h + 2) = 36h + 1. \qquad \square$$

Theorem 4.7.9. *It is not possible that Σ is tricolourable with 8 colours.*

Proof From previous Theorems and, in particular from Theorem 4.7.8, there exist four classes $\Sigma_1 = (X_1, B_1)$, $\Sigma_2 = (X_2, B_2)$, $\Sigma_3 = (X_3, B_3)$, $\Sigma_4 = (X_4, B_4)$, such that:

$$|X_1| + |X_2| + |X_3| + |X_4| \leq 36h + 1.$$

It follows:

$$(8h + 1 + k_1) + (8h + 1 + k_2) + (8h + 1 + k_3) + (8h + 1 + k_4) \leq 36h + 1,$$

from which:

$$32h + 4 + (k_1 + k_2 + k_3 + k_4) \leq 36h + 1,$$

and therefore:

$$k_1 + k_2 + k_3 + k_4 \leq 4h - 3.$$

But, for Theorem 4.7.6:

$$|X_1 \cup X_2 \cup X_3 \cup X_4| \geq 32h - 2 \cdot (k_{i1} + k_{i2} + k_{i3} + k_{i4} + 1) \geq$$

$$\geq 32h - 2 \cdot [(4h - 3) + 1] = 24h + 4$$

and this is a contradiction. □

So, we have the following conclusive result:

Theorem 4.7.10. $\overline{\chi}'_3(1 + 24h) = 7$.

Proof The statement follows from Theorems 4.3.6, 4.7.9. □

4.8 Tricolourings with four colours

In this section we will consider tricolourings for 4-cycle systems which use 4 colours.

Lemma 4.8.1. *There exist 4-tricolourable 4-cycle systems of order 17.*

Proof Let $\Sigma = (\mathbb{Z}_{17}, \mathcal{B})$ be the C_4-system defined on $\mathbb{Z}_{17}$ as follows:

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4,$$

where

$$\mathcal{B}_1 = \{(13, 11, 6, 12), (13, 14, 6, 15), (13, 16, 6, 7), (7, 0, 8, 15), (11, 0, 12, 16),$$
$$(9, 0, 10, 15),\ (14, 10, 9, 11), (14, 12, 8, 7), (16, 9, 8, 10)\},$$

$$\mathcal{B}_2 = \{(13, 4, 0, 6), (13, 1, 14, 8), (13, 9, 12, 10), (2, 0, 5, 1), (1, 3, 2, 10),$$
$$(2, 12, 7, 5), (7, 9, 4, 10), (11, 3, 16, 7), (3, 15, 16, 8), (15, 12, 5, 11), (9, 14, 4, 6)\},$$

$$\mathcal{B}_3 = \{(12, 11, 10, 3), (12, 1, 8, 4), (8, 2, 4, 11), (5, 10, 6, 8), (3, 6, 5, 9),$$
$$(9, 1, 11, 2),\ (7, 3, 5, 4), (6, 1, 7, 2)\},$$

$$\mathcal{B}_4 = \{(13, 0, 14, 2), (13, 3, 14, 5), (15, 0, 16, 14), (15, 1, 16, 2),$$
$$(15, 5, 16, 4),\ (0, 1, 4, 3)\}.$$

If we assign the colour A_i, for each $i = 1, 2, 3, 4$, to the blocks belonging to B_i, we obtain a tricolouring of Σ with 4 colours. $\square$

Theorem 4.8.2. *There exist* 4*-tricolourable* 4*-cycle systems of order* v*, for every admissible order* $v = 1 + 8k$*.*

Proof *i*) If $v = 9$, $v = 17$, $v = 1 + 24h$ for each positive integer h, then the statement follows from Lemmas 4.4.1, 4.8.1 and Theorem 4.6.1, respectively.

ii) Let $v = 9 + 24h$, with $h > 0$. Let $A = \{a_1, a_2, \ldots, a_{8h}\}$, $B = \{b_1, b_2, \ldots, b_{8h}\}$, $C = \{c_1, c_2, \ldots, c_{8h}\}$, be any tree sets such that

$$A \cap B = A \cap C = B \cap C = \emptyset.$$

Let $D = \mathbb{Z}_9$, where $x \notin A \cup B \cup C$, for every $x \in D$.
Following *Construction 2*, we define a 4-cycle system $\Sigma = (X, \Gamma)$ of order $v = 9 + 24h$. Using the same symbolism of Theorem 4.5.2, let $\Sigma' = (\mathbb{Z}_9, D')$ be a 4-cycle system of order 9 and $A_1 = A$, $A_2 = B$, $A_3 = C$. Further, let $\Sigma_1 = (A \cup \{1\}, B_1), \Sigma_2 = (B \cup \{7\}, B_2), \Sigma_3 = (C \cup \{3\}, B_3)$ be $k = 3$ 4-cycle systems of order $w = 1 + 8h$.
Observe that here, the 4-cycle system $\Sigma' = (\mathbb{Z}_9, D')$ is exactly the system define in Lemma 4.4.1.
Finally, let

$$F_1 = \{\{0, 8\}, \{6, 7\}, \{2, 3\}, \{4, 5\}\},$$
$$F_2 = \{\{1, 2\}, \{4, 5\}, \{3, 6\}, \{0, 8\}\},$$
$$F_3 = \{\{1, 7\}, \{2, 6\}, \{0, 4\}, \{5, 8\}\}.$$

Then, define a block-colouring of Σ, let $f : \Gamma \to \Omega$, Ω set of colours, as follows:

$$f(\square) = \alpha, \quad \forall\ \square \in B_1 \cup B_2 \cup B_3 \cup [A, \{0, 8\}] \cup [B, \{4, 5\}] \cup [C, \{2, 6\}];$$

$$f(\square) = \beta, \quad \forall\ \square \in [A, B] \cup [A, \{6, 7\}] \cup [A, \{4, 5\}] \cup [B, \{1, 2\}];$$

$$f(\square) = \gamma, \quad \forall\ \square \in [B, C] \cup [B, \{3, 6\}] \cup [B, \{0, 8\}] \cup [C, \{1, 7\}];$$

$$f(\square) = \delta, \quad \forall\ \square \in [A, C] \cup [A, \{2, 3\}] \cup [C, \{5, 8\}] \cup [C, \{0, 4\}].$$

For the colouring of the blocks of Σ', f assign to them the colour described in Lemma 4.4.1, putting $A_1 = \alpha, A_2 = \beta, A_3 = \gamma, A_4 = \delta$.

We can verify that the mapping f defines a tricolouring of Σ with 4 colours.

iii) Let $v = 17 + 24h$, with $h > 0$. We follows the same construction of the case *ii)* and use the same symbolism. In this case, instead of Σ', we consider a 4-cycle system $\Sigma'' = (\mathbb{Z}_{17}, D'')$ of order 17. Observe that here, the 4-cycle system $\Sigma'' = (\mathbb{Z}_{17}, D'')$ is exactly the system defined in Lemma 4.8.1, further $A_1 = A \cup \{1\}, A_2 = B \cup \{8\}, A_3 = C \cup \{9\}$ and:

$$F_1 = \{\{2,3\},\{4,5\},\{6,7\},\{8,9\},\{10,11\},\{0,16\},\{12,13\},\{14,15\}\},$$
$$F_2 = \{\{1,9\},\{2,3\},\{4,5\},\{6,7\},\{10,11\},\{0,16\},\{12,13\},\{14,15\}\},$$
$$F_3 = \{\{0,1\},\{2,3\},\{4,5\},\{6,7\},\{8,10\},\{11,12\},\{13,14\},\{15,16\}\}.$$

Then, define a block-colouring of Σ, let $f : \Gamma \to \Omega$, Ω set of colours, as follows:

$$f(\square) = \alpha, \quad \forall \ \square \in [A,B] \cup [A,\{6,7\}] \cup [A,\{8,9)\}] \cup [A,\{10,11\}]$$
$$\cup [B,\{0,16\}] \cup [B,\{14,15\}] \cup [B,\{12,13\}];$$
$$f(\square) = \beta, \quad \forall \ \square \in B_1 \cup B_2 \cup B_3 \cup [A,\{0,16\}] \cup [A,\{12,13\}] \cup [A,\{14,15\}]$$
$$\cup [B,\{6,7\}] \cup [B,\{10,11\}] \cup [C,\{2,3\}] \cup [C,\{4,5\}];$$
$$f(\square) = \gamma, \quad \forall \ \square \in [B,C] \cup [C,\{6,7\}] \cup [C,\{8,10\}] \cup [C,\{11,12\}]$$
$$\cup [B,\{1,9\}] \cup [B,\{2,3\}] \cup [B,\{4,5\}];$$
$$f(\square) = \delta, \quad \forall \ \square \in [A,C] \cup [A,\{2,3\}] \cup [A,\{4,5\}] \cup [C,\{0,1\}]$$
$$\cup [C,\{13,14\}] \cup [C,\{15,16\}].$$

Finally, f assign to the blocks of Σ'' the same colours defined in Lemma 4.8.1, putting $A_1 = \alpha, A_2 = \beta, A_3 = \gamma, A_4 = \delta$.

We can verify that the mapping f defines a tricolouring of Σ with 4 colours.
□

4.9 Quadricolourings

In this section, we will consider quadricolourings.

Lemma 4.9.1. *If Σ is a 4-quadricolourable $4CS(1+8k)$, then $k \equiv 0 \pmod 4$.*

Proof Let $\Sigma = (V, \mathcal{B})$ be a 4-quadricolourable $4CS(1+8k)$ and let $\phi : \mathcal{B} \to \{1, 2, 3, 4\}$ be a 4-quadricolouring of Σ.
Let $\mathcal{B}_1$ be the set of all the blocks coloured by 1 and let $|\mathcal{B}_1| = a$. For each vertex, there exist $\frac{1+8k-1}{8} = k$ blocks coloured by 1 and each block contains 4 vertices.
Then: $4a = k(1+8k)$ and so: $k \equiv 0 \pmod 4$. □

Theorem 4.9.2. *The lower 4-chromatic index $\chi'_4(v)$ for 4-cycle systems is 4 if and only if $v \equiv 1$ (mod 32).*

Proof The necessary condition is in Lemma 4.9.1. Let $\Sigma = (\mathbb{Z}_{32h+1}, \mathcal{B})$ be the $4CS(32h+1)$ with starter blocks $\{(0, i, 16h+1, 4h+i) \mid 1 \leq i \leq 4h\}$. If we assign the colour j to the blocks obtained for $i = jh+1, jh+2, \ldots, jh+h$ and $j = 0, 1, 2, 3$ and to all their translated, we define a 4-quadricolouring of Σ. □

Theorem 4.9.3. *Any $4CS(9)$ is 9-quadricolourable. For every $k = 6, 7, 8, 9$ there exist k-quadricolourable $4CS(9)$s. There is not any 5-quadricolourable $4CS(9)$.*

Proof In any $4CS(9)$ there exists a 9-quadricolouring, assigning nine different colours to the nine blocks. If $\Sigma = (\mathbb{Z}_9, \mathcal{B})$ is the 4-cycle system where:

$$\mathcal{B} = \{B_1 = (8, 0, 1, 3), B_2 = (7, 2, 5, 4), B_3 = (0, 4, 2, 6),$$
$$B_4 = (1, 5, 3, 7), B_5 = (1, 4, 3, 2), B_6 = (5, 7, 8, 6),$$
$$B_7 = (0, 5, 8, 2), B_8 = (3, 6, 7, 0), B_9 = (1, 6, 4, 8)\},$$

then we can verify that there exist in Σ:

- an 8-quadricolouring by $\phi(B_1) = \phi(B_2) = 1, \phi(B_i) = i$, for $i > 2$;
- a 7-quadricolouring by $\phi(B_1) = \phi(B_2) = 1, \phi(B_3) = \phi(B_4) = 2, \phi(B_i) = i$ for $i > 4$;
- a 6-quadricolouring by $\phi(B_1) = \phi(B_2) = 1, \phi(B_3) = \phi(B_4) = 2, \phi(B_5) = \phi(B_6) = 3, \phi(B_i) = i$, for $i > 6$.

Let $\Gamma = (\mathbb{Z}_9, \mathcal{B}')$ be a 5-quadricolourable 4-cycle system and let A, B, C, D, E be the corresponding colours.

Observe that four colours (suppose A, B, C, D) are associated with two blocks, which involve 8 distinct vertices (leaving out only one vertex). Assume that $1, 2, 3, 4$ do not appear in any block coloured with A, B, C and D, respectively, and the block $(1, 2, 3, 4)$ is coloured with E.
Let $X = \{1, 2, 3, 4\}$, $Y = \{5, 6, 7, 8, 0\}$.
Denote by A_i, B_i, C_i, D_i the blocks coloured with A, B, C, D (respectively) and containing exactly i elements of X, for $i = 1, 2$. Assume that $\{2, 4\}$ is in A_2 and $\{1, 3\}$ in B_2 and so $3 \in A_1$, $4 \in B_1$.

Now, denote by $P_3(\Lambda)$ the path P_3 generated by the elements of Y contained in the block Λ_1, coloured by the colour Λ. Observe that no element of Y can be the center of two of these paths [without loss of generality, if $A_1 = (5, 7, 3, 6)$, $B_1 = (5, 0, 4, 8)$, then $\{8, 0\}$ is an edge of A_2 with repetition of a pair between $\{4, 8\}$ and $\{4, 0\}$; the other cases are immediate]. Further, for every $y \in Y$, if $y \in C_2$, then $y \notin D_2$; otherwise, y should be the center of $P_3(A)$ and $P_3(B)$.
Assume $5, 6, \in C_2$ and $7, 8 \in D_2$ and observe that 0 cannot be the center of any $P_3(\Lambda)$. So, $\{0, 3\}$,$\{0, 4\}$ are contained in A_1,B_1, respectively, and we have: $C_2 = (5, p, 6, 4)$, $D_2 = (7, q, 8, 3)$, $C_1 = (7, r, 0, 8)$, $D_1 = (5, s, 0, 6)$, where $\{p, q\}$=$\{r, s\}$=$\{1, 2\}$.
It follows $p = s$, with a contradiction. □

Remark By Theorems 4.9.2 and 4.9.3, it follows that $\Omega_4(9) = \{6, 7, 8, 9\}$.

Theorem 4.9.4. *For the upper* 4*-chromatic index* $\overline{\chi}'_4(8k+1)$ *of* $4CS(8k+1)$ *the following relations hold:*

- $\overline{\chi}'_4 = 9$*, if* $k = 1$*;*
- $\overline{\chi}'_4 \leq 13$*, if* $k = 2$*;*
- $\overline{\chi}'_4 \leq 14$*, if* $k = 3, 4, 5$*;*
- $\overline{\chi}'_4 \leq 15$*, if* $k \geq 6$.

Proof For $k = 1$ the proof is in Theorem 4.9.3. Let $\Sigma = (V, \mathcal{B})$ be a $4CS(1+8k)$, $k > 1$, and let $\phi : \mathcal{B} \to \mathcal{C}$ be a h-quadricolouring of Σ. Let $c \in \mathcal{C}$ and let $x \in V$ be an element incident with blocks of colour c. There are k blocks of colour c incident with x. Thus, there are at least $1+2k$ elements in V incident with blocks of colour c. Then: $h(1+2k) \leq 4v = 4+32k$. Hence: $h \leq \left\lfloor \frac{32k+4}{2k+1} \right\rfloor$, and so: $\overline{\chi}'_4(1+8k) \leq 13$, for $k = 2$, $\overline{\chi}'_4(1+8k) \leq 14$, for $k = 3, 4, 5$, $\overline{\chi}'_4(1+8k) \leq 15$, for $k \geq 6$. □

Chapter 5

Embeddings of K_4-designs

5.1 Introduction and prelimaries

Definition 11. *Let G_1 be a subgraph of G_2 and let V and W be two sets such that $|V| = v$, $|W| = w$, $V \subseteq W$. Denote by $(V, \mathcal{B})$ a G_1-design of order v and index λ_1, and by $(W, \mathcal{C})$ a G_2-design of order n and index λ_2. $(V, \mathcal{B})$ is* embedded *into $(W, \mathcal{C})$ if there is an injective mapping*

$$f : \mathcal{B} \to \mathcal{C}$$

such that B is subgraph of $f(B)$ for every $B \in \mathcal{B}$.

Example 1. Every affine plane of order n is embedded into some projective plane.

Example 2. Figure 5.1 shows a P_2-design $(V, \mathcal{B})$ of order 3 embedded into a balanced P_3-design $(W, \mathcal{C})$ of order 5: $V = \{0, 1, 2\}$, $W = \{0, 1, \ldots, 4\}$, $\mathcal{B} = \{[0, 1], [1, 2], [0, 2]\}$ and $\mathcal{C} = \{[0, 1, 4], [1, 2, 0], [2, 3, 1], [3, 4, 2], [4, 0, 3]\}$.

.

Example 3. A balanced P_3-design $(V, \mathcal{B})$ of order 5 strictly embedded into a 4-cycle system $(W, \mathcal{C})$ of order 9: $V = \{0, 1, \ldots, 4\}$, $W = \{0, 1, \ldots, 8\}$, $\mathcal{B} = \{[0, 4, 1], [2, 0, 3], [0, 1, 2], [4, 2, 3], [1, 3, 4]\}$ and $\mathcal{C} = \{(0, 4, 1, 6),$
$(2, 0, 3, 7), (0, 1, 2, 5), (4, 2, 3, 6), (1, 3, 4, 5), (7, 0, 8, 1), (6, 2, 8, 5), (5, 3, 8, 7),$
$(7, 4, 8, 6)\}$.

In this chapter we wish to consider the minimum embedding of an $S_3(2, 4, u)$ into an $S_\lambda(2, 4, u + w)$, $\lambda \geq 3$. In particular, we will prove the following result:

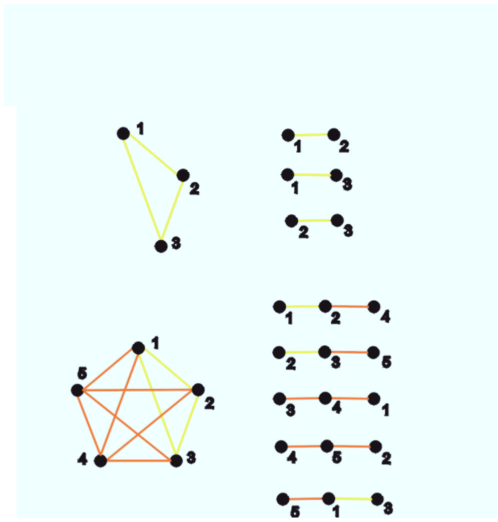


Figure 5.1: A P_2 design of order 3 embedded into a balanced P_3 design of order 5

Main Theorem. *Let $u \equiv 0, 1 \pmod 4$ and $\lambda \geq 3$. Every $S_3(2, 4, u)$ can be embedded into an $S_\lambda(2, 4, u + w)$ of minimum order $u + w$ if and only if the conditions in Table 1 are satisfied.*

Table 1

$\lambda \pmod 6, \lambda \geq 3$	$u \pmod{12}, u \geq 4$	w
3	$0, 1, 4, 5, 8, 9$	0
$2, 4$	$1, 4$	0
	$0, 9$	1
	$5, 8$ ($u \geq 17$ for $\lambda = 4, 8$)	2^a
$1, 5$	$1, 4$	0
	0	1
	5	8^b
	8	5^c
	9 ($u \geq 21$ for $\lambda = 5$)	4
0	$\forall$	0
$\lambda = 4$	$u = 5, 8$	$11, 14$
$\lambda = 8$	$u = 5, 8$	$5, 2$
$\lambda = 5$	$u = 9$	≥ 7

[a] with possible exceptions for $\lambda = 4$ and $u = 29, 32, 41, 44, 53, 56, 65$
[b] with possible exceptions for $\lambda = 5$ and $u = 29, 53$
[c] with possible exceptions for $\lambda = 5$ and $u = 32, 44$

A *pairwise balanced design* $PBD(v, K)$ of order v with block-sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set of cardinality v and $\mathcal{B}$ is a family of subsets of V (*blocks*) such that $|B| \in K$ for every $B \in \mathcal{B}$ and every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$.

We recall the existence of some 4-GDD and $PBD(v, K)$ we need in the following.

Lemma 5.1.1. *[11] There exists a 4-GDD of type*

- $u^1 1^t$ *for each* $u \equiv 4, 10 \pmod{12}$, $t \equiv 0, 9 \pmod{12}$, $t \geq 2u + 1$;
- $u^1 1^t$ *for each* $u \equiv 1, 7 \pmod{12}$, $t \equiv 0, 3 \pmod{12}$, $t \geq 2u + 1$.

Lemma 5.1.2. *[11] There exists a $PBD(v, \{4, 5\})$ for each $v \equiv 0, 1 \pmod 4$, $v \neq 8, 9, 12$.*

Lemma 5.1.3. *[10] A $PBD(v, \{4, 7^*\})$, that is a pairwise balanced design on v point with blocks of sizes 4 and exactly one block of size 7 exists if and only if $v \equiv 7, 10 \pmod{12}$, $v \neq 10, 19$.*

Lemma 5.1.4. *Let* $u \equiv 0, 1 \pmod 4$ *and* $0 \leq w < 2u+1$*,* $\lambda > 3$*. If there exists an* $S_\lambda(2,4,u+w)$ *which embeds an* $S_3(2,4,u)$ *then*

$$3\lambda\ w^2 - \lambda\ w(2u+3) + (\lambda-3)u(u-1) \geq 0.$$

Proof. Let $u \equiv 0, 1 \pmod 4$, $W = \{a_i : i \in \mathbb{Z}_w\}$ and $V = \mathbb{Z}_u \cup W$. Suppose we embed an $S_3(2,4,u)$ $(\mathbb{Z}_u, \mathcal{C})$ into a $S_\lambda(2,4,u+w)$ $(V, \mathcal{B})$. Simple counting arguments show that $|\mathcal{C}| = \frac{u(u-1)}{4}$, $|\mathcal{B}| = \frac{\lambda\ (u+w)(u+w-1)}{12}$ and every vertex of V occurs in $\lambda\frac{u+w-1}{3}$ blocks of $\mathcal{B}$. Since $w < 2u+1$, the vertices of W occur in at least $\lambda\left[\frac{w(u+w-1)}{3} - \frac{w(w-1)}{2}\right]$ blocks of $\mathcal{B} \setminus \mathcal{C}$. Then necessarily we must have

$$\lambda\left[\frac{w(u+w-1)}{3} - \frac{w(w-1)}{2}\right] \leq \lambda\frac{(u+w)(u+w-1)}{12} - \frac{u(u-1)}{4}$$

which is equivalent to

$$3\lambda\ w^2 - \lambda\ w(2u+3) + (\lambda-3)u(u-1) \geq 0.$$

□

Lemma 5.1.5. *Let* $u \equiv 0, 1 \pmod 4$*,* $\lambda \geq 3$ *and* $w \geq \frac{u-1}{2}$*. If there exists an* $S_\lambda(2,4,u+w)$ *which embeds an* $S_3(2,4,u)$ *then*

$$\lambda\ w^2 - \lambda\ w(2u+1) + 3(\lambda-3)u(u-1) \geq 0.$$

Proof. Let $u \equiv 0, 1 \pmod 4$ and $W = \{a_i : i \in \mathbb{Z}_w\}$. Suppose we embed an $S_3(2,4,u)$ $(\mathbb{Z}_u, \mathcal{C})$ into a $S_\lambda(2,4,u+w)$ $(\mathbb{Z}_u \cup W, \mathcal{B})$. Simple counting arguments show that $|\mathcal{C}| = \frac{u(u-1)}{4}$, $|\mathcal{B}| = \frac{\lambda\ (u+w)(u+w-1)}{12}$ and every vertex of $\mathbb{Z}_u$ occurs in $u-1$ blocks of $\mathcal{C}$ and

$$\lambda\frac{u+w-1}{3} - (u-1) = \frac{(\lambda-3)(u-1)+\lambda w}{3}$$

blocks of $\mathcal{B} \setminus \mathcal{C}$. Since $w \geq \frac{u-1}{2}$, the vertices of $\mathbb{Z}_u$ occur in at least

$$\frac{(\lambda-3)u(u-1)+\lambda uw}{3} - \frac{(\lambda-3)u(u-1)}{2} = \frac{2\lambda uw - (\lambda-3)u(u-1)}{6}$$

blocks of $\mathcal{B} \setminus \mathcal{C}$. Then necessarily we must have

$$\frac{2\lambda uw - (\lambda-3)u(u-1)}{6} \leq \lambda\frac{(u+w)(u+w-1)}{12} - \frac{u(u-1)}{4}$$

which is equivalent to

$$\lambda\ w^2 - \lambda\ w(2u+1) + 3(\lambda-3)u(u-1) \geq 0.$$

□

Appliyng Lemmas 5.1.4 and 5.1.5 with $u = 5, 8, 9$ and the spectrum of $S_\lambda(2, 4, u)$, we obtain the following

Corollary 5.1.6. *If there exists an*

- $S_\lambda(2, 4, 5 + w)$ *which embeds an* $S_3(2, 4, 5)$*, then* $\lambda \geq 10$ *for* $w = 2$*,* $w \geq 11$ *for* $\lambda = 4$ *and* $w \geq 5$ *for* $\lambda = 8$*;*
- $S_\lambda(2, 4, 8+w)$ *which embeds an* $S_3(2, 4, 8)$*, then* $\lambda \geq 6$ *for* $w = 2$ *and* $w \geq 14$ *for* $\lambda = 4$*;*
- $S_\lambda(2, 4, 9 + w)$ *which embeds an* $S_3(2, 4, 9)$*, then* $\lambda \geq 6$ *for* $w = 4$ *and* $w \geq 7$ *for* $\lambda = 5$*.*

5.2 Proof of Main Theorem

The necessary part of the Main Theorem follows easily from the necessary and sufficient conditions for the existence of an $S_3(2, 4, u)$ and an $S_\lambda(2, 4, u + w)$ and from Corollary 5.1.6. It is easy to see that the sufficiency of Main Theorem for $\lambda = 3, 4, 5, 6, 7, 8, 10$ implies its sufficiency for every $\lambda \geq 3$, with $\lambda = a + 6k, a = 0, 1, 2, 3, 4, 5$. The minimum embedding is obtained:

- for $a = 0, 1, 2$ and $k \geq 1$, by pasting the blocks of an $S_{a+6}(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$ to the blocks of an $S_{6(k-1)}(2, 4, u + w)$.
- for $a = 3$ and $k \geq 1$, by pasting the blocks of the given $S_3(2, 4, u)$ to the blocks of an $S_{6k}(2, 4, u)$.
- for $a = 4$, $u \neq 5$ and $k \geq 1$, by pasting the blocks of an $S_8(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$ to the blocks of an $S_{6k-4}(2, 4, u + w)$,
- for $a = 4$, $u = 5$ and $k \geq 2$, by pasting the blocks of an $S_{10}(2, 4, 5+w)$ which embeds the given $S_3(2, 4, 5)$ to the blocks of an $S_{6k-6}(2, 4, 5 + w)$
- for $a = 5$ and $k \geq 1$, by pasting the blocks of an $S_7(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$ to the blocks of an $S_{6k-2}(2, 4, u + w)$.

5.2.1 $\lambda = 4$

For $u \equiv 1, 4 \pmod{12}$ the proof of the Main Theorem follows by pasting an $S(2, 4, u)$ to the given $S_3(2, 4, u)$. For $u = 5$ and $u = 8$ the proof follows from Corollary 5.1.6 and cases 6, 9 in the Appendix.

Theorem 5.2.1. *If $u \equiv 0, 9 \pmod{12}$, $u \geq 9$ then every $S_3(2,4,u)$ can be embedded into an $S_4(2,4,u+1)$.*

Proof Let $(\mathbb{Z}_u, \mathcal{C})$ be an $S_3(2,4,u)$. Construct a 4-GDD of type $4^1 1^u$ on $\mathbb{Z}_u \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$ having $\{\infty_0, \infty_1, \infty_2, \infty_3\}$ as group of size 4 and $\mathcal{B}$ as the block-set. Let $\bar{\mathcal{B}}$ be the block-set obtained from $\mathcal{B}$ by replacing,for each $i \in \mathbb{Z}_4$, ∞_i with ∞. It is easy to check that $(\mathbb{Z}_u \cup \{\infty\}, \mathcal{C} \cup \bar{\mathcal{B}})$ is the required design. □

Theorem 5.2.2. *If $u \equiv 5, 8 \pmod{12}$, $u \geq 17$ and $u \neq 29, 32, 41, 44, 53, 56, 65$, then every $S_3(2,4,u)$ can be embedded into an $S_4(2,4,u+2)$.*

Proof. For $u = 17, 20$, see cases 7,8 in Appendix. For $u \geq 68$, write $u = x+17+12t$, $t \geq 4$ and $x = 0, 3$. Now let $X = \{a_0, a_1, ..., a_{16}\}$ (or $X = \{a_0, a_1, ..., a_{19}\}$ for $x = 3$), $U = \mathbb{Z}_{u-17} \cup X$ (or $U = \mathbb{Z}_{u-20} \cup X$ for $x = 3$) and $(U, \mathcal{D})$ be an $S_3(2,4,u)$. Construct a 4-GDD of type $25^1 1^{u-17}$ (or of type $28^1 1^{u-20}$ for $x = 3$) on $U \cup \{\infty_i, \overline{\infty_i} : i \in \mathbb{Z}_4\}$ having $X \cup \{\infty_i, \overline{\infty_i} : i \in \mathbb{Z}_4\}$ as group of size 25 (or 28 for $x = 3$) and $\mathcal{B}$ as the block-set. Let $\bar{\mathcal{B}}$ be the block-set obtained from $\mathcal{B}$ by replacing, for each $i \in \mathbb{Z}_4$, ∞_i with ∞_1 and $\overline{\infty_i}$ with ∞_2. Place on $X_1 = X \cup \{\infty_1, \infty_2\}$ an $S_4(2,4,19)$ $(X_1, \mathcal{B}_1)$ which embeds an $S_3(2,4,17)$ $(X, \mathcal{C}_1)$ on X (see cases 7, 8 in Appendix). It is easy to check that $(U \cup \{\infty_1, \infty_2\}, \mathcal{D} \cup \bar{\mathcal{B}} \cup (\mathcal{B}_1 \setminus \mathcal{C}_1)$ is the required design. □

5.2.2 $\lambda = 5$

For $u \equiv 1, 4 \pmod{12}$ the proof of the Main Theorem follows by pasting an $S_2(2,4,u)$ to the given $S_3(2,4,u)$ and for $u \equiv 0 \pmod{12}$, $u \geq 12$, by pasting an $S(2,4,u+1)$ to an $S_4(2,4,u+1)$ which embeds the given $S_3(2,4,u)$. So we suppose $u \equiv 5, 8, 9 \pmod{12}$. For $u = 9$ the proof follows from Corollary 5.1.6.

Theorem 5.2.3. *If $u \equiv 5 \pmod{12}$, $u \neq 29, 53$, then every $S_3(2,4,u)$ can be embedded into an $S_5(2,4,u+8)$.*

Proof For $u = 5, 17, 41$, see cases 10, 13 and 15 in Appendix. For $u \geq 65$ write $u = 5 + 12t$, $t \geq 5$. Now let $X = \{a_0, a_1, a_2, a_3, a_4\}$, $U = \mathbb{Z}_{u-5} \cup X$ and $(U, \mathcal{D})$ be an $S_3(2,4,u)$. Construct a 4-GDD of type $25^1 1^{u-5}$ (see Lemma 5.1.1) on $\mathbb{Z}_{u-5} \cup \{a_{ij} : (i,j) \in \mathbb{Z}_5 \times \mathbb{Z}_2\} \cup \{b_{ij} : (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_5\}$ and a 4-GDD of type $25^1 1^{u-5}$ on $\mathbb{Z}_{u-5} \cup \{\infty_{ij} : (i,j) \in \mathbb{Z}_5 \times \mathbb{Z}_5\}$. For each $j \in \mathbb{Z}_2$, replace a_{ij} with a_i, for each $k \in \mathbb{Z}_5$, replace ∞_{ik} with ∞_i and b_{ik} with b_i and denote by $\bar{\mathcal{B}}$ be the block-set so obtained. On $\{a_0, a_1, a_2, a_3, a_4\} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\} \cup \{b_0, b_1, b_2\}$, place an $S_5(2,4,13)$ $(V_1, \mathcal{B}_1)$ which embeds an $S_3(2,4,5)$ $(X, \mathcal{C}_1)$ on $\{a_0, a_1, a_2, a_3, a_4\}$ (see case 10 in Appendix). Let $V = U \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\} \cup \{b_0, b_1, b_2\}$, $\mathcal{C} = \mathcal{B}_1 \setminus \mathcal{C}_1$, $\mathcal{B} = \mathcal{D} \cup \bar{\mathcal{B}} \cup \mathcal{C}$. It is easy to check that (V, B) is the required design. □

Theorem 5.2.4. *If $u \equiv 8 \pmod{12}$, $u \neq 32, 44$, then every $S_3(2,4,u)$ can be embedded into an $S_5(2,4,u+5)$.*

Proof For $u = 8, 20$, see cases 11 and 14 in Appendix. For $u \geq 56$ write $u = 8 + 12t$, $t \geq 4$. Now let $X = \{a_i, i \in \mathbb{Z}_8\}$, $U = \mathbb{Z}_{u-8} \cup X$ and $(U, \mathcal{D})$ be an $S_3(2,4,u)$. Construct a 4-GDD of type $19^1 1^{u-8}$ (see Lemma 5.1.1) on $\mathbb{Z}_{u-8} \cup \{a_{ij} : (i,j) \in \mathbb{Z}_8 \times \mathbb{Z}_2\} \cup \{\infty_{4j} : j \in \mathbb{Z}_3\}$ and a 4-GDD of type $22^1 1^{u-8}$ on $\mathbb{Z}_{u-8} \cup \{\infty_{ij} : (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_5\} \cup \{\infty_{4j} : j = 3, 4\}$. For each $j \in \mathbb{Z}_2$, replace a_{ij} with a_i, for each $k \in \mathbb{Z}_5$, replace ∞_{ik} with ∞_i and denote by $\bar{\mathcal{B}}$ be the block-set so obtained. On $X \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$, place an $S_5(2,4,13)$ $(V_1, \mathcal{B}_1)$ which embeds an $S_3(2,4,8)$ $(X, \mathcal{C}_1)$ on X. Let $V = U \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$, $\mathcal{C} = \mathcal{B}_1 \backslash \mathcal{C}_1$ $\mathcal{B} = \mathcal{D} \cup \bar{\mathcal{B}} \cup \mathcal{C}$. It is easy to check that (V, B) is the required design. □

Theorem 5.2.5. *If $u \equiv 9 \pmod{12}$, $u \geq 21$ then every $S_3(2,4,u)$ can be embedded into an $S_5(2,4,u+4)$.*

Proof Let $(\mathbb{Z}_u, \mathcal{D})$ be an $S_3(2,4,u)$. For $u = 21$, see case 12 in Appendix. For $u \geq 33$ write $u = 9 + 12t$, $t \geq 2$. Take a 4-GDD of type $16^1 1^u$ (see Lemma 5.1.1) on $\mathbb{Z}_u \cup \{\infty_{ij} : (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4\}$ having $G = \{\infty_{ij} : (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4\}$ as group of size 16 and $\mathcal{B}_1$ as the block-set. Let $\bar{\mathcal{B}}$ be the block-set obtained from $\mathcal{B}_1$ by replacing, for each $j \in \mathbb{Z}_4$, $\infty_{i,j}$ with ∞_i. Put in $\mathcal{C}$ the blocks of an $S_4(2,4,4)$ on $\{\infty_0, \infty_1, \infty_2, \infty_3\}$ and the blocks of an $S(2,4,13+12t)$ on $V = U \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. Let $\mathcal{B} = \mathcal{D} \cup \bar{\mathcal{B}} \cup \mathcal{C}$. It is easy to check that $(V, \mathcal{B})$ is the required design. □

5.2.3 $\lambda = 6$

The proof of the Main Theorem follows by doubling the solution for $\lambda = 3$. The following result will be used in this chapter.

Theorem 5.2.6. *If $u \equiv 5, 8 \pmod{12}$, then every $S_3(2,4,u)$ can be embedded into an $S_6(2,4,u+1)$.*

Proof For $u = 5, 8, 17$ see cases 16, 18 and 19 in Appendix. For $u \geq 20$, write $u = x + 5 + 12t$, $t \geq 2$ and $x = 0, 3$. Let $(U, \mathcal{D})$ be an $S_3(2,4,u)$ where $U = \mathbb{Z}_{u-5} \cup \{a_0, a_1, a_2, a_3, a_4\}$. Construct a 4-GDD of type $7^1 1^u$ on $U \cup \{\infty_0, \infty_1\}$ having $\{a_0, a_1, a_2, a_3, a_4\} \cup \{\infty_0, \infty_1\}$ as the group of size 7. Replace, for each $i \in \mathbb{Z}_2$, ∞_i with ∞ and repeat the blocks so obtained three times. Develop (mod 5) the base blocks $\{\infty, a_0, a_1, a_2\}$, $\{\infty, a_0, a_1, a_3\}$. The result is an $S_6(2,4,u+1)$ on $V = U \cup \{\infty\}$ which embeds the $S_3(2,4,u)$ $(U, \mathcal{D})$. □

5.2.4 $\lambda = 7$

For $u \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ and for $u \neq 9, 29, 32, 44, 53$ the proof of the Main Theorem follows by pasting an $S_2(2, 4, u+w)$ to an $S_5(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$. For $u = 9, 29, 32, 44, 53$ see cases 22, 23, 24, 26, 25 in Appendix.

5.2.5 $\lambda = 8$

For $u \equiv 0, 1, 4, 9 \pmod{12}$ the proof of the Main Theorem follows by doubling the solution for $\lambda = 4$. So we suppose $u \equiv 5, 8 \pmod{12}$. For $u = 5$ the proof follows from Corollary 5.1.6, by embedding the given $S_3(2, 4, 5)$ into an $S_6(2, 4, 10)$ (see case 17 in Appendix) and by adding the blocks of an $S_2(2, 4, 10)$.

Theorem 5.2.7. *If* $u \equiv 5 \pmod{12}$, $u \geq 17$ *then every* $S_3(2, 4, u)$ *can be embedded into an* $S_8(2, 4, u + 2)$.

Proof Let $U = \mathbb{Z}_{u-7} \cup \{a_i, i \in \mathbb{Z}_7\}$. Embed an $S_3(2, 4, u)$ on U into an $S_6(2, 4, u+1)$ on $U \cup \{\infty_0\}$. Construct on $U \cup \{c_0, c_1, c_2, c_3\} \cup \{\infty_0\}$ a $PBD(10 + 12t, \{4, 7^*\})$ having $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ as the block of size 7 and $\{c_0, c_1, c_2, c_3\}$ as a block of size 4. Replace, for each $i \in \mathbb{Z}_4$, c_i with ∞ and repeat the blocks so obtained twice, after removing the block of size 7 and the block $\{c_0, c_1, c_2, c_3\}$. Place on $\{a_i, i \in \mathbb{Z}_7\}$ an $S_2(2, 4, 7)$. The result is an $S_8(2, 4, u + 2)$ on $V = U \cup \{\infty_0, \infty\}$ which embeds an $S_3(2, 4, u)$ on U. □

Theorem 5.2.8. *If* $u \equiv 8 \pmod{12}$, $u \geq 8$ *then every* $S_3(2, 4, u)$ *can be embedded into an* $S_8(2, 4, u + 2)$.

Proof For $u = 8$ see case 27 in Appendix. For $u \geq 20$ embed an $S_3(2, 4, u)$ on $\mathbb{Z}_u$ into an $S_6(2, 4, u + 1)$ on $\mathbb{Z}_u \cup \{\infty_0\}$. Construct a 4-GDD of type $4^1 1^{u+1}$ on $\mathbb{Z}_u \cup \{\infty_0\} \cup \{a_0, a_1, a_2, a_3\}$ having $\{a_0, a_1, a_2, a_3\}$ as the group of size 4. Replace, for each $i \in \mathbb{Z}_4$, a_i with ∞ and repeat the blocks so obtained twice. The result is an $S_8(2, 4, u + 2)$ on $V = \mathbb{Z}_u \cup \{\infty_0, \infty\}$ which embeds an $S_3(2, 4, u)$ on $\mathbb{Z}_u$. □

5.2.6 $\lambda = 10$

For $u = 5$ see case 28 in Appendix. For $u \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ and $u \neq 5$ the proof of the Main Theorem follows by pasting an $S_2(2, 4, u+w)$ to an $S_8(2, 4, u+w)$ which embeds an $S_3(2, 4, u)$.

5.3 Applications for other designs

In this section,we shall use the Main Theorem to give new results on E_2-designs. Let E_2 be the graph $[a, b; c, d]$ having vertices $\{a, b, c, d\}$ and edges $\{a, b\}$, $\{c, d\}$. An E_2-design of order u and index 1, $E_2(u, 1)$, exists if and only if $u \equiv 0, 1 \pmod 4$.

Lemma 5.3.1. *Let $u \equiv 0, 1 \pmod 4$. If there exists an $S_\lambda(2, 4, u + w)$ which embeds an $E_2(u, 1)$ then $\lambda \geq 3$.*

Proof. Let $u \equiv 0, 1 \pmod 4$. Suppose we embed an $E_2(u, 1)$ $(U, \mathcal{C})$ into an $S_\lambda(2, 4, u+w)$ $(V, \mathcal{B})$. Counting the number of edges of λK_u not covered by blocks of $\mathcal{C}$ we obtain $\lambda \frac{u(u-1)}{2} \geq 6 \frac{u(u-1)}{4}$ and hence $\lambda \geq 3$. □

Lemma 5.3.2. *If $u \equiv 0, 1 \pmod 4$, $u \geq 4$ then there is an $S_3(2, 4, u)$ which embeds an $E_2(u, 1)$.*

Proof. For $u = 4, 5, 8, 9, 12$ see cases 1,2,3,4 and 5 in Appendix. For $u \geq 13$, take a $PBD(u, \{4, 5\})$ (see Lemma 5.1.2) and place on each block an $S_3(2, 4, k)$ wich embeds an $E_2(k, 1)$, with $k = 4, 5$. □

Now using the results of the Main Theorem and Lemma 5.3.2 we obtain the following new results for an $E_2(u, 1)$.

Theorem 5.3.3. *Let $u \equiv 0, 1 \pmod 4$ and $\lambda \geq 3$. Then there exists a minimum embedding of an $E_2(u, 1)$ into an $S_\lambda(2, 4, u + w)$ if and only if the conditions in Table 1 are satisfied.*

Appendix to Chapter 5

In this appendix we list some minimum embeddings of an $S_3(2,4,u)$ $(U,\mathcal{C})$ into an $S_\lambda(2,4,u+w)$ $(V,\mathcal{B})$, $V=U\cup W$, for small values of u. *Only* for $\lambda=3$ we list five minimum embeddings of an $E_2(u,1)$ into an $S_3(2,4,u)$. In these cases we list the blocks of an $E_2(u,1)$-design using square brackets (braces). For example, $[x,y;z,t]$ is the block of an $E_2(u,1)$-design having vertices x,y,z,t and edges $\{x,y\}$ and $\{z,t\}$.

1. $\lambda=3$, $u=4$, $w=0$. Let $U=\mathbb{Z}_4$. Blocks: $[0,1;2,3]$, $[0,3;1,2]$, $[0,2;1,3]$.

2. $\lambda=3$, $u=5$, $w=0$. Let $U=\mathbb{Z}_5$. Develop (mod 5) the base block $[0,1;2,4]$.

3. $\lambda=3$, $u=8$, $w=0$. Let $U=\mathbb{Z}_7\cup\{\infty\}$. Develop (mod 7) the base blocks $[0,1;3,6]$,
   $[\infty,3;0,2]$.

4. $\lambda=3$, $u=9$, $w=0$. Let $U=\mathbb{Z}_8\cup\{\infty\}$. Develop (mod 8) the base blocks: $[0,1;4,7]$,$[\infty,3;0,2]$. Add the following blocks: $[0,4;2,6]$, $[1,5;3,7]$.

5. $\lambda=3$, $u=12$, $w=0$. Let $U=\mathbb{Z}_{11}\cup\{\infty\}$. Develop (mod 11) the base blocks $[0,1;4,10]$, $[\infty,6;0,4]$, $[0,3;4,6]$.

6. $\lambda=4$, $u=5$, $w=11$. Let $V=\mathbb{Z}_5\cup\{a_i : i\in Z_{11}\}$. Embed an $S_3(2,4,5)$ on $\mathbb{Z}_5$ into an $S_3(2,4,16)$ on V. Paste an $S(2,4,16)$ on V. The result is an $S_4(2,4,16)$ on V which embeds an $S_3(2,4,5)$ on $\mathbb{Z}_5$.

7. $\lambda=4$, $u=17$, $w=2$. Let $U=\{i,i' : i\in\mathbb{Z}_8\}\cup\{\infty\}$ and $V=U\cup\{a,b\}$. Take on U an $S_3(2,4,17)$. Put

   $C=\{\{\infty,1,7\},\{\infty,2,0\},\{\infty,3,5\},\{\infty,4,6\},\{1,1',5'\},\{2,2',6'\},\{3,3',7'\},$ $\{4,4',0'\},\{1,2',7'\},\{2,3',0'\},\{3,4',5'\},\{4,1',6'\},\{5,2',3'\},\{6,3',4'\},\{7,4',1'\},$ $\{0,1',2'\},\{5,5',6'\},\{6,6',7'\},\{7,7',0'\},\{0,0',5'\}\}$.

   $D=\{\infty,1',3'\},\{\infty,2',4'\},\{\infty,5',7'\},\{\infty,6',0'\},\{1,4',6'\},\{2,1',7'\},$ $\{3,2',0'\},\{4,3',5'\},\{1,3',0\},\{2,4',5\},\{3,1',6\},\{4,2',7\},\{1,0',6\},\{2,5',7\},$ $\{3,6',0\},\{4,7',5\},\{5,1',0'\},\{6,2',5'\},\{7,3',6'\},\{0,4',7'\}\}$.

   Take the blocks $\{a,x,y,z\}$ for any $\{x,y,z\}\in C$ and $\{b,x,y,z\}$ for any $\{x,y,z\}\in D$. At last add the blocks: $\{1,2,3,4\},\{5,6,7,0\}$,
   $\{a,b,1,5\}$, $\{a,b,2,6\},\{a,b,3,7\},\{a,b,4,0\}$. The result is an $S_4(2,4,19)$ on V which embeds an $S_3(2,4,17)$ on U.

8. $\lambda=4$, $u=20$, $w=2$. Let $U=\{i,i' : i\in\mathbb{Z}_{10}\}$ and $V=U\cup\{a,b\}$. Take on U an $S_3(2,4,20)$. Put

$C = \{\{0', 2, 4\}, \{1', 1, 3\}, \{2', 6, 8\}, \{3', 5, 7\}, \{4', 6, 3\}, \{5', 1, 8\},$
$\{6', 4, 5\}, \{7', 7, 2\}, \{9, 1, 6\}, \{9, 2, 8\}, \{9, 3, 5\}, \{9, 4, 7\}, \{0, 0', 5'\},$
$\{0, 1', 6'\}, \{0, 2', 7'\}, \{0, 3', 4'\}, \{8', 4, 4'\}, \{8', 2, 2'\}, \{8', 5, 5'\}, \{8', 7, 1'\},$
$\{9', 0', 3\}, \{9', 3', 6\}, \{9', 6', 8\}, \{9', 7', 1\}\}.$

$D = \{\{0', 6, 7\}, \{1', 5, 8\}, \{2', 1, 4\}, \{3', 2, 3\}, \{4', 7, 8\}, \{5', 3, 4\},$
$\{6', 1, 2\}, \{7', 5, 6\}, \{9, 0', 6'\}, \{9, 1', 7'\}, \{9, 2', 4'\}, \{9, 3', 5'\}, \{0, 1, 7\},$
$\{0, 2, 5\}, \{0, 3, 8\}, \{0, 4, 6\}, \{8', 3', 1\}, \{8', 7', 3\}, \{8', 0', 8\}, \{8', 6', 6\},$
$\{9', 2, 4'\}, \{9', 5, 2'\}, \{9', 4, 1'\}, \{9', 7, 5'\}\}.$

Take the blocks $\{a, x, y, z\}$ for any $\{x, y, z\} \in C$ and $\{b, x, y, z\}$ for any $\{x, y, z\} \in D$. At last add the blocks: $\{9, 0, 8', 9'\}, \{0', 1', 2', 3'\}, \{4', 5', 6', 7'\},$ $\{0', 4', 1, 5\}, \{1', 5', 2, 6\}, \{2', 6', 3, 7\}, \{3', 7', 4, 8\}, \{a, b, 0', 7'\},$ $\{a, b, 1', 4'\}, \{a, b, 2', 5'\}, \{a, b, 3', 6'\}$. The result is an $S_4(2, 4, 22)$ on V which embeds an $S_3(2, 4, 20))$ on U.

9. $\lambda = 4$, $u = 8$, $w = 14$. Let $U = \{a_i : i \in \mathbb{Z}_8\}$, $W = \mathbb{Z}_{14}$ and $V = U \cup W$. Take on U an $S_3(2, 4, 8)$. The edges of K_{14} may be factored into a set of 7 disjoint classes P_1, P_2,...,P_7 where $(i, j) \in P_k$ if and only of $i - j \equiv k \pmod{14}$. For $i \in Z_{14}$, let $T_0 = \{i, 6 + i, 5 + i\}$, $T_1 = \{i, 2 + i, 5 + i\}$, $T_2 = \{i, 2 + i, 6 + i\}$, $T_3 = \{i, 3 + i, 4 + i\}$ be four sets of 14 triangles covering respectively P_1, P_2,...,P_6 repeated twice times. For $i = 0, 1, 2, 3$, put $T_{i+4} = T_i$. For $i = 0, 1, ..., 7$, construct the blocks $\{a_i, x, y, z\}$, $\{x, y, z\} \in T_i$. Let F_0, F_2,...,F_6 be the 1-factors of a 1-factorization of the complete graph K_8 on U. For $i = 0, 1, \ldots, 6$, construct the blocks $\{i, i+7, x, y\}$, $\{x, y\} \in F_i$. The result is an $S_4(2, 4, 22)$ on V which embeds an $S_3(2, 4, 8)$ on U.

10. $\lambda = 5$, $u = 5$, $w = 8$. Let $U = \{a_i : i \in \mathbb{Z}_5\}$, $W = \mathbb{Z}_8$ and $V = U \cup W$. Take on U an $S_3(2, 4, 5)$. For $i \in \mathbb{Z}_5$, develop (mod 8) the base block $\{a_i, 0, 1, 3\}$. Add the blocks: $\{a_0, a_1, 0, 4\}$, $\{a_0, a_1, 1, 5\}$, $\{a_0, a_2, 2, 6\}$, $\{a_0, a_2, 3, 7\}$, $\{a_0, a_4, 0, 4\}$, $\{a_0, a_4, 1, 5\}$, $\{a_0, a_3, 2, 6\}$, $\{a_0, a_3, 3, 7\}$, $\{a_1, a_3, 0, 4\}$, $\{a_1, a_3, 1, 5\}$
$\{a_1, a_2, 2, 6\}$, $\{a_1, a_2, 3, 7\}$, $\{a_2, a_3, 0, 4\}$, $\{a_2, a_3, 1, 5\}$, $\{a_1, a_4, 2, 6\}$
$\{a_1, a_4, 3, 7\}$, $\{a_2, a_4, 0, 4\}$, $\{a_2, a_4, 1, 5\}$, $\{a_3, a_4, 2, 6\}$, $\{a_3, a_4, 3, 7\}$. The result is an $S_5(2, 4, 13)$ on V which embeds an $S_3(2, 4, 5)$ on U.

11. $\lambda = 5$, $u = 8$, $w = 5$. Let $U = \mathbb{Z}_8$, $W = \{a, b, c, d, e\}$ and $V = U \cup W$. Take on U an $S_3(2, 4, 8)$. Add the blocks:
$\{a, b, c, 1\}, \{a, 1, 2, 3\}, \{b, 1, 4, 5\}, \{c, 6, 7, 8\}, \{a, b, 1, 4\}, \{a, b, 2, 6\},$
$\{a, b, 3, 8\}, \{a, b, 5, 7\}, \{a, c, 1, 6\}, \{a, c, 2, 3\}, \{a, c, 4, 8\}, \{a, c, 5, 7\},$
$\{b, c, 1, 7\}, \{b, c, 2, 6\}, \{b, c, 3, 4\}, \{b, c, 5, 8\}, \{a, d, 2, 4\}, \{a, d, 3, 5\},$
$\{a, d, 4, 8\}, \{a, d, 2, 5\}, \{a, d, 6, 7\}, \{a, e, 1, 8\}, \{a, e, 6, 8\}, \{a, e, 5, 6\},$
$\{a, e, 3, 7\}, \{a, e, 4, 7\}, \{b, d, 1, 6\}, \{b, d, 3, 6\}, \{b, d, 4, 7\}, \{b, d, 3, 8\},$

$\{b, d, 8, 5\}, \{b, e, 6, 4\}, \{b, e, 2, 7\}, \{b, e, 3, 7\}, \{b, e, 2, 8\}, \{b, e, 2, 5\}$, $\{c, d, 1, 2\}, \{c, d, 4, 6\}, \{c, d, 2, 7\}, \{c, d, 3, 5\}, \{c, d, 7, 8\}, \{c, e, 1, 3\}$, $\{c, e, 4, 5\}, \{c, e, 5, 6\}, \{c, e, 3, 4\}, \{c, e, 2, 8\}, \{d, e, 1, 5\}, \{d, e, 1, 7\}$, $\{d, e, 1, 8\}, \{d, e, 2, 4\}, \{d, e, 3, 6\}$. The result is an $S_5(2, 4, 13)$ on V which embeds an $S_3(2, 4, 8)$ on U.

12. $\lambda = 5$, $u = 21$, $w = 4$. Let $U = \mathbb{Z}_{21}$ and $V = \mathbb{Z}_5 \cup \{a_0, a_1, a_2, a_3\}$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 21)$ on $\mathbb{Z}_{21}$. Take on U a resolvable $S_2(2, 3, 21)$ having the resolution classes R_j, $j = 0, 1, ..., 19$. For each $i = 0, 1, 2, 3$, place the blocks $\{a_i, x, y, z\}$, $\{x, y, z\} \in \bigcup_{j=0}^{4} \mathcal{R}_{5i+j}$. Add the blocks of an $S_5(2, 4, 4)$ on $\{a_0, a_1, a_2, a_3\}$ and the result is an $S_5(2, 4, 25)$ on V which embeds an $S_3(2, 4, 21)$ on U.

13. $\lambda = 5$, $u = 17$, $w = 8$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 17)$ having $U = (\mathbb{Z}_8 \times \{0, 1\}) \cup \{\infty\}$ as point-set and let V be the set $(\mathbb{Z}_8 \times \{0, 1, 2\}) \cup \{\infty\}$.
Let us develop (mod 8) the following 24 base blocks:
$\{0_0, 2_0, 4_0, 6_0\}, \{0_0, 4_0, 1_2, 5_2\}, \{0_0, 3_0, 7_2, \infty\}, \{0_1, 2_2, 5_2, \infty\}, \{0_1, 3_2, 5_2, \infty\}$, $\{0_0, 0_1, 0_2, 1_2\}, \{0_0, 0_1, 0_2, 2_2\}, \{0_0, 1_1, 0_2, 1_2\}, \{0_0, 1_1, 0_2, 2_2\}, \{0_0, 2_1, 0_2, 1_2\}$, $\{0_0, 2_1, 1_2, 3_2\}, \{0_0, 3_1, 2_2, 6_2\}, \{0_0, 3_1, 4_2, 7_2\}, \{0_0, 4_1, 4_2, 7_2\}, \{0_0, 4_1, 6_2, 7_2\}$, $\{0_0, 5_1, 3_2, 7_2\}, \{0_0, 6_1, 2_2, 4_2\}, \{0_0, 6_1, 3_2, 6_2\}, \{0_0, 7_1, 2_2, 5_2\}, \{0_1, 1_1, 3_1, 5_2\}$, $\{0_1, 1_1, 4_1, 5_2\}, \{0_0, 1_0, 3_0, 6_2\}, \{0_0, 1_0, 5_2, 6_2\}, \{0_0, 5_1, 7_1, 3_2\}$.
We get 2 blocks from the first base block, 4 blocks from the second base block, and 8 blocks from each one of the other twenty-two base blocks. Add these 182 blocks to the 68 of $\mathcal{C}$ and denote by $\mathcal{B}$ the set containing all these 250 blocks. The result is an $S_5(2, 4, 25)$ $(V, \mathcal{B})$ which embeds an $S_3(2, 4, 17)$ $(U, \mathcal{C})$.

14. $\lambda = 5$, $u = 20$, $w = 5$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 20)$ having $U = \mathbb{Z}_5 \times \{0, 1, 2, 3\}$ as point-set and let V be the set $\mathbb{Z}_5 \times \{0, 1, 2, 3, 4\}$.
Let us develop (mod 5) the following 31 base blocks: $\{0_3, 0_2, 0_1, 4_0\}$, $\{0_4, 0_3, 1_3, 2_3\}, \{0_4, 0_2, 1_2, 2_2\}, \{0_4, 0_1, 1_1, 2_1\}, \{0_4, 0_0, 1_0, 2_0\}, \{0_4, 1_4, 0_3, 2_3\}$, $\{0_4, 1_4, 0_2, 2_2\}, \{0_4, 1_4, 0_1, 2_1\}, \{0_4, 1_4, 0_0, 2_0\}, \{0_4, 1_4, 0_3, 0_2\}, \{0_4, 2_4, 0_3, 0_1\}$, $\{0_4, 2_4, 0_3, 0_0\}, \{0_4, 2_4, 0_2, 0_1\}, \{0_4, 2_4, 0_2, 0_0\}, \{0_4, 2_4, 0_1, 0_0\}, \{0_4, 1_3, 1_2, 1_1\}$, $\{0_4, 1_3, 2_2, 3_1\}, \{0_4, 1_3, 2_2, 3_1\}, \{0_4, 2_3, 4_2, 1_1\}, \{0_4, 2_3, 1_2, 3_0\}, \{0_4, 2_3, 3_2, 4_0\}$, $\{0_4, 3_3, 3_2, 4_0\}, \{0_4, 3_3, 3_2, 1_0\}, \{0_4, 4_3, 4_1, 1_0\}, \{0_4, 4_3, 4_1, 2_0\}, \{0_4, 4_3, 4_1, 4_0\}$, $\{0_4, 3_3, 1_1, 2_0\}, \{0_4, 1_2, 2_1, 4_0\}, \{0_4, 2_2, 4_1, 2_0\}, \{0_4, 4_2, 2_1, 1_0\}, \{0_4, 4_2, 2_1, 3_0\}$.
We get 5 blocks from each base block. Add these 155 blocks to the 95 of $\mathcal{C}$ and denote by $\mathcal{B}$ the set containing all these 250 blocks. The result is an $S_5(2, 4, 25)$ $(V, \mathcal{B})$ which embeds an $S_3(2, 4, 20)$ $(U, \mathcal{C})$.

15. $\lambda = 5$, $u = 41$, $w = 8$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 41)$ having $U = (\mathbb{Z}_8 \times \{0, 1, 2, 3, 4\}) \cup \{\infty\}$ as point-set and let V be the set $(\mathbb{Z}_8 \times \{0, 1, 2, 3, 4, 5\}) \cup$

$\{\infty\}$. Let us develop (mod 8) the following 72 base blocks:
$\{0_5,0_4,1_4,2_4\},\{0_5,0_4,2_4,5_4\},\{0_5,0_3,1_3,2_3\},\{0_5,0_3,2_3,5_3\},\{0_5,0_2,1_2,2_2\},$
$\{0_5,0_2,2_2,5_2\},\{0_5,0_1,1_1,2_1\},\{0_5,0_1,2_1,5_1\},\{0_5,0_0,1_0,2_0\},\{0_5,0_0,2_0,5_0\},$
$\{0_5,1_5,0_4,4_4\},\{0_5,1_5,0_3,4_3\},\{0_5,1_5,0_2,4_2\},\{0_5,1_5,0_1,4_1\},\{0_5,1_5,0_0,4_0\},$
$\{0_5,2_5,0_4,0_3\},\{0_5,2_5,1_4,0_2\},\{0_5,2_5,1_4,1_1\},\{0_5,2_5,1_4,1_0\},\{0_5,3_5,1_3,1_2\},$
$\{0_5,3_5,1_3,1_1\},\{0_5,3_5,1_3,1_0\},\{0_5,3_5,1_2,1_1\},\{0_5,3_5,1_2,1_0\},\{0_5,4_5,1_1,1_0\},$
$\{0_5,1_4,2_3,1_2\},\{0_5,2_4,1_3,3_2\},\{0_5,2_4,3_3,4_2\},\{0_5,2_4,4_3,3_2\},\{0_5,3_4,2_3,3_1\},$
$\{0_5,3_4,5_3,2_1\},\{0_5,3_4,6_3,2_1\},\{0_5,3_4,7_3,4_1\},\{0_5,4_4,2_3,2_0\},\{0_5,4_4,7_3,2_0\},$
$\{0_5,5_4,3_3,2_0\},\{0_5,4_4,2_2,5_1\},\{0_5,4_4,2_2,6_1\},\{0_5,5_4,2_2,3_1\},\{0_5,5_4,4_2,2_1\},$
$\{0_5,5_4,7_2,4_0\},\{0_5,6_4,3_2,5_0\},\{0_5,6_4,7_2,3_0\},\{0_5,6_4,3_1,7_0\},\{0_5,6_4,4_1,7_0\},$
$\{0_5,7_4,7_1,3_0\},\{0_5,3_3,3_2,5_1\},\{0_5,4_3,4_2,5_1\},\{0_5,5_3,5_2,7_1\},\{0_5,4_3,5_2,7_0\},$
$\{0_5,7_3,7_2,4_0\},\{0_5,4_3,5_2,6_0\},\{0_5,7_3,6_2,4_0\},\{0_5,3_3,7_1,5_0\},\{0_5,5_3,4_1,6_0\},$
$\{0_5,5_3,6_1,6_0\},\{0_5,5_2,4_1,3_0\},\{0_5,4_2,3_1,5_0\},\{0_5,7_2,6_1,3_0\},\{0_4,5_3,3_2,3_0\},$
$\{0_4,5_3,3_1,4_0\},\{0_4,4_3,3_1,2_0\},\{0_4,3_2,4_1,2_0\},\{0_4,4_2,2_1,3_0\},\{0_3,1_2,7_1,4_0\},$
$\{0_5,0_4,0_0,\infty\},\{0_5,0_2,0_1,\infty\},\{0_5,0_3,0_1,\infty\},\{0_4,0_3,0_2,\infty\},\{0_5,4_5,0_0,\infty\},$
$\{0_5,2_5,4_5,6_5\},\{0_3,1_2,1_1,4_0\}$.

We get 2 blocks from the first base block and 8 blocks from each one of the seventy-one other base blocks. Add these 570 blocks to the 410 of $\mathcal{C}$ and denote by $\mathcal{B}$ the set containing all these 980 blocks. The result is an $S_5(2,4,49)$ $(V,\mathcal{B})$ which embeds an $S_3(2,4,41)$ $(U,\mathcal{C})$.

16. $\lambda=6$, $u=5$, $w=1$. Let $U=\mathbb{Z}_5$ and $V=\mathbb{Z}_5\cup\{\infty\}$. Let $(U,\mathcal{C})$ be an $S_3(2,4,5)$ on $\mathbb{Z}_5$. Develop (mod 5) the base blocks: $\{\infty,0,1,2\},\{\infty,0,2,3\}$. The result is an $S_6(2,4,6)$ on V which embeds an $S_3(2,4,5)$ on $\mathbb{Z}_5$.

17. $\lambda=6$, $u=5$, $w=5$. Let $U=\mathbb{Z}_5\times\{0\}$, $W=\mathbb{Z}_5\times\{1\}$ and $V=U\cup W$. Let $(U,\mathcal{C})$ be an $S_3(2,4,5)$ on U. Develop (mod 5) the base blocks: $\{0_0,1_0,0_1,1_1\}$, $\{0_0,1_0,2_1,3_1\}$, $\{0_0,1_0,3_1,4_1\}$, $\{0_0,2_0,0_1,2_1\}$, $\{0_0,2_0,4_1,1_1\}$, $\{0_0,2_0,1_1,3_1\}$, $\{0_0,0_1,2_1,3_1\}$, $\{0_0,0_1,1_1,4_1\}$. The result is an $S_6(2,4,10)$ on V which embeds the given $S_3(2,4,5)$ on U.

18. $\lambda=6$, $u=8$, $w=1$. Let $U=\mathbb{Z}_8$ and $V=\mathbb{Z}_8\cup\{\infty\}$. Take on U an $S_3(2,4,8)$ and add the blocks $\{\infty,0,1,4\}$, $\{\infty,1,2,5\}$, $\{\infty,2,3,6\}$, $\{\infty,3,0,7\}$, $\{\infty,4,5,0\}$, $\{\infty,5,6,1\}$, $\{\infty,6,7,2\}$, $\{\infty,7,4,3\}$, $\{\infty,0,1,6\}$, $\{\infty,1,2,7\}$, $\{\infty,2,3,4\}$, $\{\infty,3,0,5\}$, $\{\infty,4,5,2\}$, $\{\infty,5,6,3\}$, $\{\infty,6,7,0\}$, $\{\infty,7,4,1\},\{0,1,2,3\},\{4,5,6,7\}$ $\{0,2,4,6\},\{1,3,5,7\},\{0,2,5,7\},\{1,3,4,6\}$. The result is an $S_6(2,4,9)$ on V which embeds an $S_3(2,4,8)$ on U.

19. $\lambda=6$, $u=17$, $w=1$. Let $U=\mathbb{Z}_{17}$ and $V=\mathbb{Z}_{17}\cup\{\infty\}$. Take an $S_3(2,4,17)$ on U. Develop (mod 17) the base blocks: $\{\infty,0,6,7\},\{\infty,0,2,7\},\{0,4,6,9\}$, $\{0,1,3,12\}$, $\{0,4,7,8\}$. The result is an $S_6(2,4,18)$ on V which embeds an $S_3(2,4,17)$ on $\mathbb{Z}_{17}$.

20. $\lambda = 6$, $u = 8$, $w = 2$. Let $U = \mathbb{Z}_8$ and $V = \mathbb{Z}_8 \cup \{a, b\}$. Take on U an $S_3(2,4,8)$ and add the blocks
$\{a,b,0,2\},\{a,b,0,2\},\{a,b,1,3\},\{a,b,1,3\},\{a,b,4,6\},\{a,b,5,7\},$
$\{a,0,1,4\},\{a,1,2,5\},\{a,2,3,6\},\{a,3,0,7\},\{a,0,4,5\},\{a,0,6,7\},$
$\{a,1,4,7\},\{a,1,6,5\},\{a,2,5,7\},\{a,2,4,6\},\{a,3,4,5\},\{a,3,6,7\},$
$\{b,0,1,6\},\{b,1,2,7\},\{b,2,3,4\},\{b,3,0,5\},\{b,0,5,6\},\{b,0,4,7\},$
$\{b,1,4,6\},\{b,1,5,7\},\{b,2,4,5\},\{b,2,6,7\},\{b,3,4,7\},\{b,3,5,6\},$
$\{0,1,2,3\}$. The result is an $S_6(2,4,10)$ on V which embeds an $S_3(2,4,8)$ on U.

21. $\lambda = 6$, $u = 29$, $w = 7$. Let $U = \mathbb{Z}_{29}$, $W = \{a_i : i \in Z_7\}$ and $V = U \cup W$. Take an $S_3(2,4,29)$ and develop (mod 29) the base blocks: $\{a_0,0,1,3\}$, $\{a_0,0,4,11\}$, $\{a_1,0,10,24\}$, $\{a_1,0,11,12\}$, $\{a_2,0,5,7\}$, $\{a_2,0,5,8\}$, $\{a_3,0,10,23\}$, $\{a_3,0,12,25\}$, $\{a_4,0,8,14\}$, $\{a_4,0,9,18\}$, $\{a_5,0,12,13\}$, $\{a_5,0,8,10\}$, $\{a_6,0,14,20\}$, $\{a_6,0,4,7\}$. Add the blocks of an $S_6(2,4,7)$ on W. The result is an $S_6(2,4,36)$ on V which embeds an $S_3(2,4,29)$ on U.

22. $\lambda = 7$, $u = 9$, $w = 4$. Let $U = \mathbb{Z}_3 \times \{0,1,2\}$, $W = \{a_i : i \in \mathbb{Z}_4\}$ and $V = U \cup W$. Take on U an $S_3(2,4,9)$. Develop (mod 3) the base blocks:
$\{a_1,a_2,0_0,1_0\},\{a_1,a_2,0_1,0_2\},\{a_1,a_3,0_1,1_1\},$
$\{a_1,a_3,0_0,0_2\},\{a_1,a_0,0_2,1_2\},\{a_1,a_0,0_0,0_1\},\{a_2,a_3,0_0,1_2\},$
$\{a_2,a_3,0_1,1_2\},\{a_2,a_0,0_0,1_1\},\{a_2,a_0,0_1,2_2\},\{a_3,a_0,0_0,2_1\},$
$\{a_3,a_0,0_0,2_2\}$. Take on U a resolvable $S_2(2,3,9)$ having the resolution classes R_j, $j = 0,1,...,7$. For each $i = 0,1,2,3$, place the blocks $\{a_i,x,y,z\}$, $\{x,y,z\} \in \bigcup_{j=0}^{1} \mathcal{R}_{2i+j}$. Add the blocks of an $S(2,4,13)$ on V and the result is an $S_7(2,4,13)$ on V which embeds an $S_3(2,4,9)$ on U.

23. $\lambda = 7$, $u = 29$, $w = 8$. Let $U = \mathbb{Z}_{29}$, $W = \{a_i : i \in \mathbb{Z}_8\}$ and $V = U \cup W$. Take on $U \cup \{a_i : i \in \mathbb{Z}_7\}$ an $S_6(2,4,36)$ which embeds an $S_3(2,4,29)$ on U (see case 21 in Appendix). Construct on $U \cup \{a_i : i \in Z_7\} \cup \{\infty_i : i \in \mathbb{Z}_7\}$ a 4-GDD of type $7^1 1^{36}$ having $\{\infty_i : i \in \mathbb{Z}_7\}$ as a group of size 7. Replace, for each $i \in \mathbb{Z}_7$, ∞_i with a_7 and take the blocks so obtained. The result is an $S_7(2,4,37)$ on V which embeds an $S_3(2,4,29)$ on U.

24. $\lambda = 7$, $u = 32$, $w = 5$. Let $U = \mathbb{Z}_{31} \cup \{\infty\}$, $W = \{a_i : i \in \mathbb{Z}_5\}$ and $V = U \cup W$. Take an $S_3(2,4,32)$ on U. Develop (mod 31) the base blocks: $\{\infty,0,11,12\},\{a_0,0,7,9\}$, $\{a_0,0,5,8\},\{a_1,0,13,27\}$, $\{a_1,0,14,16\}$, $\{a_2,0,13,20\}$, $\{a_2,0,6,14\}$, $\{a_3,0,3,10\}$, $\{a_3,0,12,13\}$, $\{a_4,0,1,3\}$, $\{a_4,0,12,20\}$, $\{0,5,9,15\}$, $\{0,5,9,15\}$. Add the blocks of an $S(2,4,37)$ on V. The result is an $S_7(2,4,37)$ on V which embeds an $S_3(2,4,32)$ on U.

25. $\lambda = 7$, $u = 53$, $w = 8$. Let $U = \mathbb{Z}_{48} \cup \{b_i : i \in \mathbb{Z}_5\}$, $W = \{a_i : i \in \mathbb{Z}_8\}$ and $V = U \cup W$. Construct an $S_3(2,4,53)$ on U. Give weight 7 to every point of W and weight 4 to every point of $\{b_i : i \in \mathbb{Z}_5\}$, costruct on V four 4-GDD of type $19^1 1^{48}$. On $\{a_i, i \in \mathbb{Z}_8\} \cup \{b_i : i \in \mathbb{Z}_5\}$, place an $S_7(2,4,13)$ $(V, \mathcal{B})$ which embeds an $S_3(2,4,5)$ $(Y, \mathcal{C})$ on $\{b_i : i \in Z_5\}$. Delete the blocks of $\mathcal{C}$ and take the blocks so obtained. The result is an $S_7(2,4,61)$ on V which embeds an $S_3(2,4,53)$ on U.

26. $\lambda = 7$, $u = 44$, $w = 5$. Let $U = \mathbb{Z}_{39} \cup \{b_i : i \in \mathbb{Z}_5\}$, $W = \{a_i : i \in \mathbb{Z}_5\}$ and $V = U \cup W$. Take an $S_3(2,4,44)$ on U. Give weight 6 to every point of W and weight 3 to every point of $Y = \{b_i : i \in \mathbb{Z}_5\}$, costruct on V two 4-GDD of type $19^1 1^{39}$ and a 4-GDD of type $7^1 1^{39}$. On $\{a_i : i \in \mathbb{Z}_5\} \cup \{b_i : i \in \mathbb{Z}_5\}$, place an $S_6(2,4,10)$ $(X, \mathcal{B})$ which embeds an $S_3(2,4,5)$ $(Y, \mathcal{C})$ on $\{b_i : i \in Z_5\}$ (see case 17). Delete the blocks of $\mathcal{C}$ and take the blocks so obtained. Finally paste an $S(2,4,49)$ on V. The result is an $S_7(2,4,49)$ on V which embeds an $S_3(2,4,44)$ on U.

27. $\lambda = 8$, $u = 8$, $w = 2$. The result follows by pasting an $S_2(2,4,10)$ to an $S_6(2,4,10)$ which embeds an $S_3(2,4,8)$ on U (see case 20).

28. $\lambda = 10$, $u = 5$, $w = 2$. Let $U = \mathbb{Z}_5$ and $V = \mathbb{Z}_5 \cup \{\infty_1, \infty\}$. Construct on $\mathbb{Z}_5 \cup \{\infty_1\}$ an $S_6(2,4,6)$ which embeds an $S_3(2,4,5)$ on $\mathbb{Z}_5$ (see case 16). Take on $\mathbb{Z}_5 \cup \{\infty_1\}$ an $S_4(2,3,6)$ having block set $\mathcal{B}$ and form the blocks $\{\infty, x, y, z\}$, for each $\{x, y, z\} \in \mathcal{B}$. The result is an $S_{10}(2,4,7)$ on V which embeds an $S_3(2,4,5)$ on U.

Chapter 6

Embeddings of kite designs

6.1 Introduction and definitions

In this chapter we study the minimum embedding of a KS(u, λ) into a KS$(u+w, \mu)$.

To begin with, note what follows:

1. If $(V, \mathcal{B})$ is a KS$(u + w, \mu)$ embedding a KS(u, λ) $(U, \mathcal{C})$, then $\mathcal{C} \subseteq \mathcal{B}$ and replacing $\mathcal{C}$ with $\mathcal{C}'$, where $\mathcal{C}'$ is any decomposition of K_u into kites, gives a KS$(u + w, \mu)$ embedding $(U, \mathcal{C}')$, hence proving the existence of a KS(u, λ) embedded into a KS$(u+w, \mu)$ will imply that any KS(u, λ) can be embedded into a KS$(u + w, \mu)$.
2. Taking the union of a KS(u, ν) and a KS(u, λ) (clearly, when they both exist) gives a KS(u, λ) embedded into a KS$(u, \lambda + \nu)$;
3. If there exists a KS(u, λ) embedded into a KS$(u + w, \mu)$ and $u + w$ is an admissible order for the existence of a KS of index ν, then any KS(u, λ) can be embedded into a KS$(u + w, \mu + \nu)$.

To obtain our results we will make a massive use of the *difference method.* Let D_u denote the following set with elements from $\mathbb{Z}_u$:

$$D_u = \begin{cases} d : 1 \leq d \leq \frac{u}{2} & \text{if } u \text{ is even;} \\ d : 1 \leq d \leq \frac{u-1}{2} & \text{if } u \text{ is odd.} \end{cases}$$

The elements of D_u are called *differences* of $\mathbb{Z}_u$. For any $d \in D_u$, if $d \neq \frac{u}{2}$, then we can form a single 2-factor $\{\{i, d+i\} : i \in \mathbb{Z}_u\}$, if u is even and $d = \frac{u}{2}$, then we can form a 1-factor $\{\{i, \frac{u}{2} + i\} : 0 \leq i \leq \frac{u}{2} - 1\}$. It is also worth remarking that

2-factors obtained from distinct differences are disjoint from each other and from the 1-factor.

Let $W = \{\infty_1, \infty_2, \ldots, \infty_w\}$, $W \cap \mathbb{Z}_u = \emptyset$. Denote by $\langle \mathbb{Z}_u \cup W, \{d_1, d_2, \ldots, d_t\}\rangle$ the graph G with vertex set $V(G) = \mathbb{Z}_u \cup W$ and edge set $E(G) = \{\{x, y\} : x - y \text{ or } y - x \equiv d_i \pmod{u}, \text{ for some } i \in \{1, 2, \ldots, t\}\} \cup \{\{\infty, j\} : \infty \in W,\ j \in \mathbb{Z}_u\}$. When $W = \emptyset$, we simply write $\langle \mathbb{Z}_u, \{d_1, d_2, \ldots, d_t\}\rangle$.

Lemma 6.1.1. [55] *For any difference $d \in D_u \setminus \{\frac{u}{2}\}$ such that the integer $r = \frac{u}{gcd(u,d)}$ is even, the graph $\langle \mathbb{Z}_u \cup \{\infty\}, \{d\}\rangle$ can be decomposed into kites.*

Lemma 6.1.2. [55] *Let $u \equiv 0 \pmod 8$. The graph $\langle \mathbb{Z}_u \cup \{\infty_1, \infty_2\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Lemma 6.1.3. *For any difference $d \in D_u \setminus \{\frac{u}{2}\}$, the graph $\langle \mathbb{Z}_u, \{d\}\rangle \cup 3K_{u,1}$, where $K_{u,1}$ is the star based on $\mathbb{Z}_u \cup \{\infty\}$, can be decomposed into kites.*

Proof Consider the kites $(i, d+i, \infty)$-$(2d+i)$, $i \in \mathbb{Z}_u$. □

Lemma 6.1.4. *For any two distinct differences $d_1, d_2 \in D_u \setminus \{\frac{u}{2}\}$, the graph $\langle \mathbb{Z}_u, \{d_1, d_2\}\rangle \cup 2K_{u,1}$, where $K_{u,1}$ is the star based on $\mathbb{Z}_u \cup \{\infty\}$, can be decomposed into kites.*

Proof Consider the kites $(\infty, d_1 + i, i)$-$(d_2 + i)$, $i \in \mathbb{Z}_u$. □

We quote the following known result ([55], [56]) for later use.

Theorem 6.1.5. *Any KS(u, λ) can be embedded into a KS(v, λ) if and only if $v \geq \frac{5}{3}u + 1$ or $v = u$, and u, v are admissible orders.*

6.2 Minimum embedding of a KS$(u, 2)$ into a KS$(u + w, 3)$

In this section we determine the minimum embedding of a KS$(u, 2)$ into a KS$(u + w, 3)$. Since a KS$(u, 2)$ exists if and only if $u \equiv 0, 1 \pmod 4$ and a KS$(u + w, 3)$ exists if and only if $u + w \equiv 0, 1 \pmod 8$, $w = 0$ when $u \equiv 0, 1 \pmod 8$. If $u \equiv h \pmod 8$, with $h \in \{4, 5\}$, then $w \geq 8 - h$; here we prove that $w = 8 - h$ for every $u \equiv h \pmod 8$ and $h \in \{4, 5\}$.

Proposition 6.2.1. *For $u = 8k + h$ and $h = 4, 5$, any KS$(u, 2)$ can be embedded into a KS$(u + w, 3)$, $w = 8 - h$.*

Proof For $u = 4$, it follows from Theorem 6.1.5. For $u = 5, 12, 13$, see Cases $1, 2, 3$ in Appendix. Let $k \geq 2$ and $(U, \mathcal{K})$ be a KS$(u, 2)$, $u = 8k + h$ and $h = 4, 5$; without loss of generality, we can assume $U = \mathbb{Z}_{8k} \cup H$, where $H = \{a_s : s = 1, 2, \ldots, h\}$. Let $W = \{\infty_1, \infty_2, \ldots, \infty_w\}$ and take a KS$(h + w, 3)$ $(H \cup W, \mathcal{K}_1)$ which embeds a KS$(h, 2)$ $(H, \mathcal{K}_1^*)$. Consider the collection $\mathcal{K}_2$ of kites obtained by translating the $k - 2$ base blocks

$(4k-1,\ 2k+2,\ 0) - (2k-2),$
$(4k-2,\ 2k+3,\ 0) - (2k-4),$
$\ldots$
$(3k+2,\ 3k-1,\ 0) - 4.$

The result is a decomposition of $\langle \mathbb{Z}_{8k}, D \rangle$, where $D = D_{8k} \setminus \{1, 2, 2k-1, 2k, 2k+1, 3k, 3k+1, 4k\}$. Handle the remaining differences as follows and say $\mathcal{K}_3$ the resulting collection of kites: by Lemma 6.1.2 arrange the differences 1 and 4k with the vertices a_1 and a_2; by Lemma 6.1.1 the differences $2k-1$ and $2k+1$ with a_3 and a_4, respectively, and by Lemma 6.1.3 the differences $2k$, $3k$, and $3k+1$ with ∞_1, ∞_2, and ∞_3, respectively; finally, if $h = 4$, by Lemma 6.1.3 arrange 2 with ∞_4, while if $h = 5$, by Lemma 6.1.1 arrange 2 with a_5. Then $(\mathbb{Z}_{8k} \cup H \cup W, \mathcal{K} \cup (\mathcal{K}_1 \setminus \mathcal{K}_1^*) \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a KS$(u + w, 3)$ which embeds the given KS$(u, 2)$. □

6.3 Minimum embedding of a KS$(u, 4)$ into a KS$(u + w, 5)$

In this section we determine the minimum embedding of a KS$(u, 4)$ into a KS$(u + w, 5)$. Since a KS$(u, 4)$ exists for every $u \geq 4$ and a KS$(u+w, 5)$ exists if and only if $u + w \equiv 0, 1 \pmod 8$, $w = 0$ when $u \equiv 0, 1 \pmod 8$. If $u \equiv h \pmod 8$, with $h \in \{2, 3, 4, 5, 6, 7\}$, then $w \geq 8 - h$; here we prove that $w = 8 - h$ for every $u \equiv h \pmod 8$ and $h \in \{2, 3, 4, 5, 6, 7\}$.

Lemma 6.3.1. *There exists a decomposition of* $4(K_8 \setminus K_2)$ *into kites.*

Proof Consider the following kites on $\mathbb{Z}_6 \cup \{a, b\}$: $(a, 1+i, i) - b$ twice; $(b, 2+i, i) - (3+i)$ and $(3+i, 1+i, i) - (2+i)$ for $i \in \mathbb{Z}_6$; $(2i, 2+2i, 1+2i) - (3+2i)$ for $i = 0, 1, 2$. □

Proposition 6.3.2. *For* $u = 8k + 2$, $u \geq 10$, *any KS*$(u, 4)$ *can be embedded into a KS*$(u + 6, 5)$.

Proof For $k = 1, 2$, see Cases $4, 5$ in Appendix. Let $k \geq 3$, $H = \{a, b\}$, $W = \{\infty_j : j \in \mathbb{Z}_6\}$, and $(\mathbb{Z}_{8k} \cup H, \mathcal{K})$ be a KS$(u, 4)$. By Lemma 6.3.1 decompose $4(K_8 \setminus K_2)$ on $H \cup W$ (with H as hole) into kites and say $\mathcal{K}_1$ the resulting set of kites together with those ones of a KS$(8, 1)$ on the vertex set $H \cup W$. Consider the collection $\mathcal{K}_2$ of kites obtained by translating the $k - 3$ base blocks

$(4k - 2, 2k + 3, 0) - (2k - 4)$,
$(4k - 3, 2k + 4, 0) - (2k - 6)$,
...
$(3k + 2, 3k - 1, 0) - 4$.

The result is a decomposition of $\langle \mathbb{Z}_{8k}, D\rangle$, where $D = D_{8k} \setminus \{1, 2, 2k - 3, 2k - 2, 2k - 1, 2k, 2k + 1, 2k + 2, 3k, 3k + 1, 4k - 1, 4k\}$. Handle the remaining differences as follows and say $\mathcal{K}_3$ the resulting collection of kites: arrange the vertices a, b with the differences $1, 4k$ by using Lemma 6.1.2 and the infinity points with the 10 differences left, say d_j, d'_j, for $j \in \mathbb{Z}_6 \setminus \{5\}$, in the blocks $(i, i + d_j, \infty_j) - (i + 1)$, $(i, \infty_j, i + d'_j) - \infty_5$, $j \in \mathbb{Z}_6 \setminus \{5\}$, $i \in \mathbb{Z}_{8k}$. Then $(\mathbb{Z}_{8k} \cup H \cup W, \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a KS$(u + 6, 5)$ which embeds the given KS$(u, 4)$. □

Proposition 6.3.3. *For $u = 8k + 3$, $u \geq 11$, any KS$(u, 4)$ can be embedded into a KS$(u + 5, 5)$.*

Proof For $k = 1, 2$, see Cases $7, 8$ in Appendix. Let $k \geq 3$, $H = \{a_i : i \in \mathbb{Z}_4\}$, $W = \{\infty_j : j \in \mathbb{Z}_5\}$, and $(\mathbb{Z}_{8k-1} \cup H, \mathcal{K})$ be a KS$(u, 4)$. Take a KS$(9, 5)$ $(H \cup W, \mathcal{K}_1)$ which embeds a KS$(4, 4)$ $(H, \mathcal{K}_1^*)$ (see Case 6 in Appendix). Consider the collection $\mathcal{K}_2$ of kites obtained by translating the $k - 3$ base blocks

$(4k - 3, 2k + 4, 0) - (2k - 6)$,
$(4k - 4, 2k + 5, 0) - (2k - 8)$,
...
$(3k + 1, 3k, 0) - 2$.

The result is a decomposition of $\langle \mathbb{Z}_{8k-1}, D\rangle$, where $D = D_{8k-1} \setminus \{2k - 5, 2k - 4, 2k - 3, 2k - 2, 2k - 1, 2k, 2k + 1, 2k + 2, 2k + 3, 4k - 2, 4k - 1\}$. Handle the remaining differences as follows and say $\mathcal{K}_3$ the resulting collection of kites: arrange ∞_4 with the differences $2k - 5$, $2k - 4$, $2k - 3$ by using Lemmas 6.1.3 and 6.1.4 and the vertices a_j, ∞_j, for $j = 0, 1, 2, 3$, with the 8 differences left, say d_j, d'_j, for $j = 0, 1, 2, 3$, in the blocks $(i, d_j + i, \infty_j) - (1 + i)$, $(i, \infty_j, d'_j + i) - a_j$, $j = 0, 1, 2, 3$ and $i \in \mathbb{Z}_{8k-1}$. Then $(\mathbb{Z}_{8k-1} \cup H \cup W, \mathcal{K} \cup (\mathcal{K}_1 \setminus \mathcal{K}_1^*) \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a KS$(u + 5, 5)$ which embeds the given KS$(u, 4)$. □

Proposition 6.3.4. *For $u = 8k+4$, any KS$(u,4)$ can be embedded into a KS$(u+4,5)$.*

Proof For $k = 0$, it follows from Theorem 6.1.5. For $k = 1,2$, see Cases 9, 10 in Appendix. Let $k \geq 3$, $H = \{a_1, a_2, a_3, a_4\}$, $W = \{\infty_1, \infty_2, \infty_3, \infty_4\}$, and $(\mathbb{Z}_{8k} \cup H, \mathcal{K})$ be a KS$(u,4)$. Take a KS$(8,5)$ $(H \cup W, \mathcal{K}_1)$ which embeds a KS$(4,4)$ $(H, \mathcal{K}_1^*)$. Consider the collection $\mathcal{K}_2$ of kites obtained by translating the $k-3$ base blocks

$(4k-1,\ 2k+2,\ 0) - (2k-2)$,
$(4k-2,\ 2k+3,\ 0) - (2k-4)$,
$\dots$
$(3k+3, 3k-2, 0) - 6$.

The result is a decomposition of $\langle \mathbb{Z}_{8k}, D \rangle$, where $D = D_{8k} \setminus \{1,2,3,4,2k-1,2k,2k+1,3k-1,3k,3k+1,3k+2,4k\}$. Handle the remaining differences as follows and say $\mathcal{K}_3$ the resulting collection of kites: by Lemma 6.1.2, arrange the differences 1 and $4k$ with the vertices a_1 and a_2; by Lemma 6.1.1, $2k-1$ and $2k+1$ with a_3 and a_4, respectively, and arrange the remaining differences with the infinity vertices in the blocks $(i, 3k-1+i, \infty_3)-(1+i)$, $(i, 2k+i, \infty_1)-(1+i)$, $(i, 3+i, \infty_2)-(1+i)$, $(i, 3k+1+i, \infty_4)-(1+i)$, $(\infty_1, i, 3k+i)-\infty_4$, $(\infty_3, i, 4+i)-\infty_4$, $(\infty_2, i, 3k+2+i)-(3k+i)$, for $i \in \mathbb{Z}_{8k}$. Then $(\mathbb{Z}_{8k} \cup H \cup W, \mathcal{K} \cup (\mathcal{K}_1 \setminus \mathcal{K}_1^*) \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a KS$(u+4,5)$ which embeds the given KS$(u,4)$. □

Proposition 6.3.5. *For every $u = 8k+5$, any KS$(u,4)$ can be embedded into a KS$(u+3,5)$.*

Proof For $k = 0,1,2$, see Cases 11, 12, 13 in Appendix. Let $k \geq 3$, $H = \{a_1, a_2, a_3, a_4, a_5\}$, $W = \{\infty_1, \infty_2, \infty_3\}$, and $(\mathbb{Z}_{8k} \cup H, \mathcal{K})$ be a KS$(u,4)$. Take a KS$(8,5)$ $(H \cup W, \mathcal{K}_1)$ which embeds a KS$(5,4)$ $(H, \mathcal{K}_1^*)$. Consider the collection $\mathcal{K}_2$ of kites obtained by translating the $k-3$ base blocks

$(4k-1,\ 2k+2,\ 0) - (2k-2)$,
$(4k-2,\ 2k+3,\ 0) - (2k-4)$,
$\dots$
$(3k+3, 3k-2, 0) - 6$.

The result is a decomposition of $\langle \mathbb{Z}_{8k}, D \rangle$, where $D = D_{8k} \setminus \{1,2,3,4,2k-1,2k,2k+1,3k-1,3k,3k+1,3k+2,4k\}$. Handle the remaining differences as follows and say $\mathcal{K}_3$ the resulting collection of kites: by Lemma 6.1.2, arrange the differences 1 and $4k$ with the vertices a_1 and a_2; by Lemma 6.1.1, arrange $4, 2k$ with a_3, a_4, respectively. Arrange the differences 2, $2k-1$, $2k+1$ with a_5 in the blocks $(i, 2k-$

$1+i, 2k+1+i) - a_5,\ i \in \mathbb{Z}_{8k}$, and the remaining differences with the infinity vertices in the blocks $(i, 3+i, \infty_1) - (1+i),\ (i, 3k+1+i, \infty_2) - (1+i),\ (i, 3k-1+i, \infty_3) - (1+i),\ (\infty_1, i, 3k+i) - \infty_3,\ (\infty_2, i, 3k+2+i) - \infty_3,\ i \in \mathbb{Z}_{8k}\}$. Then $(\mathbb{Z}_{8k} \cup H \cup W, \mathcal{K} \cup (\mathcal{K}_1 \setminus \mathcal{K}_1^*) \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a KS$(u+3, 5)$ which embeds the given KS$(u, 4)$. □

Proposition 6.3.6. *For $u = 8k+6$ any KS$(u, 4)$ can be embedded into a KS$(u+2, 5)$.*

Proof For $k = 0, 1, 2$, see Cases $14, 15, 16$ in Appendix. Let $k \geq 3$, $H = \{a_1, a_2, \ldots, a_6\}$, $W = \{\infty_1, \infty_2\}$, and $(\mathbb{Z}_{8k} \cup H, \mathcal{K})$ be a KS$(u, 4)$. Take a KS$(8, 5)$ $(H \cup W, \mathcal{K}_1)$ which embeds a KS$(6, 4)$ $(H, \mathcal{K}_1^*)$. Consider the collection $\mathcal{K}_2$ of kites obtained by translating the $k-3$ base blocks

$(4k-1,\ 2k+2,\ 0) - (2k-2)$,
$(4k-2,\ 2k+3,\ 0) - (2k-4)$,
...
$(3k+3, 3k-2, 0) - 6$.

The result is a decomposition of $\langle \mathbb{Z}_{8k}, D \rangle$, where $D = D_{8k} \setminus \{1, 2, 3, 4, 2k-1, 2k, 2k+1, 3k-1, 3k, 3k+1, 3k+2, 4k\}$. Handle the remaining differences as follows and say $\mathcal{K}_3$ the resulting collection of kites: by Lemma 6.1.2, arrange the differences 1 and $4k$ with the vertices a_1 and a_2; by Lemma 6.1.1, arrange 2, 4, $2k-1$, $2k$ with a_3, a_4, a_5, a_6, respectively; finally, by using Lemmas 6.1.3 and 6.1.4 arrange the six difference left with ∞_1, and ∞_2. Then $(\mathbb{Z}_{8k} \cup H \cup W, \mathcal{K} \cup (\mathcal{K}_1 \setminus \mathcal{K}_1^*) \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a KS$(u+2, 5)$ which embeds the given KS$(u, 4)$. □

Proposition 6.3.7. *For $u = 8k+7$, any KS$(u, 4)$ can be embedded into a KS$(u+1, 5)$.*

Proof Let $(\mathbb{Z}_u, \mathcal{K})$ be a KS$(u, 4)$, $u = 8k+7$. Consider the collection $\mathcal{K}_1$ of kites obtained by translating the k base blocks

$(4k, 2k+1, 0) - 2k$
$(4k-1,\ 2k+2,\ 0) - (2k-2)$,
...
$(3k+1, 3k, 0) - 2$.

the result is a decomposition of $\langle \mathbb{Z}_{8k+7}, D \rangle$, where $D = D_u \setminus \{4k+1, 4k+2, 4k+3\}$. Consider the set of kites $\mathcal{K}_2 = \{(\infty, i, 4k+1+i) - (-4+i), (i, 4k+3+i, \infty) - (1+i) : i \in \mathbb{Z}_u\}$. Then $(\mathbb{Z}_u \cup \{\infty\}, \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2)$ is a KS$(u+1, 5)$ which embeds the given KS$(u, 4)$. □

Proposition 6.3.8. *For $u = 8k + h$, with $2 \leq h \leq 7$, $u \geq 4$, any KS$(u, 4)$ can be embedded into a KS$(u + w, 5)$, where $w = 8 - h$.*

Proof It follows from Propositions 6.3.2, 6.3.3, 6.3.4, 6.3.5, 6.3.6, 6.3.7. □

6.4 Minimum embedding of a KS$(u, 4)$ into a KS$(u + w, 6)$

In this section we determine the minimum embedding of a KS$(u, 4)$ into a KS$(u + w, 6)$. Since a KS$(u, 4)$ exists for every $u \geq 4$ and a KS$(u + w, 6)$ exists if and only if $u + w \equiv 0, 1 \pmod 4$, $w = 0$ when $u \equiv 0, 1 \pmod 4$. If $u \equiv h \pmod 4$, with $h \in \{2, 3\}$, then $w \geq 4 - h$; here we prove that $w = 4 - h$ for every $u \equiv h \pmod 4$ and $h \in \{2, 3\}$.

Proposition 6.4.1. *For $u = 4k + 2$, $u \geq 6$, any KS$(u, 4)$ can be embedded into a KS$(u + 2, 6)$.*

Proof For $k = 2p + 1$ the thesis follows from Proposition 6.3.6. Let $k = 2p$ and let $(\mathbb{Z}_{8p} \cup \{a, b\}, \mathcal{B})$ be a KS$(8p + 2, 4)$. Consider the kites obtained by translating the $2p - 2$ base blocks

$(0, 4p - 3, 4p - 1) - (4p + 3)$,
$(0, 4p - 5, 4p - 2) - (4p + 4)$,
...
$(0, 3, 2p + 2) - 6p$.

Now handle the remaining differences as follows: by applying Lemma 6.1.2 twice, arrange the differences 1 and $4p$ with the vertices a and b; arrange the infinity points with the four differences left in the kites $(i, 2+i, \infty_0) - (1+i)$, $(i, 2p+i, \infty_1) - (1+i)$, $(\infty_1, i, 2p+1+i) - \infty_0$, $(\infty_0, i, 4p-1+i) - \infty_1$, $i \in \mathbb{Z}_{8p}$. Finally, consider the kites $(a, \infty_1, \infty_0) - b$, $(b, \infty_1, \infty_0) - a$, $(a, \infty_1, \infty_0) - b$, $(b, \infty_0, \infty_1) - a$, $(a, \infty_0, \infty_1) - b$, $(b, \infty_0, \infty_1) - a$, $(\infty_1, b, a) - \infty_0$, $(\infty_0, a, b) - \infty_1$ to obtain a KS$(8p + 4, 6)$ on $\mathbb{Z}_{8p} \cup \{a, b, \infty_0, \infty_1\}$ which embeds $(\mathbb{Z}_{8p} \cup \{a, b\}, \mathcal{B})$. □

Proposition 6.4.2. *For $u = 4k + 3$, $u \geq 7$, any KS$(u, 4)$ can be embedded into a KS$(u + 1, 6)$.*

Proof For $k = 2p + 1$, the thesis follows from Proposition 6.3.7. Let $k = 2p$ and $(\mathbb{Z}_u, \mathcal{K})$ be a KS$(8p + 3, 4)$. Consider the kites obtained by translating the $2p$ base blocks

$(0, 4p-1, 4p) - (4p+2)$,
$(0, 4p-3, 4p-1) - (4p+3)$,
...
$(0, 1, 2p+1) - (6p+1)$,

together with the kites $(i, 4p+1+i, \infty) - (1+i)$, $i \in \mathbb{Z}_{8p+3}$, twice repeated, to obtain a KS$(8p+4, 6)$ on $\mathbb{Z}_u \cup \{\infty\}$ which embeds $(\mathbb{Z}_u, \mathcal{B})$. □

6.5 Main theorem

Theorem 6.5.1. *The minimum value of w such that a KS(u, λ) can be embedded into a KS$(u+w, \mu)$ is:*

λ	$u \geq 4$	$\mu \geq \lambda$	w
any	$0, 1 \pmod 8$	any	0
even	$4, 5 \pmod 8$	even	0
$0 \pmod 4$	$2, 3 \pmod 4$	$0 \pmod 4$	0
$0 \pmod 4$	$4k+h, h=2,3$	$2 \pmod 4, \mu \geq 3\lambda/2$	$4-h$
$0 \pmod 4$	$8k+h,\ 2 \leq h \leq 7$	odd, $\mu \geq 5\lambda/4$	$8-h$
$2 \pmod 4$	$8k+h, h=4,5$	odd, $\mu \geq 5\lambda/4$	$8-h$

Proof The conclusion is trivial in the first three cases.
If $\lambda = 4l$ and $u = 4k+h,\ h = 2, 3$, then for every even $\mu = 6l + 2q$ take l copies of a KS$(u, 4)$ embedded into a KS$(u+w, 6)$ from Propositions 6.4.1 and 6.4.2 so to obtain a KS$(u, 4l)$ which is embedded into a $KS(u+w, 6l+2q)$.
If $\lambda = 4l$ and $u = 8k+h,\ 2 \leq h \leq 7$, then for every odd $\mu = 5l + q$ take l copies of a KS$(u, 4)$ embedded into a KS$(u+w, 5)$ from Proposition 6.3.8 so to obtain a KS$(u, 4l)$ which is embedded into a $KS(u+w, 5l+q)$.
If $\lambda = 4l+2$ and $u = 8k+h,\ h = 4, 5$, then for every odd $\mu = 5l+q+3$ embed a KS$(u, 4l)$ into a $KS(u+w, 5l+q)$ and then paste a KS$(u, 2)$ embedded into a KS$(u+w, 3)$ from Proposition 6.2.1 so to obtain a KS$(u, 4l+2)$ which is embedded into a $KS(u+w, 5l+q+3)$. □

6.6 Conclusion

Taking account that if $(U \cup W, \mathcal{B})$ is a $KS(u+w, \mu)$ which embeds a $KS(u, \lambda)$ $(U, \mathcal{C})$, then each block in $\mathcal{B} \setminus \mathcal{C}$ contains at most three pairs of $U \times W$, it follows that

$$\mu \frac{uw}{3} \leq \mu \frac{(u+w)(u+w-1)}{8} - \lambda \frac{u(u-1)}{8}.$$

We formulate the following

Conjecture: *For every fixed triples of parameters u, λ, and μ, with $\mu \geq \lambda$, any $KS(u, \lambda)$ can be embedded into a $KS(u + \bar{w}, \mu)$, where $\bar{w}$ is the minimum admissible value for the existence of a $KS(u+\bar{w}, \mu)$ such that the above inequality is satisfied.*

Theorem 6.5.1 proves the conjecture, fixed any pair of parameters u and λ, for every $\mu \geq \lambda$, with the exception of a finite set of values:

1. for $\lambda < \mu < 5\lambda/4$, when $u \not\equiv 0, 1 \pmod 8$, μ is odd, and λ is even;
2. for $\lambda < \mu < 3\lambda/2$, when $u \equiv 2, 3 \pmod 4$, $\mu \equiv 2 \pmod 4$, and $\lambda \equiv 0 \pmod 4$.

Appendix to Chapter 6

In this appendix we list some embeddings of a $KS(u, \lambda)$ into a $KS(u + w, \mu)$.

1. $\lambda = 2$, $u = 5$, $\mu = 3$, $w = 3$

   Add the following blocks to a KS(5,2) on $\mathbb{Z}_5$: $(0, 1, \infty_3) - 3$, $(2, 4, \infty_3) - 0$, $(2, 3, \infty_3) - 4$, $(3, 4, \infty_1) - 0$, $(0, 4, \infty_1) - 1$, $(0, 2, \infty_2) - 3$, $(1, 3, \infty_2) - 4$, $(1, 4, \infty_2) - 2$, $(\infty_1, \infty_2, \infty_3) - 4$, $(4, \infty_2, \infty_1) - 2$, $(\infty_2, \infty_3, 0) - \infty_1$, $(\infty_2, \infty_3, 1) - \infty_1$, $(2, \infty_3, \infty_1) - 3$, $(3, \infty_1, \infty_3) - 1$, $(1, 2, \infty_1) - \infty_2$, $(0, 3, \infty_2) - 2$.

2. $\lambda = 2$, $u = 12$, $\mu = 3$, $w = 4$

   Let $(\mathbb{Z}_{12}, \mathcal{K})$ be a KS(12,2). Consider the following sets of blocks: $\mathcal{K}_1 = \{(0, 4, 8) - 2, (1, 5, 9) - 3, (2, 6, 10) - 4, (3, 7, 11) - 5, (0, 6, \infty_3) - 5, (\infty_4, \infty_1, \infty_2) - \infty_3\}$, $\mathcal{K}_2 = \{(i, 1+i, \infty_4) - (2+i) : i \in \mathbb{Z}_{12}\}$, $\mathcal{K}_3 = \{(i, 2+i, \infty_1) - (1+i) : i \in \mathbb{Z}_{12}\}$, $\mathcal{K}_4 = \{(i, 3+i, \infty_3) - (4+i) : i \in \mathbb{Z}_{12}\}$, $\mathcal{K}_5 = \{(i, 5+i, \infty_2) - (6+i) : i \in \mathbb{Z}_{12}\}$. Replace in $\mathcal{K}_4$ the kites with tails $\{\infty_3, 0\}, \{\infty_3, 5\}, \{\infty_3, 6\}$ by $(8, 11, \infty_3) - \infty_4, (4, \infty_3, 1) - 7, (2, 5, \infty_3) - \infty_1$. If $(W, \mathcal{K}_5)$, where $W = \{\infty_1, \infty_2, \infty_3, \infty_4\}$, is a KS(4,2), then $(\mathbb{Z}_{12} \cup W, \mathcal{K} \cup (\cup_{i=1}^{5} \mathcal{K}_i))$ is a KS(16,3) which embeds $(\mathbb{Z}_{12}, \mathcal{K})$.

3. $\lambda = 2$, $u = 13$, $\mu = 3$, $w = 3$

   Add the following sets of blocks to a KS(13,2) on $\mathbb{Z}_{13}$: $\mathcal{K}_1 = \{(9, 3, 0) - 6, (4, 1, 10) - 0, (3, 7, 10) - 6\}$, $\mathcal{K}_2 = \{(7, 11, 1) - \infty_1, (8, 12, 2) - \infty_1, (5, 11, 2) - \infty_0, (6, 12, 3) - \infty_2, (7, 4, 0) - \infty_1, (8, 5, 1) - \infty_2, (9, 6, 2) - \infty_2, (8, 11, 4) -$

$\infty_0, (12,9,5) - \infty_0\}$, $\mathcal{K}_3 = \{(i, 1+i, \infty_0) - (2+i) : i \in \mathbb{Z}_{13}\}$, $\mathcal{K}_4 = \{(i, 2+i, \infty_1) - (1+i) : i \in \mathbb{Z}_{13}\}$, $\mathcal{K}_5 = \{(i, 5+i, \infty_2) - (1+i) : i \in \mathbb{Z}_{13}\}$. Replace the tails of $\mathcal{K}_2$ in $\cup_{i=3}^{5} \mathcal{K}_i$ by the tails $\{\infty_j, \infty_{1+j}\}$, $j \in \mathbb{Z}_3$, three times repeated.

4. $\lambda = 4$, $u = 10$, $\mu = 5$, $w = 6$

   Add the following set of blocks to a KS$(10,4)$ on $\mathbb{Z}_{10}$: $\{(i, 1+j+i, \infty_j) - (5+i) : j = 0,1,2,3, i \in \mathbb{Z}_{10}\} \cup \{(1+i, 6+i, \infty_4) - i : i = 0,1,2,3,4\} \cup \{(\infty_j, \infty_{1+j}, i) - \infty_5 : 0 \le j = 0,1,2, i \le 4\} \cup \{(\infty_4, i, \infty_3) - \infty_0, (\infty_5, i, \infty_4) - \infty_1, (\infty_5, \infty_0, i) - \infty_4 : 0 \le i \le 4\} \cup \{(\infty_j, \infty_{2+j}, i) - \infty_5 : j = 1,2,4,5 \le i \le 9\} \cup \{(\infty_j, \infty_{2+j}, i) - \infty_4 : j = 0,5,5 \le i \le 9\} \cup \{(\infty_3, i, \infty_5) - \infty_2 : 5 \le i \le 9\}$.

5. $\lambda = 4$, $u = 18$, $\mu = 5$, $w = 6$

   Add the following set of kites to a KS$(18,4)$ on $\mathbb{Z}_{18}$: $\{(i, 1+j+i, \infty_j) - (6+i) : j = 0,1,3,4, i \in \mathbb{Z}_{18}\} \cup \{(i, 7+i, \infty_2) - (8+i), (i, 8+i, \infty_5) - (1+i) : i \in \mathbb{Z}_{18}\} \cup \{(i, \infty_j, \infty_{1+j}) - (10+i) : j \in \mathbb{Z}_6, 0 \le i \le 4\} \cup \{(i, \infty_j, \infty_{2+j}) - (5+i) : j \in \mathbb{Z}_6, 5 \le i \le 9\} \cup \{(15, \infty_{3+j}, \infty_j) - 17, (16, \infty_j, \infty_{3+j}) - 17 : j = 0,1,2\} \cup \{(i, 3+i, 9+i) - (15+i) : i = 0,1,\ldots,5\} \cup \{(i, 3+i, 6+i) - (9+i) : i = 6,7,8\} \cup \{(\infty_j, \infty_{3+j}, i) - (3+i) : j = 0,1,2, i = 15,16,17\}$.

6. $\lambda = 4$, $u = 4$, $\mu = 5$, $w = 5$

   Given a KS$(4,4)$ on $U = \{a,b,c,d\}$, consider the set of blocks on $\mathbb{Z}_5 \cup U$: $\{(i, 1+i, a) - (2+i), (c, i, 2+i) - b, (i, 1+i, c) - (2+i), (i, 2+i, d) - (1+i), (a, i, 2+i) - b, (i, 1+i, b) - (2+i) : i \in \mathbb{Z}_5\}$. Now replace the kites $(a,0,2) - b$, $(a,1,3) - b$, $(c,0,2) - b$ by the kites $(0,2,a) - b$, $(1,3,a) - c$, $(0,2,c) - b$ and add the blocks $(2,d,0) - 4$, $(3,d,1) - 0$, $(2,4,d) - a$, $(0,3,d) - b$, $(1,4,d) - c$, $(0,1,2) - b$, $(1,2,3) - b$, $(3,4,2) - b$, $(0,3,4) - 1$ to obtain a KS$(9,5)$ which embeds the given KS$(4,4)$.

7. $\lambda = 4$, $u = 11$, $\mu = 5$, $w = 5$

   Add the following set of blocks to a KS$(11,4)$ on $\mathbb{Z}_{11}$: $\{(i, 1+j+i, \infty_j) - (6+i) : j \in \mathbb{Z}_5, i \in \mathbb{Z}_{11}\} \cup \{(\infty_j, i, \infty_{1+j}) - (i+8), (\infty_j, 4+i, \infty_{2+j}) - (8+i) : j \in \mathbb{Z}_5, i = 0,1,2\} \cup \{(\infty_j, i, \infty_{1+j}) - \infty_{3+j} : j \in \mathbb{Z}_5, i = 3,7\}$.

8. $\lambda = 4$, $u = 19$, $\mu = 5$, $w = 5$

   Add to a KS$(19,4)$ on $\mathbb{Z}_{19}$ the blocks of a KS$(5,2)$ on $\{\infty_j : j \in \mathbb{Z}_5\}$ together with the blocks: $(i, 5+j+i, \infty_j) - (1+i)$, for $j \in \mathbb{Z}_5$ and $i \in \mathbb{Z}_{19}$; $(i, 3+i, \infty_0) - \infty_{1+i}$, $(4+i, 7+i, \infty_0) - \infty_{1+i}$, $(8+i, 11+i, \infty_0) - \infty_{1+i}$, for $i = 0,1,2,3$; $(i, 4+i, \infty_1) - \infty_{2+i}$, $(3+i, 7+i, \infty_1) - \infty_{2+i}$, $(6+i, 10+i, \infty_1) - \infty_{2+i}$, for $i = 0,1,2$; $(\infty_0, 3+i, i) - \infty_4$, for $i = 12, 13, \ldots, 18$; $(\infty_1, 4+i, i) - \infty_4$, for

$i = 9, 10, \ldots, 18$; $(\infty_2, 1+i, i) - \infty_4$, for $i = 0, 1, \ldots, 11$; $(\infty_3, 2+i, i) - \infty_4$, for $i = 0, 1, \ldots, 8$; $(13, 12, \infty_2) - \infty_3$, $(14, 13, \infty_2) - 0$, $(15, 14, \infty_2) - 18$, $(16, 15, \infty_2) - \infty_4$, $(17, 16, \infty_2) - \infty_4$, $(18, 17, \infty_2) - \infty_4$, $(11, 9, \infty_3) - \infty_4$, $(12, 10, \infty_3) - \infty_4$, $(13, 11, \infty_3) - \infty_4$, $(15, 13, \infty_3) - 14$, $(17, 15, \infty_3) - 12$, $(18, 16, \infty_3) - \infty_2$, $(0, 17, \infty_3) - \infty_2$, $(\infty_3, 16, 14) - 12$, $(\infty_3, 1, 18) - 0$.

9. $\lambda = 4$, $u = 12$, $\mu = 5$, $w = 4$

   Add the following set of blocks to a KS(12, 4) on $\mathbb{Z}_{12}$: $\{(i, 2+i, \infty_1) - (1+i), (i, 3+i, \infty_2) - (1+i), (i, 4+i, \infty_3) - (1+i), (i, 5+i, \infty_4) - (1+i) : i \in \mathbb{Z}_{12}\} \cup \{(\infty_1, \infty_2, i) - (1+i), (6+i, \infty_1, i) - \infty_4, (4+i, \infty_3, \infty_4) - i : 1 \leq i \leq 4\} \cup \{(\infty_1, \infty_3, i) - \infty_2, (\infty_2, \infty_4, i) - (1+i) : 5 \leq i \leq 9\} \cup \{(i, \infty_2, \infty_3) - (2+i)) : 0 \leq i \leq 2\} \cup \{(0, 6, \infty_1) - \infty_4, (5, 11, \infty_1) - \infty_4, (0, \infty_2, \infty_1) - \infty_4, (10, \infty_1, \infty_4) - 0, (11, \infty_1, \infty_4) - 10, (\infty_2, 11, 10) - \infty_3, (0, 1, \infty_3) - 11, (0, \infty_4, 11) - \infty_3, (3, \infty_3, \infty_2) - 10, (4, \infty_3, \infty_2) - 11, (9, \infty_4, \infty_3) - 10\}$.

10. $\lambda = 4$, $u = 18$, $\mu = 5$, $w = 4$

    Let $(\mathbb{Z}_{20}, \mathcal{K})$ be a KS(20, 4) and consider the following sets of kites: $\mathcal{K}_1 = \{(i, 4+i, \infty_1) - (i+1), (i, 5+i, \infty_2) - (1+i), (i, 6+i, \infty_3) - (1+i), (i, 7+i, \infty_4) - (1+i), (\infty_1, i, 8+i) - \infty_2, (\infty_3, i, 3+i) - \infty_2 : i \in \mathbb{Z}_{20}\}$, $\mathcal{K}_2 = \{(i, 1+i, 10+i) - (11+i), (2+i, \infty_4, i) - (11+i) : 0 \leq i \leq 9\}$, $\mathcal{K}_3 = \{(\infty_1, \infty_3, \infty_2) - \infty_4, (11, 13, \infty_4) - 10, (14, \infty_4, 12) - 10, (13, 15, \infty_4) - 12, (14, 16, \infty_4) - \infty_1, (15, 17, \infty_4) - \infty_3, (16, 18, \infty_4) - 19, (17, \infty_4, 19) - 1, (18, 0, \infty_4) - 1\}$. If $(W, \mathcal{K}_4)$, where $W = \{\infty_1, \infty_2, \infty_3, \infty_4\}$, is a KS(4, 4), then $(\mathbb{Z}_{20} \cup W, \mathcal{K} \cup (\cup_{i=1}^{4} \mathcal{K}_i))$ is a KS(24, 5) which embeds $(\mathbb{Z}_{20}, \mathcal{K})$.

11. $\lambda = 4$, $u = 5$, $\mu = 5$, $w = 3$

    Add the following set of blocks to a KS(5, 4) on $\mathbb{Z}_5$: $\{(\infty_1, \infty_2, i) - \infty_3, (\infty_1, \infty_3, i) - \infty_2, (\infty_2, \infty_3, i) - \infty_1 : i \in \mathbb{Z}_5\} \cup \{(0, 1, \infty_1) - 2, (0, 2, \infty_2) - 3, (0, 3, \infty_3) - 1, (0, 4, \infty_1) - 3, (1, 2, \infty_2) - 0, (1, 3, \infty_3) - 4, (\infty_1, 1, 4) - \infty_2, (\infty_1, 3, 2) - \infty_3, (2, 4, \infty_3) - 0, (3, 4, \infty_2) - 1\}$.

12. $\lambda = 4$, $u = 13$, $\mu = 5$, $w = 3$

    Add the following set of blocks to a KS(13, 4) on $\mathbb{Z}_{13}$: $\{(i, 1+i, \infty_1) - (2+i), (i, 2+i, \infty_2) - (1+i), (i, 3+i, \infty_3) - (1+i), (\infty_1, i, 4+i) - \infty_2 : i \in \mathbb{Z}_{13}\} \cup \{(\infty_2, 5, 11) - 4, (\infty_2, 7, 0) - 6, (\infty_1, \infty_3, \infty_2) - 10, (\infty_1, \infty_3, \infty_2) - 1, (\infty_1, \infty_3, \infty_2) - 2, (\infty_1, \infty_3, \infty_2) - 3, (\infty_1, \infty_3, \infty_2) - 9, (\infty_3, 0, 5) - 12, (\infty_3, 1, 6) - 12, (\infty_3, 2, 7) - 1, (\infty_3, 3, 8) - 2, (\infty_3, 4, 9) - 3, (\infty_3, 5, 10) - 4, (\infty_3, 11, 6) - \infty_2, (\infty_3, 7, 12) - \infty_2, (\infty_3, 0, 8) - \infty_2, (\infty_3, 9, 1) - 8, (\infty_3, 10, 2) - 9, (\infty_3, 11, 3) - 10, (\infty_3, 12, 4) - \infty_2\}$.

13. $\lambda = 4$, $u = 21$, $\mu = 5$, $w = 3$

Add the following set of blocks to a KS$(21, 4)$ on $\mathbb{Z}_{21}$: $\{(i, 1+i, 3+i) - (7+i), (1+i, 6+i, \infty_1) - i, (1+i, 7+i, \infty_2) - i, (i, 8+i, \infty_3) - (1+i), (\infty_1, i, 9+i) - \infty_3, (\infty_2, i, 10+i) - \infty_3 : i \in \mathbb{Z}_{21}\}$. Now replace the tails $\{\infty_1, i\}$, $0 \le i \le 5$, and $\{\infty_2, i\}$, $i = 0, 1, 6$, by the tails $\{\infty_1, \infty_2\}, \{\infty_1, \infty_3\}, \{\infty_2, \infty_3\}$, three times repeated. Finally, add the following set of kites: $\{(7+i, 14+i, i) - \infty_1 : 0 \le i \le 5\} \cup \{(13, 20, 6) - \infty_2, (\infty_1, \infty_3, \infty_2) - 1, (\infty_1, \infty_3, \infty_2) - 0\}$.

14. $\lambda = 4$, $u = 6$, $\mu = 5$, $w = 2$

Add the following blocks to a KS$(6, 4)$ on $\mathbb{Z}_6$: $(0, \infty_1, \infty_2) - 3$, $(1, \infty_1, \infty_2) - 3$, $(2, \infty_1, \infty_2) - 4$, $(3, \infty_2, \infty_1) - 5$, $(4, \infty_1, \infty_2) - 2$, $(0, 1, \infty_1) - 5$, $(0, 2, \infty_1) - 5$, $(0, 3, \infty_1) - 5$, $(0, 4, \infty_1) - 5$, $(0, 5, \infty_2) - 2$, $(1, 2, \infty_1) - 4$, $(1, 3, \infty_1) - 4$, $(1, 4, \infty_2) - 2$, $(1, 5, \infty_2) - 0$, $(2, 3, \infty_1) - 1$, $(2, 4, \infty_1) - 3$, $(2, 5, \infty_2) - 1$, $(3, 4, \infty_2) - 0$, $(3, 5, \infty_2) - 0$, $(4, 5, \infty_2) - 1$.

15. $\lambda = 4$, $u = 14$, $\mu = 5$, $w = 2$

Add the following blocks to a KS$(14, 4)$ on $\mathbb{Z}_{14}$: $(i, 7+i, \infty_1) - \infty_2$ for $i = 0, 1, 2, 3, 4$; $(\infty_1, 4+4i, 3+4i) - (2+4i)$, $(4+4i, 5+4i, \infty_1) - (2+4i)$, and $(5+4i, 6+4i, \infty_1) - (3+4i)$ for $i = 0, 1, 2$; $(\infty_1, 2+i, i) - (3+i)$, $(i, 4+i, \infty_2) - (1+i)$ and $(\infty_2, 5+i, i) - (6+i)$, for $i \in \mathbb{Z}_{14}$; $(5, 12, \infty_1) - 0$, $(6, 13, \infty_1) - 1$, $(\infty_1, 2, 1) - 0$.

16. $\lambda = 4$, $u = 22$, $\mu = 5$, $w = 2$

Add the following blocks to a KS$(22, 4)$ on $\mathbb{Z}_{22}$: $(i, 11+i, \infty_1) - \infty_2$ for $i = 0, 1, 2, 3, 4$; $(\infty_1, 8+4i, 7+4i) - (6+4i)$, $(9+4i, 8+4i, \infty_1) - (6+4i)$ and $(9+4i, 10+4i, \infty_1) - (7+4i)$ for $i = 0, 1, 2, 3$; $(3+i, 5+i, i) - (4+i)$, $(\infty_1, 6+i, i) - (7+i)$, $(i, 8+i, \infty_2) - (1+i)$ and $(\infty_2, 9+i, i) - (10+i)$ for $i \in \mathbb{Z}_{22}$; $(5, 16, \infty_1) - 0$, $(6, 17, \infty_1) - 1$, $(\infty_1, 2, 1) - 0$, $(7, 18, \infty_1) - 2$, $(8, 19, \infty_1) - 3$, $(\infty_1, 4, 3) - 2$, $(9, 20, \infty_1) - 4$, $(10, 21, \infty_1) - 5$, $(\infty_1, 6, 5) - 4$.

Chapter 7

Embedding of paths systems into kite systems

7.1 Basic lemmas

The embedding of path systems into kite systems for $\mu = \lambda = 1$ are studied in [26, 68]. In this chapter we solve the embedding problem of a $P_k(u, \lambda)$ into a $KS(u, \mu)$, with $k = 3$ (Section 7.2), $k = 2, 4$ (Section 7.3). We will prove the following

Main theorem: *There exists a* $KS(u, \mu)$ *which embeds an* $P_k(u, \lambda)$ *if and only if* u, λ, μ *are admissible and* $\mu \geq \lceil \frac{4}{k-1} \rceil \lambda$ *for* $k = 2, 3, 4$*. When* $\mu = \lceil \frac{4}{k-1} \rceil \lambda$ *the embedding is exact.*

To obtain our results we will make use of the two following lemmas:

Lemma 7.1.1. [55] *Let* u *and* k *be integers such that* $u > 8k$*. Then there exists a cyclic partial kite system of order* u*, whose base blocks contains every difference* $d \in \{1, 2, \ldots, 4k\}$ *exactly once.*

Lemma 7.1.2. [56] *Let* u *and* k *be integers such that* $u > 4k$*. Then there exists a cyclic partial kite system of order* u*, whose base blocks contains every difference* $d \in \{1, 2, \ldots, 2k\}$ *exactly twice.*

7.2 P_3-designs

Proposition 7.2.1. *For every* $u = 8k + h$*, with* $h = 0, 1, 4, 5$*,* $u \geq 4$*, there exists a* $KS(u, 2)$ *which embeds a* $P_3(u, 1)$*.*

Proof For every $u = 8k + h$, with $h = 0, 1, 4, 5$, $u \geq 4$, construct a KS$(u, 2)$ $(U, \mathcal{B})$ as follows.
Case $h = 0$. Set $U = \mathbb{Z}_{8k-1} \cup \{\infty\}$ and place in $\mathcal{B}$ the translates of the base blocks $(2+i,\ 4k-1-i,\ 0)-(4k+1+2i)$, for $i = 0, 1, \ldots, 2k-2$, and $(1,\ \infty,\ 0)-(4k-1)$.
Case $h = 1$. Set $U = \mathbb{Z}_{8k+1}$ and place in $\mathcal{B}$ the translates of the base blocks $(1+i,\ 4k-i,\ 0)-(4k+1+2i)$, for $i = 0, 1, \ldots, 2k-1$.
Case $h = 4$. Set $U = \mathbb{Z}_{8k+3} \cup \{\infty\}$ and place in $\mathcal{B}$ the translates of the base blocks $(2+i,\ 4k+1-i,\ 0)-(4k+3+2i)$, for $i = 0, 1, \ldots, 2k-1$, and $(1,\ \infty,\ 0)-(4k+1)$.
Case $h = 5$. Set $U = \mathbb{Z}_{8k+5}$ and place in $\mathcal{B}$ the translates of the base blocks $(1+i,\ 4k+2-i,\ 0)-(4k+3+2i)$, for $i = 0, 1, \ldots, 2k$.
For every $h = 0, 1, 4, 5$, $(U, \mathcal{C})$, where $\mathcal{C}$ is the collection of copies of P_3 obtained by considering the laterals of each kite in $\mathcal{B}$, is a $P_3(u, 1)$ embedded into $(U, \mathcal{B})$.□

Proposition 7.2.2. *For every $u = 8k + h$, with $h = 2, 3, 6, 7$, $u \geq 4$, there exists a KS$(u, 4)$ which embeds a $P_3(u, 2)$.*

Proof For every $u = 8k + h$, with $h = 0, 1, 4, 5$, $u \geq 4$, construct a KS$(u, 4)$ $(U, \mathcal{B})$ as follows.
Case $h = 2$. Set $U = \mathbb{Z}_{8k+1} \cup \{\infty\}$ and place in $\mathcal{B}$ the translates of the base blocks $(0,\ 4k-1-2i,\ 4k-i)-(4k+4+i)$, for $i = 0, 1, \ldots, 2k-2$ (twice), and $(0,\ 1,\ 2k+1)-(2k+3)$, $(0,\ 1,\ 2k+1)-\infty$, $(0,\ 2,\ \infty)-1$.
Case $h = 3$. Set $U = \mathbb{Z}_{8k+3}$ and place in $\mathcal{B}$ the translates of the base blocks $(0,\ 4k-1-2i,\ 4k+1-i)-(4k+2+i)$, for $i = 0, 1, \ldots, 2k-1$ (twice), and $(0,\ 4k+2,\ 2k+1)-(6k+2)$.
Case $h = 6$. Set $U = \mathbb{Z}_{8k+5} \cup \{\infty\}$ and place in $\mathcal{B}$ the translates of the base blocks $(0,\ 4k-1-2i,\ 4k+1-i)-(4k+5+i)$, for $i = 0, 1, \ldots, 2k-1$ (twice), and $(0,\ 4k+1,\ 4k+2)-(4k+4)$, $(0,\ 4k+1,\ 4k+2)-\infty$, $(0,\ 2,\ \infty)-1$.
Case $h = 7$. Set $U = \mathbb{Z}_{8k+7}$ and place in $\mathcal{B}$ the translates of the base blocks $(0,\ 4k+2-2i,\ 4k+3-i)-(4k+4+i)$, for $i = 0, 1, \ldots, 2k$ (twice), and $(0,\ 4k+4,\ 2k+2)-(6k+5)$.
For every $h = 2, 3, 6, 7$, $(U, \mathcal{C})$, where $\mathcal{C}$ is the collection of copies of P_3 obtained by considering the laterals of each kite in $\mathcal{B}$, is a $P_3(u, 2)$ embedded into $(U, \mathcal{B})$.□

Theorem 7.2.3. *There exists a $KS(u, \mu)$ which embeds an $P_3(u, \lambda)$ if and only if u, λ, μ are admissible and $\mu \geq 2\lambda$.*

Proof The necessity is trivial. Now we prove the sufficiency. Let $\mu = 2\lambda$. For $u \equiv 0, 1 \pmod 4$, use λ copies of the $KS(u, 2)$ of Proposition 7.2.1. For $u \equiv 2, 3 \pmod 4$, use $\lambda/2$ copies of the $KS(u, 4)$ of Proposition 7.2.2. Let now $\mu > 2\lambda$; it is sufficient to embed a $P_3(u, \lambda)$ into a $KS(u, 2\lambda)$ and the resulting $KS(u, 2\lambda)$ into a $KS(u, \mu)$ by adding the blocks of a $KS(u, \mu - 2\lambda)$ on the same vertex set. □

7.3 P_4-designs and P_2-designs

Here we will study the embedding of a $P_4(u,\lambda)$ into a KS(u,μ). In order to describe a KS(u,μ) $(V,\mathcal{B})$ embedding a $P_4(u,\lambda)$ $(U,\mathcal{C})$ we always denote by $\mathcal{B}_e$ the subcollection of $\mathcal{B}$ such that $f(\mathcal{C})=\mathcal{B}_e$, where $f:\mathcal{C}\to\mathcal{B}$ is the injective function defined by $f([a,b,c,d])=(a,b,c)-d$. Note that when $\mathcal{B}_e=\mathcal{B}$, the embedding is exact.

Proposition 7.3.1. *For every $u\geq 4$ and $l\geq 1$ there exists a KS$(u,4l)$ which embeds a $P_4(u,3l)$.*

Proof It is sufficient to prove the assertion for $l=1$. If $u=2k+1$, on $\mathbb{Z}_{2k+1}$ consider the base kites $(2k-1-i,\ 2+i,\ 0)-(2k-2-2i)$, $i=0,1,\ldots,k-2$, and $(1,\ 2,\ 0)-2k$, except for the case $k\equiv 2 \pmod 3$, where $(2k-1-i,\ 2+i,\ 0)-(2k-2-2i)$, for $i=\frac{2k-4}{3}$, and $(1,\ 2,\ 0)-2k$ are replaced by $(\frac{4k+1}{3},\ \frac{2k+2}{3},\ 0)-2k$ and $(1,\ 2,\ 0)-\frac{4k+1}{3}$. If $u=2k$, then on $\mathbb{Z}_{2k-1}\cup\{\infty\}$ consider the following base kites: for $k=2$, $(2,\ \infty,\ 0)-1$ and $(\infty,\ 0,\ 1)-2$; for $k\geq 3$, $(2k-4-i,\ 3+i,\ 0)-(2k-6-2i)$, for $i=0,1,\ldots,k-4$, $(2k-2,\ 1,\ 0)-\infty$, $(2k-3,\ 2,\ 0)-\infty$, and $(\infty,\ 3,\ 0)-1$, except for the case $k\equiv 0 \pmod 3$, where $(2k-4-i,\ 3+i,\ 0)-(2k-6-2i)$,, for $i=\frac{2k-9}{3}$, and $(\infty,\ 3,\ 0)-1$ are replaced by $(\frac{4k-3}{3},\ \frac{2k}{3},\ 0)-1$ and $(\infty,\ 3,\ 0)-\frac{2k}{3}$.

Corollary 7.3.2. *There exists a KS(u,μ) which embeds a $P_2(u,\lambda)$ if and only if u,λ,μ are admissible and $\mu\geq 4\lambda$.*

Proof The necessity is trivial. Now we prove the sufficiency. P_2 is the complementary graph of P_4 respect to the kite and so by Proposition 7.3.1 we deduce the existence of a $P_2(u,\lambda)$ exactly embedded into a KS$(u,4\lambda)$ for every $\lambda\geq 1$. By adding the blocks of a KS$(u,\mu-4\lambda)$ we obtain the thesis. □

Proposition 7.3.3. *For every $u=12k+h$, with $h=0,1,4,9$ and $u\geq 4$, there exists a KS$(u,4l+2)$ which embeds a $P_4(u,3l+1)$.*

Proof By Proposition 7.3.1, it is sufficient to prove the assertion for $l=0$. For each $u=12k+h$, $h\in\{0,1,4,9\}$, construct a KS$(u,2)$ $(U,\mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.
Case $h=0$. Set $U=\mathbb{Z}_{12k-1}\cup\{\infty\}$; place in $\mathcal{B}_e$ the translates of the base blocks $(6k-1-i,\ 2+i,\ 0)-(6k-2-2i)$, for $i=0,1,\ldots,2k-2$, and $(\infty,\ 1,\ 0)-(6k-1)$, and obtain $\mathcal{B}'$ by applying Lemma 7.1.1.
Case $h=1$. Set $U=\mathbb{Z}_{12k+1}$; place in $\mathcal{B}_e$ the translates of the base blocks $(6k-1-i,\ 2+i,\ 0)-(6k-2-2i)$, for $i=0,1,\ldots,2k-2$, and $(6k,\ 1,\ 0)-(6k+1)$, and obtain $\mathcal{B}'$ by applying Lemma 7.1.1.
Case $h=4$. Set $U=\mathbb{Z}_{12k+4}$ and place in $\mathcal{B}_e$ the translates of the base blocks

$(3k-1-i,\ 2+i,\ 0)-(3k-2-2i)$, for $i=0,1,\ldots,k-2$,
$(9k+2-i,\ 3k+2+i,\ 0)-(6k+1-2i)$, for $i=0,1,\ldots,k-1$,
$(3k,\ 1,\ 0)-(9k+4)$,

along with the kites $(6k+2+i,\ 9k+3+i,\ 3k+1+i)-i$ and $(3k+1+i,\ 6k+2+i,\ i)-(9k+3+i)$, for $i=0,1,\ldots,3k$. Finally, place in $\mathcal{B}'$ the translates of $(6k+1-i,\ 4k+2+i,\ 0)-(2k-2i)$, for $i=0,1,\ldots,k-1$, and the kites $(i,\ 6k+2+i,\ 9k+3+i)-(3k+1+i)$, for $i=0,1,\ldots,3k$.
Case $h=9$. Set $U=\mathbb{Z}_{12k+8}\cup\{\infty\}$ and place in $\mathcal{B}_e$ the translates of the base blocks

$(3k+1-i,\ 1+i,\ 0)-(3k-1-2i)$, for $i=0,1,\ldots,k-1$,
$(9k+5-i,\ 3k+3+i,\ 0)-(6k+1-2i)$, for $i=0,1,\ldots,k-1$,
$(\infty,\ 0,\ 3k+1)-(9k+4)$,

along with the kites $(6k+4+i,\ 9k+6+i,\ 3k+2+i)-i$ and $(3k+2+i,\ 6k+4+i,\ i)-(9k+6+i)$, for $i=0,1,\ldots,3k+1$. Finally, place in $\mathcal{B}'$ the translates of $(6k+2-i,\ 4k+3+i,\ 0)-(2k-2i)$, for $i=0,1,\ldots,k-3$, $(5k+4,\ 5k-1,\ 0)-(2k+1)$, and $(5k+3,\ 5k,\ 0)-(6k+3)$, along with the kites $(4i,\ 4+4i,\ 2+4i)-(6+4i)$, $(1+4i,\ 5+4i,\ 3+4i)-(7+4i)$, $(0,\ 6k+4+i,\ 9k+6+i)-(3k+2+i)$, for $i=0,1,\ldots,3k+1$. □

Proposition 7.3.4. *For every $u=24k+h$, with $h=0,1,9,16$, $u\geq 9$, there exists a KS$(u,4l+3)$ which embeds a $P_4(u,3l+2)$.*

Proof By Proposition 7.3.1, it is sufficient to prove the assertion for $l=0$. For each $u=24k+h$, $h\in\{0,1,9,16\}$, construct a KS$(u,3)$ $(U,\mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.
Case $h=0$. Set $U=\mathbb{Z}_{24k-1}\cup\{\infty\}$; place in $\mathcal{B}_e$ the translates of the base blocks $(12k-2-i,\ 1+i,\ 0)-(12k-4-2i)$, for $i=0,1,\ldots,4k-2$, $(4k+1+i,\ 12k-2-i,\ 0)-(8k-2-2i)$, for $i=0,1,\ldots,4k-3$, $(8k-1,\ 8k,\ 0)-(12k-1)$, $(0,\ 12k+1,\ 12k-1)-4k$, and $(12k-1,\ 0,\ \infty)-1$, and obtain $\mathcal{B}'$ by applying Lemma 7.1.1.
Case $h=1$. Set $U=\mathbb{Z}_{24k+1}$; place in $\mathcal{B}_e$ the translates of the base blocks $(4k+1+i,\ 12k-i,\ 0)-(8k-2i)$, for $i=0,1,\ldots,4k-1$, $(12k-1-i,\ 2+i,\ 0)-(12k-2-2i)$, for $i=0,1,\ldots,4k-2$, and $(12k,\ 1,\ 0)-(12k+1)$, and obtain $\mathcal{B}'$ by applying Lemma 7.1.1.
Case $h=9$. Set $U=\mathbb{Z}_{24k+9}$; place in $\mathcal{B}_e$ the translates of the base blocks $(12k+2-i,\ 2+i,\ 0)-(12k+1-2i)$, $(4k+2+i,\ 12k+3-i,\ 0)-(8k+2-2i)$, for $i=0,1,\ldots,4k-1$, $(12k+3,\ 1,\ 0)-(12k+6)$, $(12k+5,\ 0,\ 12k+4)-(4k+1)$, along with the kites $(3i,1+3i,2+3i)-(3+3i)$ and $(3i,2+3i,4+3i)-(6+3i)$, for

$i=0,1,\dots,4k+2$, and place in $\mathcal{B}'$ the translates of $(4k-i,\ 2k+1+i,\ 0)-(2k-2i)$, for $i=0,1,\dots,k-3$, $(3k-1,\ 3k+2,\ 0)-3k$, and $(12k+3,\ 4k+1,\ 0)-(3k+1)$, along with the kites $(2+3i,6+3i,4+3i)-(8+3i)$, for $i=0,1,\dots,4k+2$.
Case $h=16$. Set $U=\mathbb{Z}_{24k+15}\cup\{\infty\}$; place in $\mathcal{B}_e$ the translates of the base blocks $(12k+5-i,\ 2+i,\ 0)-(12k+4-2i)$, $(4k+3+i,\ 12k+6-i,\ 0)-(8k+4-2i)$, for $i=0,1,\dots,4k$, $(8k+4,\ 8k+5,\ 0)-(12k+7)$, $(12k+7,\ 2,\ 0)-(12k+9)$, $(12k+6,\ \infty,\ 0)-(12k+7)$, along with the kites $(3i,1+3i,2+3i)-(3+3i)$, for $i=0,1,\dots,4k+4$, and place in $\mathcal{B}'$ the translates of $(4k-i,\ 2k+1+i,\ 0)-(2k-2i)$, for $i=0,1,\dots,k-3$, $(3k-1,\ 3k+2,\ 0)-(4k+1)$, and $(3k,\ 3k+1,\ 0)-(4k+2)$, along with the kites $(2+3i,6+3i,4+3i)-(8+3i)$ and $(3i,\ 4+3i,\ \infty)-(2+3i)$, for $i=0,1,\dots,4k+4$. □

Proposition 7.3.5. *For every $u=6k+h$, with $h=0,1,3,4$, $u\geq 4$, there exists a KS$(u,4l+4)$ which embeds a $P_4(u,3l+1)$.*

Proof By Proposition 7.3.1, it is sufficient to prove the assertion for $l=0$. For each $u=6k+h$, $h\in\{0,1,3,4\}$, construct a KS$(u,4)$ $(U,\mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.
Case $h=0$. Set $U=\mathbb{Z}_{6k-1}\cup\{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k-1-i,\ 2+i,\ 0)-(3k-2-2i)$, for $i=0,1,\dots,k-2$, and $(\infty,\ 1,\ 0)-(3k-1)$. To obtain $\mathcal{B}'$ duplicate $\mathcal{B}_e$ and apply Lemma 7.1.2 to settle the remaining differences.
Case $h=1$. Set $U=\mathbb{Z}_{6k+1}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k-1-i,\ 2+i,\ 0)-(3k-2-2i)$, for $i=0,1,\dots,k-2$, and $(3k,\ 1,\ 0)-(3k+1)$. To obtain $\mathcal{B}'$ duplicate $\mathcal{B}_e$ and apply Lemma 7.1.2 to settle the remaining differences.
Case $h=3$. Set $U=\mathbb{Z}_{6k+3}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k+2-i,\ 2+i,\ 0)-(3k+1-2i)$, for $i=0,1,\dots,k-1$, along with the kites $(3i,1+3i,2+3i)-(3+3i)$, $i=0,1,\dots,2k$, and place in $\mathcal{B}'$ the translates of the base blocks $(3k+1-i,\ 3+i,\ 0)-(3k-1-2i)$, $(2k-i,\ 2+i,\ 0)-(4k+4+2i)$, for $i=0,1,\dots,k-2$, $(2k+1,\ 2k+2,\ 0)-3k$, and $(2k,\ 4k+1,\ 0)-(k+1)$, along with the kites $(1+3i,2+3i,3+3i)-(4+3i)$ and $(2+3i,3+3i,4+3i)-(5+3i)$, $i=0,1,\dots,2k$.
Case $h=4$. Set $U=\mathbb{Z}_{6k+3}\cup\{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k+2-i,\ 2+i,\ 0)-(3k+1-2i)$, for $i=0,1,\dots,k-1$, along with the kites $(2+3i,\infty,3i)-(1+3i)$ and $(3+3i,2+3i,1+3i)-\infty$, $i=0,1,\dots,2k$, and place in $\mathcal{B}'$ the translates of the base blocks $(3k+1-i,\ 3+i,\ 0)-(3k-1-2i)$, for $i=0,1,\dots,k-2$, $(2k+1,\ 2k+2,\ 0)-3k$, and $(\infty,\ 2k+1,\ 0)-(2k+2)$, along with the kites $(2+3i,4+3i,\infty)-3i$, $i=0,1,\dots,2k$, and finally apply Lemma 7.1.2 to settle the remaining differences. □

Proposition 7.3.6. *For every $u=6k+h$, with $h=0,1,3,4$, $u\geq 4$, there exists a KS$(u,4l+4)$ which embeds a $P_4(u,3l+2)$.*

Proof By Proposition 7.3.1, it is sufficient to prove the assertion for $l = 0$. For each $u = 6k+h$, $h \in \{0, 1, 3, 4\}$, construct a KS$(u, 4)$ $(U, \mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.

Case $h = 0$. Set $U = \mathbb{Z}_{6k-1} \cup \{\infty\}$. Place in $\mathcal{B}_e$ the translates twice repeated of the base blocks $(3k-1-i,\ 2+i,\ 0)-(3k-2-2i)$, for $i = 0, 1, \dots, k-2$, and $(\infty,\ 1,\ 0)-(3k-1)$. To obtain $\mathcal{B}'$ apply Lemma 7.1.2 to settle the remaining differences.

Case $h = 1$. Set $U = \mathbb{Z}_{6k+1}$. Place in $\mathcal{B}_e$ the translates twice repeated of the base blocks $(3k-1-i,\ 2+i,\ 0)-(3k-2-2i)$, for $i = 0, 1, \dots, k-2$, and $(3k,\ 1,\ 0)-(3k+1)$. To obtain $\mathcal{B}'$ apply Lemma 7.1.2 to settle the remaining differences.

Case $h = 3$. Set $U = \mathbb{Z}_{6k+3}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k+2-i,\ 2+i,\ 0)-(3k+1-2i)$, for $i = 0, 1, \dots, k-1$, $(3k+1-i,\ 3+i,\ 0)-(3k-1-2i)$ for $i = 1, 2, \dots, k-2$, $(3, 3k+1, 0)-(3k-1)$, $(2, 0, 3)-(3k+3)$ along with the kites $(3i, 1+3i, 2+3i)-(3+3i)$, $i = 0, 1, \dots, 2k$, $(1+3i, 2+3i, 3+3i)-(4+3i)$ and $(2+3i, 3+3i, 4+3i)-(5+3i)$, $i = 0, 1, \dots, 2k$. Place in $\mathcal{B}'$ the translates of the base blocks $(2k-i,\ 2+i,\ 0)-(4k+4+2i)$, for $i = 0, 1, \dots, k-4$, $(4, 0, k-1)-(2k+1)$, $(1, 0, 2k+2)-(3k+2)$ and $(2k,\ 4k+1,\ 0)-(k+1)$.

Case $h = 4$. Set $U = \mathbb{Z}_{6k+3} \cup \{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k-1-i,\ 2+i,\ 0)-(3k-2-2i)$, for $i = 0, 1, \dots, k-2$ (twice), $(6k+1, 3k, 0)-(3k+1)$, $(3k, \infty, 0)-(3k-1)$ along with the kites $(3i, 1+3i, 2+3i)-(3+3i)$, $i = 0, 1, \dots, 2k$. Place in $\mathcal{B}'$ the translates of the base blocks $(2k-i,\ 3+i,\ 0)-(2k-2+2i)$, for $i = 0, 1, \dots, k-3$, and $(3k+1, k+2, 0)-(k+1)$, $(3k+1, \infty, 0)-2k$, along with the kites $(1+3i, 2+3i, 3+3i)-(4+3i)$ and $(2+3i, 3+3i, 4+3i)-(5+3i)$, $i = 0, 1, \dots, 2k$. □

Theorem 7.3.7. *There exists an $P_k(u, \lambda), k = 2, 4$ embedded into a $KS(u, \mu)$ if and only if u, λ, μ are admissible and $\mu \geq \frac{4}{k-1}\lambda$.*

Proof The necessity is trivial. Now we prove the sufficiency. For $k = 2$, the proof is in Corollary 7.3.2. Let $k = 4$. For $u \equiv 0, 1 \pmod 8$, we can apply Propositions 7.3.1, 7.3.3, 7.3.4. For $u \equiv 4, 5 \pmod 8$, μ is even and so we can apply Propositions 7.3.1, 7.3.3 if $\lambda \equiv 0, 1 \pmod 3$ and add, if it is necessary the blocks of a $KS(u, \mu - \lceil \frac{4}{3}\lambda \rceil)$. If $\lambda = 3l+2$ and so $u \equiv 4, 12, 13, 21 \pmod{24}$, Proposition 7.3.3 implies the existence of an $KS(u, 4)$ which embeds a $P_4(u, 2)$ and so the existence of an $KS(u, \mu)$ which embeds a $P_4(u, \lambda)$ for every even $\mu \geq 4l+4$. For $u \equiv 2, 3 \pmod 4$, it is $\mu \equiv 0 \pmod 4$ and so we can apply Propositions 7.3.1, 7.3.5, 7.3.6. □

Bibliography

[1] P. Adams, E.J. Billington and E.S. Mahmoodian *The simultaneous metamorphosis of small-wheel systems*, J. Combin. Math. Combin. Comput., **44** (2003), 209-223.

[2] F.E. Bennett and L. Zhu, *Conjugate-orthogonal latin square and related structures*, Contemporary Design Theory: A collection of Surveys (Ed. J.H. Dinitz, D.R. Stinson) J.Wiley and Sons, (1999), 41-96.

[3] J.-C. Bermond and D. Coudert. Traffic grooming in unidirectional WDM ring networks using design theory. In IEEE, editor, IEEE Conf. Commun. (ICC' 03), volume 2, pages 1402-1406, Los Alamitos, CA, 2003.

[4] J.C. Bermond, C.J. Colbourn, L. Gionfriddo, G. Quattrocchi, I. Sau, Drop Cost and Wavelength Optimal Two-Period Grooming with Ratio 4. *Siam J. Discrete Math.*, **24** (2010), 400-419.

[5] J.C. Bermond, C. Huang, A. Rosa and D. Sotteau, Decomposition of complete graphs into isomorphic subgraphs with five vertices, *Ars Combinatoria*, **10** (1980), 211-254.

[6] J.C. Bermond and J. Schönheim, G-decomposition of K_n, where G has four vertices or less, *Discrete Math.*, **19** (1977), 113–120.

[7] J.C. Bermond and D. Sotteau, Graph decompositions and G-designs, *Congressus Numerantium*, **15** (1976), 53-72.

[8] J. Bosak, Decomposition of Graphs, *Kluwer*, Dordrecht 1990.

[9] M. Buratti, Cyclic designs with block size 4 and related optimal optical orthogonal codes. *Des. Codes Cryptogr.*, **26** (2002), no 1-3, 111-125.

[10] A.E. Brouwer. *Optimal Packings of* K_4*'s into a* K_n. J. Combin. Theory (1979), Series A. 26, 278-297.

[11] *The CRC Handbook of Combinatorial Designs.* Second Edition. Edited by C.J.Colbourn and J. H. Dinitz. CRC Press Series on Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 2007.

[12] Y. Caro, Y. Rodity and J. Schönheim, On colored designs - I, *Discrete Math.*, **164** (1997), 47–65.

[13] Y. Caro, Y. Rodity and J. Schönheim, On colored designs - II, *Discrete Math.*, **138** (1995), 177–186.

[14] Y. Caro, Y. Rodity and J. Schönheim, On colored designs - III, *Discrete Math.*, **247** (2002), 51–64.

[15] C.J. Colbourn and M.J. Colbourn, Nested triple systems, *Ars Combinatoria*, **16** (1983), 27-34.

[16] C.J. Colbourn, J.H. Dinitz and A. Rosa, Bicoloring Steiner triple systems, *Electron. J. Combin.*, **6**, (1999), R25.

[17] C.J. Colbourn, A.C.H. Ling and G. Quattrocchi, Minimum embedding of Steiner triple systems into $(K_4 - e)$-designs I. *Discrete Math.*, **308** (2008), 22, 5308-5311.

[18] C.J. Colbourn, A.C.H. Ling and G. Quattrocchi, Embedding path designs into D-designs, where D is the triangle with an attached edge. *Discrete Math.*, **261** (2003), 413-434.

[19] C.J. Colbourn, A.C.H. Ling and G. Quattrocchi, Minimum embedding of P_3-designs into $(K_4 - e)$-designs, *J. Combinatorial Des.*, **11** (2003) 352-366.

[20] C.J. Colbourn and A. Rosa, Triple Systems, *Oxford Mathematical Monographs, Oxford University Press*, (1999), Clarendon Press, Oxford.

[21] C.J. Colbourn and D.R. Stinson, Edge-coloured designs with block size four, *Aequationes Math.*, **36** (1988), 230-245.

[22] C.J.Colbourn, A.C.H. Ling and G.Quattrocchi, *Embedding path designs into kite systems*, Discrete Math., **297** (2005), 38-48.

[23] C.J.Colbourn, A.C.H. Ling and G.Quattrocchi, *Minimum embedding of P_3-designs into $K_4 - e$-designs*, J. Combinatorial Des., **11** (2003) 352-366.

[24] C. J. Colbourn, G. Quattrocchi, and V. R. Syrotiuk. Grooming for two-period optical networks. *Networks*, **52**, (2008) 307-324.

[25] Chen Kejun and Zhu Lie, *On the existence of skew Room frames of type t^u*, Ars Combinatoria, **43** (1996), 65-79.

[26] M.Gionfriddo and G.Quattrocchi, *Embedding balanced P_3-designs into P_4-designs*, Discrete Math., **308** (2008), 155-160.

[27] M.J. de Resmini, On k-sets of type $(m,\, n)$ in a Steiner system $S(2,\, l,\, v)$. Finite geometries and designs (Proc. Conf., Chelwood Gate, 1980), pp. 104–113, *London Math. Soc. Lecture Note* Ser., 49, Cambridge Univ. Press, Cambridge-New York, 1981.

[28] J. Doyen and R. M. Wilson, Embeddings of Steiner triple systems, *Discrete Math.*, **36** (1988), 230-245.

[29] B. Du, On complementary path decompositions of the complete multigraph, *Australasian J. Combin.*, **11** (1995), 211-213.

[30] W. T. Federer, Construction of classes of experimental designs using transversals in Latin squares and Hedayat's sum composition method, In: T. A. Bancroft, Ed., *Statistical Papers in Honor of George W. Snedecor* (1972), 91-114.

[31] L. Gionfriddo, New nesting for G-designs, case of order a prime, *Congressus Numerantium*, **145** (2000), 167–176.

[32] L. Gionfriddo, On the spectrum of nested G-designs, where G has four non-isolated vertices or less, *Australasian J. Combin.*, **24** (2001), 59–68.

[33] M. Gionfriddo and G. Quattrocchi, Embedding balanced P_3-designs in P_4-designs, *Discrete Math.*, **308** (2008), 155-160.

[34] O. Goldschmidt, D. Hochbaum, A. Levin, and E. Olinick. The sonet edge-partition problem. *Networks*, **41** (2003), 13-23.

[35] A. Granville, A. Moisiadis and R. Rees, On Complementary Decompositions of the Complete Graph, *Graphs and Combinatorics*, **5** (1989), 57-61.

[36] J.W.P. Hirschfeld, Projective geometries over finite fields, *Clarendon Press*, Oxford (1998).

[37] K. Heinrich, Graphs decompositions and designs, In: *The CRC Handbook of Combinatorial Designs* (ed. C.J. Colbourn and J.H. Dinitz) Boca Raton 1996, 361-366.

[38] D.G. Hoffmman, Kimberly S. Kirkpatrick, G-designs of order n and index λ where G has 5 vertices or less, *Australasian J. Combin.*, **18** (1998), 13-37.

[39] S. P. Hurd and D.G. Sarvate, *Pair-resolvability and tight embeddings for path designs*, J. Combin. Math. Combin. Comput. **58** (2006), 113-127.

[40] S. P. Hurd, P. Munson, D.G. Sarvate, *Minimal Enclosing of Triple Systems I: Adding one point*, Ars Combin. **68** (2003), 145-159.

[41] S. Kageyama and Y. Miao, The spectrum of nested designs with block size three and four, *Congressus Numerantium*, **114** (1996), 73-80.

[42] S. Kageyama and Y. Miao, Nested designs of superblock size five and superblock size two, *J. Statist. Plann. Inference*, **64** (1997), 125-139.

[43] S. Kageyama and Y. Miao, Nested designs of superblock size four, *J. Statist. Plann. Inference*, **73** (1998), 1-5.

[44] S. Kucukcifci, *The metamorphosis of λ-fold block designs with block size four into maximum packings of λK_n with kites*, Util. Math., **68** (2005), 165-195.

[45] Selda Kucukcifci and C.C. Lindner, *The metamorphosis of λ-fold block designs with block size four into λ-fold kite systems*, JCMCC, **40** (2002), 241-252.

[46] S. Kucukcifci, C.C. Lindner and A.Rosa *The metamorphosis of λ-fold block designs with block size four into a maximum packing of λK_n with 4-cycles*, Discrete Math., **278** (2004), 175-193.

[47] E.R. Lamken and R.M. Wilson, Decompositions of edge-colored complete graphs, *J. Combin. Theory* (A), **89** (2000), 149-200.

[48] H. Lenz and H. Zeitler, Arcs and ovals in Steiner triple systems, *Combinatorial Theory, Lecture Notes in Math.*, **969** (1982), 229–250.

[49] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, 1997, 208 pages.

[50] C.C. Lindner and C.A. Rodger, Decomposition into cycles II: Cycle systems, *Contemporary Design Theory: A collection of surveys* (eds J.H. Dinitz and D.R.Stinson), John Wiley and Son, New York (1992), 325-369.

[51] C.C. Lindner, G. Lo Faro and A. Tripodi, *The metamorphosis of λ-fold kite systems into maximum packings of λK_n with triangles*, J. Combin. Math. Combin. Comput., **56** (2006), 171-189.

[52] C.C. Lindner and D.R. Stinson, Steiner pentagon systems, *Discrete Mathematics*, **52** (1984), 67-74.

[53] C.C. Lindner and A. Street, *The metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems*, Bulletin of the ICA, **28** (2000), 7-18.

[54] G. Lo Faro and A. Tripodi, *The spectrum of Meta($K_3 + e > P_4, \lambda$) and Meta($K_3 + e > H_4, \lambda$) with any λ*, Utilitas Math., **72** (2007), 3-22.

[55] G. Lo Faro, A. Tripodi. *The Doyen-Wilson theorem for kite systems.* Discrete Math., **306** (2006), 2695-2701.

[56] G. Lo Faro, A. Tripodi. *Embeddings of λ-fold kite systems, $\lambda \geq 2$.* Australasian Journal of Combinatorics, **36** (2006), 143-150.

[57] S. Mendelsohn, G. Quattrocchi *Minimum embedding of balanced P_4-designs into 5-cycle systems*, Discrete Math., **279** (2004), 407-421.

[58] M. Meszka and A. Rosa, Embedding Steiner triple systems into Steiner systems $S(2, 4, v)$, *Discrete Math.*, **274** (2004), 199-212.

[59] S. Milici, *Minimum embedding of P_3-designs into $TS(v, \lambda)$*, Discrete Math., **308** (2008), 331-338.

[60] S. Meszka and A. Rosa *Embedding Steiner triple systems into Steiner systems* $S(2, 4, v)$, Discrete Math., **279** (2004), 199-212.

[61] S. Milici, G. Quattrocchi *Embedding handcuffed designs with block size 2 or 3 in 4-cycle systems*, Discrete Math., **208/209** (1999), 443-449.

[62] S. Milici and G. Quattrocchi, On nesting of path designs, J. Comb. Math. Comb. Comp., **32** (2000), 115-127.

[63] S. Milici, A. Rosa and V. Voloshin, Colouring Steiner systems with specified block colour patterns, *Discrete Math.*, **240** (2001), 145-260.

[64] D.A. Preece, Nested balanced incomplete block designs, *Biometrika*, **54** (1967), 479-486.

[65] G. Quattrocchi *Embedding G_1-designs into G_2-designs, a short survey. 6th Workshop on Combinatorics (Messina, 2002)*, Rend. Sem. Mat. Messina, **Ser II Suppl. 8** (2002), 129-143.

[66] G. Quattrocchi, Embedding path designs in 4-cycle systems, *Discrete Math.*, **255** (2002), 349-356.

[67] G. Quattrocchi, Colouring 4-cycle systems with specified block colour pattern: the case of embedding P_3-designs, *The Electronic J. of Combinatorics*, **8** (2001), R24.

[68] G. Quattrocchi, Embedding handcuffed designs in D-designs, where D is the triangle with attached edge, *Discrete Math*, **261** (2003), 413-434.

[69] G. Quattrocchi *Embedding path designs in 4-cycle systems*, Discrete Math., **255** (2002), 349-356.

[70] R. Rees and C.A. Rodger, Subdesigns in complementary path decompositions and incomplete two-fold designs with block size four, *Ars Combinatoria*, **35** (1993), 117-122.

[71] R.Rees and D. R. Stinson, *On the existence of incomplete designs of block size four having one hole*, Utilitas Math., **35**, (1989), 119-152

[72] C.A. Rodger, Self-complementary graph decompositions, *J. Austral. Math. Soc. (Series A)*, **53** (1992), 17-24.

[73] D.R. Stinson, The spectrum of nested Steiner triple systems, *Graphs and Combin.*, **1** (1985), 189-191.

[74] J. Yin and B. Gong, Existence of G-designs with $|V(G)| = 6$, *Lecture Notes in Pure and Appl. Math.*, **126** (1990), 201-218.