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Some selective and monotone versions of covering properties and some results on the cardinality of a topological space

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Basic notions

In this Chapter, we recall several preliminar notions. We follow standard notations and definitions from [39], [50].

The following set-theoretic notation is adopted: κ, λ and τ are cardinal numbers, α, β and γ are ordinal numbers; i and n are non-negative integers, ω is the smallest infinite ordinal and cardinal, ω_1 is the smallest uncountable ordinal and cardinal, κ^+ is the smallest cardinal greater than κ.

Let E be a set. Then $|E|$ is the cardinality of E, $\mathcal{P}(E)$ is the power set of E, $[E]^{\leq\kappa}$ is the collection of all subsets of E of cardinality $\leq \kappa$, and $[E]^n = \{A : A \subseteq E, |A| = n\}$. A subset A of a partially ordered set $(P, <)$ is *cofinal* if for every $p \in P$ there exists some $a \in A$ such that $p \leq a$. The *cofinality* of $(P, <)$ is the smallest size of a cofinal set.

The following theorem from combinatorial set-theory plays an important role in cardinal functions.

Theorem 0.0.1 (Erdös-Rado)**.** Let κ be an infinite cardinal, let E be a set with $|E| > 2^\kappa$, and suppose $[E]^2 = \bigcup_{\alpha<\kappa} P_\alpha$. Then there exists $\alpha < \kappa$ and a subset A of E with $|A| > \kappa$ such that $[A]^2 \subseteq P_\alpha$.

X always denotes a non-empty topological space. The following topological notions and conventions are used. A *cover* of a set X is a family $\{A_s\}_{s\in S}$ of subsets of X such that $\bigcup_{s\in S} A_s = X$, and that, if X is a topological space, $\{A_s\}_{s\in S}$ is an open (closed) cover of X if all sets A_s are open (closed). A cover $\mathcal{B} = \{B_t\}_{t\in T}$ *refines* another cover $\mathcal{A} = \{A_s\}_s \in S$ of the same set X if for every $t \in T$ there exists an $s \in S$ such that $B_t \subset A_s$; equivalently, we say that $\mathcal{B}$ is a *refinement* of $\mathcal{A}$. A cover $\mathcal{A}' = \{A'_s\}_{s\in S'}$ of X is a *subcover* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of X if $S' \subset S$ and $A'_s = A_s$ for every $s \in S'$. In particular, any subcover is a refinement.

A topological space X is *compact* if every open cover of X has a finite subcover.

The following theorem represents a well known characterization of a compact space.

Theorem 0.0.2. A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection (recall that a family $\mathcal{F} = \{F_s\}_{s\in S}$ of subsets of a set X has the *finite intersection property* if $\mathcal{F} \neq \emptyset$ and $F_{s_1} \cap F_{s_2} \cap \ldots \cap F_{s_\kappa} \neq \emptyset$ for every finite set $\{s_1, s_2, \ldots, s_\kappa\} \subset S$).

A topological space X is called a *countably compact space* if every countable open cover of X has a finite subcover. Thus, every compact space is countably compact.
A topological space X is said to be *Lindelöf* if every open cover of X has a countable subcover. From the definitions, it follows that every compact space is a Lindelöf space, and it is easy to see that, in the class of countably compact spaces, compactness and Lindelöfness are equivalent properties.
A common generalization of compactness and Lindelöf property is given by paracompactness: a topological space X is called a *paracompact space* if every open cover of X has a locally finite open refinement.

A *linearly ordered topological space* (LOTS) is a triple $(X, \mathcal{T}, <)$, where $<$ is a linear ordering of X and $\mathcal{T}$ is the open interval topology of the given ordering. *Generalized ordered spaces* (GO-spaces) are exactly those spaces that can be homeomorphically embedded in some LOTS.

A topological space X is called *topologically homogeneous*, briefly *homogeneous*, if for all points $x, y \in X$ there is a homeomorphism $f : X \to X$ such that $f(x) = y$. In a topologically homogeneous space X, a topological property which holds at one point of X also holds at all other points (see for example [11, 21, 38]).

A space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ which has a countable base of open neighborhoods in X [4]. All metrizable spaces, and all locally compact Hausdorff spaces, as well as all Čech-complete spaces, are of countable type.

For every Tychonoff space X, by a *remainder* of a space X we understand the subspace $bX \setminus X$ of a compactification bX of X. A space X is a *p-space* [4] if in any (in some) compactification bX of X there exists a countable family $\{\gamma_n : n \in \omega\}$ of families γ_n of open subsets of bX such that $x \in \bigcap\{St(x, \gamma_n) : x \in \omega\} \subset X$, for each $x \in X$. Every p-space is of countable type and every metrizable space is a p-space. *Paracompact p-spaces* [4] are preimages of metrizable space under perfect mappings. A *Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping.

The following standard examples in topology are frequently mentioned in the thesis:

- the linearly ordered space ω_1 of all countable ordinals. ω_1 is countably compact, not paracompact, neither metacompact;

- the Alexandroff duplicate of X. For every space X, consider a disjoint copy X_1 of X and the union $A(X) = X \cup X_1$. Let $\pi : X \to X_1$ be the natural bijection. Topologize $A(X)$ by letting sets of the form $U \cup \pi(U \setminus \{x\})$, where U is a neighborhood of x in X, be basic neighborhoods of a point $x \in X$ and declaring points of X_1 to be isolated;

- the Isbell-Mrówka space Ψ. Let $\mathcal{A}$ be a maximal infinite family of infinite subsets of ω such that the intersection of any two members of $\mathcal{A}$ is finite, and let $D = \{\omega_A\}_{A\in\mathcal{A}}$ a set of distinct points. Ψ is the space $\omega \cup D$, where the points from ω are isolated and a basic neighborhood for ω_A consists of the sets of the form $\{\omega_A\} \cup (A \setminus F)$, where F is a finite subset of A. Ψ is Hausdorff, locally compact, pseudocompact, but it is not countably compact;

- $C_p(X, Y)$, the set of all continuous function of X to Y with the pointwise convergence topology. In particular, if $Y = \mathbb{R}$, we simply write $C_p(X)$;

- The Baire space ω^ω, which is homeomorphic to the space of irrationals $\mathbb{P}$.

Below, we recall the cardinal functions that we will use in the thesis (see [49, 50, 53]). Every cardinal function is required to take on only infinite cardinals values.

A *net* for a topological space X is a collection $\mathcal{N}$ of subsets of X such that every open set in X is the union of elements of $\mathcal{N}$. The *net weight* of X, denoted $nw(X)$, is the smallest infinite cardinality of a net for X. A π-base for X is a collection $\mathcal{V}$ of non-empty open sets in X such that if R is any non-empty open set in X, then $V \subseteq R$ for some $V \in \mathcal{V}$. The *π-weight* of X, denoted $\pi - w(X)$, is the smallest infinite cardinality of a π-base for X. The *diagonal degree* of a T_1 space X, denoted $\Delta(X)$, is the smallest infinite cardinal κ such that X has a collection $\{\mathcal{V}_\alpha : 0 \leq \alpha < \kappa\}$ of open covers such that $\bigcap_{\alpha<\kappa} St(p, \mathcal{V}_\alpha) = \{p\}$ for each $p \in X$, equivalently $\Delta(X)$ is the smallest infinite cardinal κ such that the diagonal Δ of X is the intersection of κ open sets in $X \times X$. Thus, if $\Delta(X) = \omega$, one says that X has a G_δ-diagonal. Let X be a topological space, let $\mathcal{V}$ be a collection of non-empty open sets in X, let $p \in X$. Then $\mathcal{V}$ is a *local π-base for p* if for each open neighbourhoods R of p, one has $V \subseteq R$ for some $V \in \mathcal{V}$. If in addition one has $p \in V$ for all $V \in \mathcal{V}$, then $\mathcal{V}$ is a *local base for p*. Finally, if $p \in V$ for all $V \in \mathcal{V}$, and $\bigcap\{V : V \in \mathcal{V}\} = \{p\}$, then $\mathcal{V}$ is a *pseudo-base* for p. The *character* of X at p, $\chi(p, X)$, is the smallest infinite cardinality of a local base for p, the *π-character* of X at p, $\pi\chi(p, X)$, is the smallest infinite cardinality of a local π-base for p, the *pseudocharacter* of X at p, $\psi(p, X)$, is the smallest infinite cardinality of a pseudo-base for p. Taking the suprema of these functions, we obtain the definitions of the *character* $\chi(X)$, the *π-character* $\pi\chi(X)$ and the *pseudocharacter* $\psi(X)$ of X.

The *tighness* $t(X)$ of a space X is the least cardinal number κ with the property that if $A \subset X$ and $x \in \overline{A}$, then there exists some set $B \in [A]^{\leq\kappa}$ such that $x \in \overline{B}$.

Introduction

As the title suggestes, this thesis consists of two chapters: in the first one (Chapter 1), we investigate some selective and monotone versions of covering properties, in the second chapter (Chapter 2), we introduce some bounds on the cardinality of a topological space. Both of the topics are active areas of research in General Topology. In particular, many important topological properties can be defined or characterized in terms of selection principles, by monotonicity, a lot of classical results can be generalized, and the problem to find new cardinal inequalities which bound the size of a topological space is a well known problem in literature.

The importance of the Theory of selection principles does not concern only General Topology, but encompasses several branchs of Mathematics, for example Game Theory, Ramsey Theory, Analysis, and so on.

Selection principles appear for the first time in measure theory and basis theory in metric space. In 1924, K. Menger [66] defined the *Menger basis property* for metric spaces: "A metric space (X, d) has the Menger basis property if there is for each base $\mathcal{B}$ of X a sequence $(B_n : n \in \omega)$ of sets from the basis such that $\lim_{n\to\infty} diam_d(B_n) = 0$, and $\{B_n : n \in \omega\}$ is a cover for X". In 1925, W. Hurewicz proved that the metric space (X, d) has the Menger basis property if and only if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \omega)$ of finite sets such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$, and $\bigcup_{n\in\omega} \mathcal{V}_n$ is an open cover of X. This property is called *Menger property*. In 1938, F. Rothberger in his study of Borel measure zero introduced the following statement: "For each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there is a sequence $(U_n : n \in \omega)$ such that for every n, $U_n \in \mathcal{U}_n$, and $\{U_n : n \in \omega\}$ is an open cover of X". F. Rothberger pointed out that if a metric space has this property (called *Rothberger property*), then it has Borel measure zero, and the converse does not hold. These two selection principles, Menger property and Rothberger property, are the classical selection principles. Also, there is a third one, introduced by W. Hurewicz in 1925, but this property could be included in the Rothberger-type. Later, several covering properties were defined in terms of selection principles.

In Section 1.1, we study two selective properties: the *selective absolute star-Lindelöfness*, defined by S. Bhowmik in [25], and the *selective strong star-Menger* property. These properties are both between absolute countable compactness and absolute star-Lindelöfness. In particular, selectively absolute star-Lindelöfness is weaker than selective strong star-Menger, and an example of a selectively absolute star-Lindelöf space which is not selectively strongly star Menger, is given.

It is easy to distinguish selective absolute star-Lindelöfness from absolute countable compactness. Whereas, it turns out to be not so simple to find an example to distinguish selective absolute star-Lindelöfness from absolute star-Lindelöf. In order to do this, we construct "ad hoc" a space (Example 1.1.4). Moreover, we investigate which spaces have the selective absolute star Lindelöf property. We obtain that every star-Lindelöf space of countable fan tightness with respect to dense subspaces, every space with a countable π-base and every absolute star-Lindelöf space having cardinality less than $\mathfrak{d}$ are selectively absolutely star-Lindelöf spaces. We also investigate the behaviour of selective absolute star Lindelöfness with respect to closed subspaces, dense subspaces, and finite union.

M. V. Matveev and M. Bonanzinga, respectively, showed that both absolute countable compactness and ablosute star-Lindelöfness are preserved by varpseudo-open maps (see Definition 1.1.2). Motivated by the parallelism between these properties with selective absolute star-Lindelöfness, we study if the continuous images of selective absolute star-Lindelöf spaces are selective absolute star-Lindelöf. It turns out that, also in this case, the property is preserved by varpseudo-open maps. We conclude the study of the selective star-Lindelöf property, showing that the product of a selective absolute star-Lindelöf countable metacompact space with convergent sequence is a selectively absolutely star-Lindelöf space and that the product of a selectively absolutely star-Lindelöf space and a compact Hausdorff first countable space need not be selectively absolutely star-Lindelöf.

The last part of Section 1.1, is devoted to the study of the selective strong star Menger property. In particular, it is proved that this property is a weaker form of Menger property, and that in the class of paracompact Hausdorff spaces, the gap between several Menger-type properties (Menger, selective strongly star Menger, absolute strong star Menger, selective strong star Menger and star Menger) disappears. Furthermore, motivated by the parallelism between the selective strong star Menger property with the selective absolute star-Lindelöfness, we study its behaviour with respect to particular subspaces, continuous images and finite union. Unfortunately, we can not say anything about the product of a selectively strongly star Menger space with a compact first countable space.

In Section 1.2, we define and study several properties related to monotone normality. Recall that a space X is *monotonically normal* if and only if for each open set $U \subset X$ and $x \in U$, one can assign an open set U_x containing x satisfying the following condition:

$$U_x \cap V_y \neq \emptyset \Rightarrow x \in V \text{ or } y \in U.$$

We give a characterization of monotone normality in terms of monotone function. This provides a monotone version of the well known Urysohn's Lemma. Another important characterization of normality was given in terms of stars. It is known that, for a space X the properties X is normal, every two-element open cover of X has an open star-refinement and every finite open cover of X has a finite open star-refinement are equivalent. The monotone version of this result does not hold. In fact, we define 2-*monotone star-normality* and *finitely-monotone star-normality* and prove that every 2-monotonically star-normal space is monotonically normal and that monotone normality and finite-monotone star-normality are not equivalent conditions. An important result by M. E. Rudin, I. Stares and J. E. Vaughan states that every monotonically normal space has the property (a). In order to give a monotone version of this result, we define a monotone version of property (a). Indeed, depending on the way in which the monotone operator works, we can give four monotone versions of property (a). It is nevertheless shown that none of these definitions satisfies our original request (actually, we show the senselessness of two of them), but it turns out that one of these, property **ma**, has interesting relations with other monotone covering properties: every space having property **ma** is monotonically star closed and discrete. Finally, motivated by the well-known results which states that in the class of Hausdorff space, "countable compactness together with property (a)", is equivalent to absolute countable compactness, we define *property monotone absolute countable compact* and prove the monotone version of the previous result.

A long standing problem in General Topology is to give bounds to the cardinality of a topological space, and has its roots in the following question, asked by P. Alexandroff and P. Urysohn in 1923: *does every compact first countable Hausdorff space have cardinality at most* 2^ω*?* The solution to the problem was obtained in 1969 by A. V. Arhangel'skii, who proved, by using cardinal functions, his beautiful inequality "$|X| \leq 2^{L(X)\chi(X)}$", for every Hausdorff space X. In 1967, A. Hajnal and I. Juhasz had already used cardinal invariants to prove two inequalities which are now regarded as fundamental to the theory of cardinal functions: (1) $|X| \leq 2^{c(X)\chi(X)}$ for every Hausdorff space X and (2) $|X| \leq 2^{s(X)\psi(X)}$ for every T_1 space X. A long list of generalizations or variations of the inequalities of Hajnal and Juhasz and of the Arhangel'skii's inequality has been obtained in literature.

In Chapter 2, we continue this line of investigation and proved several new bounds to the cardinality of a topological space.

In Section 2.1 we give three variations of the Hajnal and Juhasz's inequality $|X| \leq 2^{c(X)\chi(X)}$, for every Hausdorff X:

- $|X| \leq 2^{n-Uc(X)\chi(X)}$, where X is n-Urysohn, $n \in \omega$ and $n - Uc(X)$ is the n-Urysohn cellularity of X,
- $|X| \leq 2^{n-c(X)\chi(X)}$, where X is n-Hausdorff, $n \in \omega$ and $n - c(X)$ is the n-cellularity of X,
- $|X| \leq 2^{Uc(X)\pi\chi(X)}$, if X is a power homogeneous Urysohn space.

Clearly, for every space X, $Uc(X) \leq c(X)$ and, if $Uc(X) \leq \kappa$, then $n - c(X) \leq \kappa$ for every $n \in \omega$.

In Section 2.2, we consider the inequality of A. Charlsworth $|X| \leq psw(X)^{L(X)\psi(X)}$, for every T_1 space X and give an analogous of this inequality in Hausdorff case. It turns out that, in the case of Hausdorff separation property, the point separating weight and the Lindelöf degree should be replaced, respectively, by the *Hausdorff point separating weight* (see Definition 2.2.1) and the almost Lindelöf degree with respect to closed subspace. Moreover, we naturally raised the question concerning whether the Lindelöf degree can be replaced with the weakly Lindelöf degree with respect to closed sets: we should add the very strong hypothesis that the space is almost regular and has a π-base consisting of open subsets having a compact closure.

We end the treatment taking into account cardinal inequalities in topological groups.

Topological Algebra is an active area of research in General Topology. A lot of results in this connection have been obtained by A. V. Arhangel'skii, who began a systematic study of properties of topological groups, their subspaces, and cardinal invariants in such spaces. An important problem in topology is to study how properties of a space X are related to the properties of some or all remainders of X. An extensive study by A. V. Arhangel'skii shows that the remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. Of course, there is an important exception to this rule: the case of locally compact topological groups. Indeed, every locally compact non-compact topological group has a remainder consisting of exactly one point. In particular, A. V. Arhangel'skii and J. van Mill recently studied the problem concerning whether a topological group has a first-countable remainder and proved that if G is a non locally-compact group with a compactification bG such that $Y = bG \setminus G$ is first countable, then $\chi(G) \leq \omega_1$ and $|G| \leq 2^{\omega_1}$. This result only solves the countable case of a much more general problem. Motivated by this, we study topological groups with a remainder, in some compactification, of character κ, where κ is an infinite cardinal. We show that the method of A. V. Ahangel'skii and J. van Mill works also in the case of a general infinite cardinal κ: if G is a non locally-compact group with a remainder in some compactification of character κ, then the character of the space G does not exceed κ^+ and the size of G is at most $2^{(\kappa^+)}$.

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1 Some selective and monotone versions of covering properties

In this Chapter, we are taking into account some covering properties and their generalizations in two different areas of interest: selectivity and monotonicity. In particular, in the first Section, we study two properties that lie between absolute countable compactness [64] and absolute star-Lindelöfness [24, 25]: *selective absolute star-Lindelöf* and *selective strong star-Menger*, which represent, respectively, selective versions of absolute star Lindelöfness and strongly star Menger property. In the second Section, we investigate several properties close to monotone normality [62].

Some of the results that we discuss are included in [18], [27] and [29]. In the treatment the following definitions play a notable role.

Let $\mathcal{A} = \{A_s\}_{s\in S}$ be a cover of a set X, the *star of a set $M \subset X$ with respect to $\mathcal{A}$* is the set $St(M, \mathcal{A}) = \bigcup\{A_s : M \cap A_s \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathcal{A}$ is called the *star of the point x with respect to $\mathcal{A}$* and is denoted by $St(x, \mathcal{A})$. We say that a cover $\mathcal{B} = \{B_t\}_{t\in T}$ of a set X is a *star refinement* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of the same set X if for every $t \in T$ there exists an $s \in S$ such that $St(B_t, \mathcal{B}) \subset A_s$; if for every $x \in X$ there exists an $s \in S$ such that $St(x, \mathcal{B}) \subset A_s$, then we say that $\mathcal{B}$ is a *barycentric refinement* of $\mathcal{A}$. Many important topological properties are defined or can be characterized in terms of stars with respect to open covers.

The following theorem gives the star characterization of countable compactness.

Theorem 1.0.1. [40, 75] A Hausdorff space X is countably compact if and only if is *starcompact*, *i.e.* for every open cover $\mathcal{U}$ there exists a finite subset $F \subset X$ such that $St(F, \mathcal{U}) = X$.

This criterion of countable compactness motivated M. V. Matveev to introduce the following definition.

Definition 1.0.1. [64] A space X is *absolutely countably compact* (briefly *acc*) if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subset X$ there exists a finite subset $F \subset D$ such that $St(F, \mathcal{U}) = X$.

Clearly, this property is stronger than countable compactness. A natural generalization of absolute countable compactness is given by the following definition.

Definition 1.0.2. [65] A space X is an *(a) space*, or has *property (a)* if for every open cover $\mathcal{U}$ and every dense $D \subset X$ there is a closed in X and discrete $F \subset D$ such that $St(F, \mathcal{U}) = X$.

In countably compact space closed and discrete means finite. So, property (a) is the extension of acc outside of the class of countably compact spaces:

$$acc \Leftrightarrow (a) + cc,$$

where cc means countable compact space.

If in the definition of star compactness, “finite subset" is replaced by “countable subset", we obtain a natural generalization of star compactness.

Definition 1.0.3. A topological space X is *star Lindelöf* (see [84] where it was called strongly 1-star-Lindelöf) if for every open cover $\mathcal{U}$ of X there exists a countable subset C of X such that $St(C, \mathcal{U}) = X$.

In [24, 25], M. Bonanzinga introduced a property which is stronger than star Lindelöfness and represents the countable version of absolute star compactness.

Definition 1.0.4. [24, 25] A space X is *absolutely star-Lindelöf* (briefly, *a-star-Lindelöf*) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there is a countable set $C \subset D$ such that $St(C, \mathcal{U}) = X$.

Every countably compact a-star-Lindelöf space is acc.

Several important classes of topological spaces can be characterized using *selection principles*.
Selection principles appear for the first time in measure theory and basis theory in metric space. In 1924, K. Menger [66] introduced a metric notion that in 1925 W. Hurewicz [51] pointed out that is equivalent to the following topological notion.

Definition 1.0.5. [51] A space X is said to have the *Menger property* (or is *Menger*) if for every sequence $(\mathcal{U}_n : n = 1, 2, \ldots)$ of open covers of X there is a sequence $(\mathcal{V}_n : n = 1, 2, \ldots)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and such that $\bigcup_{n \in \omega} \mathcal{V}_n$ is a cover of X.

If a topological space is Menger, then it has the Lindelöf property.

The Menger property, together with the *Hurewicz property* [51] and the *Rothberger property* [72], represents the classical selection principles.

All selection principles are defined in terms of the possibility to *select* from a given sequence of open covers of some sort, an open cover of some, possibly different, sort. In [77], M. Scheepers, who pioneered the systematic study of selection principles, defined in wider terms selection principles by schemas of the sort where classes $\mathcal{A}$ and $\mathcal{B}$ of covers are given, as well as some procedure Π for generating from a sequence $(\mathcal{U}_n : n = 1, 2, \ldots)$ of covers from $\mathcal{A}$, a cover from $\mathcal{B}$. A space is said to have property $\Pi(\mathcal{A}, \mathcal{B})$ if for every sequence $(\mathcal{U}_n : n = 1, 2, \ldots)$ of covers from $\mathcal{A}$, one can build, using procedure Π a cover of X which belongs to $\mathcal{B}$.

The importance of selection principles theory lies in its deep connection with several branches of Mathematics (for example Set Theory and General Topology, Game Theory, Ramsey Theory and so on). So there was a lot of papers about selection principles, see for example [19, 20, 30, 54, 55, 56, 58, 76, 77, 78, 79].

In this Section, we study two properties defined in terms of selection principles. In particular, we study the selective absolute star-Lindelöf property, introduced by S. Bhowmik in [23] (where it was called selective star-Lindelöfness) as a generalization of star-Lindelöfness. In fact, we note that selective absolute star-Lindelöfness is between acc property [64] and absolute star-Lindelöfness [24, 25]. We study the general properties of selective absolute star-Lindelöfness, give several examples, study the behaviour of selective absolute star-Lindelöfness with respect to particular subspaces, continuous images and finite union, prove that the product of a selective absolute star-Lindelöf countable metacompact space with convergent sequence is a selectively absolutely star-Lindelöf space and that the product of a selectively absolutely star-Lindelöf space and a compact Hausdorff first countable space need not be selectively absolutely star-Lindelöf. Moreover, we introduce the selective strong star-Menger property. It is a selective version of strong star-Menger property [33, 57]. We study its connection with absolute countable compactness and selective absolute star-Lindelöfness, give examples distinguishing these properties and study the behaviour of selective strong star-Menger property with respect to subspaces, continuous images and finite union.

The importance and interest of monotonicity in topology find their most meaningful representative in *monotone normality* [17, 37, 41, 62, 67, 68, 73, 75], which was the first monotone version of a topological notion that was defined. Monotone normality was used unnamed for the first time by C. J. R. Borges in 1966. In 1971 was named by P. L. Zenor and in 1973 appears the first study about monotone normality by D. J. Lutzer, R. W. Heath and P. L. Zenor. In a general way, a space X is *monotonically* $\mathcal{P}$ if X has $\mathcal{P}$ in a manner that respect set inclusion. Recall the original definition of monotone normality.

Definition 1.0.6. [62] A space X is *monotonically normal* if for each pair (H, K) of disjoint closed subsets of X, one can assign an open set $r(H, K)$ such that

(1) $H \subset r(H,K) \subset \overline{r(H,K)} \subset X \setminus K$;

(2) if $H_1 \subset H_2$ and $K_1 \supset K_2$ then $r(H_1,K_1) \subset r(H_2,K_2)$.

The function r is called a *monotone normality operator for* X and witnessing the normality of the space in a monotonic manner.

A considerable number of characterizations of monotone normality spaces have been obtained. In particular, the following is very usefull.

Proposition 1.0.1. [74] A topological space X is *monotonically normal* provided X is T_1 and, for all open U in X and $x \in U$, there is an $H(x,U)$ such that:

(a) $H(x,U) \cap H(y,V) \neq \emptyset$ implies either $x \in V$ or $y \in U$ (normality);

(b) $U \subseteq W$ implies $H(x,U) \subseteq H(x,W)$ (monotonicity).

Equivalently, one can request in proposition above only (a) and get another H satysfying also (b), see [44, Theorem 5.19].

Metric spaces and linearly ordered spaces (as well as their subspaces) are monotonically normal and monotonically normal spaces are countably paracompact. But there are no implications between paracompactness and monotone normality. Compact spaces need not be monotonically normal. Also, linearly ordered spaces fail to be paracompact if and only if they have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal. Z. Balogh and M. E. Rudin proved in [17] that monotonically normal space share the previous property with linearly ordered spaces. Moreover, in [17] the authors studied how paracompactness can fail in monotonically normal spaces.

The idea behind monotone normality applied to paracompact spaces was given by P. M. Gartside and P. J. Moody, which in [41] defined a space X to be *monotonically paracompact* if there exists a function r which assigns to every open cover $\mathcal{U}$ an open star-refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U})$ refines $r(\mathcal{V})$. They also proved that for a T_1 space X monotone paracompactness and having a continuous monotonically normal operator are equivalent conditions.

The definition of monotone paracompactness showed the way forward to monotone versions of various covering properties. The most general definition of monotone property $\mathcal{P}$, where $\mathcal{P}$ is a covering property, was given by K. P. Hart in [47].

Definition 1.0.7. [47] Let $\mathcal{P}$ be a covering property that states that every cover of class $\mathcal{A}$ has a refinement of class $\mathcal{B}$. A space X is *monotonically* $\mathcal{P}$ if one can assign to every cover $\mathcal{U}$ of class $\mathcal{A}$ a refinement $r(\mathcal{U})$ of class $\mathcal{B}$ so that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$.

Depending on the choice of the covering property $\mathcal{P}$, we can obtain its monotone version. For example, a space X is *monotonically compact* if one can assign to each open cover $\mathcal{U}$ a finite open refinement $r(\mathcal{U})$ such that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$; a space X is *monotonically Lindelöf* if one can assign to each open cover $\mathcal{U}$ a countable open refinement $r(\mathcal{U})$ such that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$.

Recently, monotonically Lindelöf spaces have been extensively studied, see for example [22, 59, 60, 61, 63]; all second countable spaces, all separable metrizable spaces, the one point Lindelöfication of the discrete space of cardinality ω_1, all separable GO spaces, some non-separable GO spaces, for example, the lexicographic square of $[0, 1]$, (consistently) some non-metrizable countable spaces are monotonically Lindelöf. On the other hand, such "good" Lindelöf spaces as the one point Lindelöfication of the discrete space of cardinality ω_2, the one point compactification of the discrete space of cardinality ω_1, or a dense countable subset in 2^{ω_1} are not monotonically Lindelöf. The Alexandroff duplicate of X is monotonically Lindelöf if and only if X is second countable (J. Vaughan, unpublished).

In [45], G. Gruenhage answering a question of M. V. Matveev proved that every monotonically compact Hausdorff space is metrizable. C. Good, R. Knight and I. Stares, in [42], defined *monotonically countably metacompact* and *monotonically countably paracompact* spaces. In [69], S. G. Popvasillev defined a space X to be *monotonically countably compact* if there is an operator r that assigns to every countable open cover $\mathcal{U}$ a finite open refinement $r(\mathcal{U})$ that covers X, and such that if $\mathcal{V}$ refines $\mathcal{U}$ then $r(\mathcal{V})$ refines $r(\mathcal{U})$. If $r(\mathcal{U})$ is only required to be point-finite instead of finite then X is called *monotonically countably metacompact*.

In this Section, motivated by the growing interest in monotonicity, we also investigate several monotone version of known results. In particular, we give a characterization of monotone normality in terms of functions and a partial solution to the problem to find monotone version of a star-characterization of normality.
Furthermore, motivated by the following result,

$$\text{monotone normality} \;\Rightarrow\; \text{property (a)},$$

we study the problem to find a monotone version of the previous implication. It turns out that we can define a monotone version of property (a) in four different ways, but none of these gives a positive answer to the problem. Further, we study some monotone version of properties involving property (a).

1.1 On selectively absolutely star Lindelöf and selectively strongly star-Menger spaces

1.1.1 Selective absolute star-Lindelöfness

Recently, S. Bhowmik [23] introduced the following notion as a variation of star-Lindelöfness and call it *selectively star-Lindelöfness*.

Definition 1.1.1. [23] A space X is *selectively absolutely star-Lindelöf* (briefly, *selectively a-star-Lindelöf*) if for any open cover $\mathcal{U}$ of X and any sequence $(D_n : n \in \omega)$ of dense subsets of X, there are finite sets $F_n \subset D_n$ $(n \in \omega)$ such that $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = X$.

This notion lies between absolute countable compactness [64] and absolute star-Lindelöfness [25]. Every Lindelöf space is obviously selectively a-star-Lindelöf.

In this Chapter, we distinguish absolute star-Lindelöfness from selective absolute star-Lindelöfness, and study the general properties of selectively absolutely star-Lindelöf spaces.

By the definitions above, we have immediately the following implications:

acc ⇒ selectively a-star-Lindelöf ⇒ a-star-Lindelöf ⇒ star-Lindelöf.

Examples

We will give examples provided that, in general, the converse of the previous implications does not hold.

Example 1.1.1. A selelctively a-star-Lindelöf space not acc.
The discrete space ω is Lindelöf and not countably compact, hence it is a selectively a-star-Lindelöf space which is not acc.

Example 1.1.2. A star-Lindelöf space not a-star-Lindelöf.
The space $\omega_1 \times (\omega_1 + 1)$ is countably compact, so it is star-Lindelöf. But it is not a-star-Lindelöf [25, p.82]. Indeed, consider the open cover $\mathcal{U} = \{\omega_1 \times \omega_1\} \cup \{[0, \alpha] \times [\alpha, \omega_1] : \alpha < \omega_1\}$ of $\omega_1 \times (\omega_1 + 1)$ and the dense subset D of all isolated points in $\omega_1 \times (\omega_1 + 1)$. Then, for any countable set $C \subset D$, we can easily see $St(C, \mathcal{U}) \neq \omega_1 \times (\omega_1 + 1)$.
Moreover, the space ω^{ω_1} gives another example of a star-Lindelöf space not a-star-Lindelöf. Indeed it is separable (hence star-Lindelöf), but not a-star-Lindelöf [25, Theorem 7.1].

Example 1.1.3. A T_2 non-regular a-star-Lindelöf space of cardinality $2^{\mathfrak{c}}$ which is not selectively a-star-Lindelöf.

Proof. We recall [25, Example 3.18]. Let $(\mathbb{I}^{\mathfrak{c}}, \mathcal{T})$ be the Cartesian product of $\mathfrak{c}$-many closed unit intervals $\mathbb{I}$, and $\mathcal{T}$ be the topology. Fix a countable dense subset D in $(\mathbb{I}^{\mathfrak{c}}, \mathcal{T})$. We define a new topology $\mathcal{T}'$ on $\mathbb{I}^{\mathfrak{c}}$ as follows: if $x \in D$, a basic open neighborhood of x is of the form $D \cap U$, where $x \in U \in \mathcal{T}$; if $x \in \mathbb{I}^{\mathfrak{c}} \setminus D$, a basic open neighborhood of x is of the form $\{x\} \cup (U \cap D)$, where $x \in U \in \mathcal{T}$. The space $X = (\mathbb{I}^{\mathfrak{c}}, \mathcal{T}')$ is T_2 non-regular, and not a-star-Lindelöf as noted in [25]. We see that X is not selectively a-star-Lindelöf. By [20, Proposition 2.3(3)], D is not selectively separable, so there is a sequence $(D_n : n \in \omega)$ of dense subsets in D such that for any finite sets $F_n \subset D_n$, $\overline{\bigcup\{F_n : n \in \omega\}}^D \neq D$. Enumerate all sequences $(F_n : n \in \omega)$ of finite sets such that $F_n \subset D_n$ as follows: $\mathbb{F} = \{(F_{\alpha,n} : n \in \omega) : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, we take a point $x_\alpha \in \mathbb{I}^{\mathfrak{c}} \setminus D$ and an open neighborhood U_α of x_α in X inductively. Let $\gamma < \mathfrak{c}$ and assume that points $\{x_\alpha : \alpha < \gamma\}$ are already taken. By the condition $\overline{\bigcup\{F_{\gamma,n} : n \in \omega\}}^D \neq D$, we can take some $U \in \mathcal{T}$ such that $U \cap (\bigcup\{F_{\gamma,n} : n \in \omega\}) = \emptyset$. We take a point $x_\gamma \in U \setminus (D \cup \{x_\alpha : \alpha < \gamma\})$ and let $U_\gamma = \{x_\gamma\} \cup (U \cap D)$. We define an open cover $\mathcal{U}$ of X as follows:

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{X \setminus Y\}, \text{ where } Y = \{x_\alpha : \alpha < \mathfrak{c}\}.$$

Let $(F_n : n \in \omega)$ be any sequence of finite sets $F_n \subset D_n$. Then $(F_n : n \in \omega) = (F_{\gamma,n} : n \in \omega)$ for some $\gamma < \mathfrak{c}$. Hence, we have $x_\gamma \notin St(\bigcup_{n\in\omega} F_n, \mathcal{U})$. □

In order to give an example of a Tychonoff a-star-Lindelöf space, not selectively a-star-Lindelöf, in view of Proposition 1.1.1 (see below), we have to find an a-star-Lindelöf space which is not of countable fan tightness with respect to dense subspaces. The following lemma follows from [19, Theorem 21, Proposition 25]. The symbol $\mathbb{D}$ is the space of the two points 0 and 1, and $C_p(X, \mathbb{D})$ is the space of all $\mathbb{D}$-valued continuous functions on X with the topology of pointwise convergence.

Lemma 1.1.1. Let X be a zero-dimensional separable metric space.
If $C_p(X, \mathbb{D})$ has countable fan tightness with respect to dense subspaces, X is Menger.

Recall the well known theorem of Cantor-Bendixon:

Theorem 1.1.1 (Cantor-Bendixon)**.** [39] Every second-countable space can be rapresented as the union of two disjoint sets, of which one is perfect and the other countable.

For functions $\varphi, \psi \in \omega^\omega$, $\varphi \leq \psi$ stands for $\varphi(n) \leq \psi(n)$ for all $n \in \omega$. The minimal cardinality of a cofinal subset in $(\omega^\omega, \leq)$ is denoted by $\mathfrak{d}$ [83].

Example 1.1.4. There exists a Tychonoff a-star-Lindelöf space of cardinality $\mathfrak{d}$ which is not selectively a-star-Lindelöf.

Proof. Let $\mathbb{P}$ be the space of irrationals. Take any non-Menger subspace $X \subset \mathbb{P}$ of cardinality $\mathfrak{d}$ (for instance, consider the Baire space ω^ω which is homeomorphic to $\mathbb{P}$, and take a cofinal subset of cardinality $\mathfrak{d}$. It is well known that any cofinal subset of ω^ω is not Menger). We may assume that X is dense-in-itself. Indeed, by the Cantor and Bendixson theorem, we can put $X = X_0 \cup X_1$, where X_0 is dense-in-itself and X_1 is countable. Then X_0 is a dense-in-itself non-Menger space of cardinality $\mathfrak{d}$. By Lemma 1.1.1, there are a sequence $(D_n : n \in \omega)$ of dense subsets of $C_p(X, \mathbb{D})$ and a point $f \in C_p(X, \mathbb{D})$ such that for any finite subsets $F_n \subset D_n$, $f \notin \overline{\bigcup\{F_n : n \in \omega\}}$. Since $C_p(X, \mathbb{D})$ is hereditarily separable and homogeneous, we can assume that each D_n is countable and $f = \mathbf{0}$, where $\mathbf{0}$ is the constant function with the value 0. Note that each D_n is dense also in $\mathbb{D}^X$, because $C_p(X, \mathbb{D})$ is dense in $\mathbb{D}^X$. For each $x \in X$, we define a function $f_x \in \mathbb{D}^X$ as follows: $f_x(y) = 1$ if $y = x$; $f_x(y) = 0$ if $y \neq x$. Since X is dense-in-itself, each f_x is discontinuous, so $\{f_x : x \in X\} \cap C_p(X, \mathbb{D}) = \emptyset$. Obviously, $\{f_x : x \in X\} \cup \{\mathbf{0}\}$ is homeomorphic to the one-point compactification of the discrete space of cardinality $\mathfrak{d}$. Enumerate D_n as $D_n = \{d_{n,m} : m \in \omega\}$. Let $\{\varphi_\gamma : \gamma < \mathfrak{d}\}$ be a cofinal subset of ω^ω. For each $\gamma < \mathfrak{d}$, we put $E_\gamma = \{d_{n,m} : n \in \omega, m \leq \varphi_\gamma(n)\}$. Fix a $\gamma < \mathfrak{d}$, and assume that points $\{x_\alpha : \alpha < \gamma\} \subset X$ are already taken. Note that the condition $\mathbf{0} \notin \overline{E}_\gamma$ implies $\overline{E}_\gamma \cap \{f_x : x \in X\}$ is finite. Hence, we can take a point $x_\gamma \in X \setminus \{x_\alpha : \alpha < \gamma\}$ such that $f_{x_\gamma} \notin \overline{E}_\gamma$. Let U_γ be an open neighborhood of f_{x_γ} in $\mathbb{D}^X$ such that $U_\gamma \cap E_\gamma = \emptyset$ and $U_\gamma \cap \{f_x : x \in X\} = \{f_{x_\gamma}\}$.
Now consider the subspace

$$Z = \{f_{x_\alpha} : \alpha < \mathfrak{d}\} \cup \left(\bigcup\{D_n : n \in \omega\}\right)$$

of $\mathbb{D}^X$. If D is a dense subset of Z, then $D \setminus \{f_{x_\alpha} : \alpha < \mathfrak{d}\}$ is a countable dense subset of Z. Therefore, Z is a-star-Lindelöf. We observe that Z is not selectively a-star-Lindelöf. Consider the open cover $\mathcal{U} = \{U_\alpha \cap Z : \alpha < \mathfrak{d}\} \cup \{\bigcup_{n\in\omega} D_n\}$ of Z and the sequence $\{D_n : n \in \omega\}$ of dense subsets of Z. If F_n is a finite subset of D_n for each $n \in \omega$, there is some $\gamma < \mathfrak{d}$ such that $\bigcup\{F_n : n \in \omega\} \subset E_\gamma$. Since U_γ is the only one member of $\mathcal{U}$ containing f_{x_γ}, we have $f_{x_\gamma} \notin St(E_\gamma, \mathcal{U})$, so $f_{x_\gamma} \notin St(\bigcup_{n\in\omega} F_n, \mathcal{U})$. Thus, Z is not selectively a-star-Lindelöf.

Additionally, we note that Z^ω is a-star-Lindelöf. Since Z is separable, so is Z^ω, hence Z^ω is star-Lindelöf. Since Z is locally countable, so is every finite power of Z. Therefore, Z^ω has countable tightness. Since a star-Lindelöf space of countable tightness is a-star-Lindelöf [25, Proposition 3.17], Z^ω is a-star-Lindelöf. $\square$

A sufficient condition

Recall that a space X has *countable fan tighness with respect to dense subspaces* [19] if for every $x \in X$ and every sequence $(D_n : n \in \omega)$ of dense subspaces of X, there are finite sets $F_n \subset D_n$ $(n \in \omega)$ such that $x \in \overline{\bigcup_{n\in\omega} F_n}$. We give a sufficient condition for a space to be selectively a-star-Lindelöf.

Proposition 1.1.1. Every star-Lindelöf space of countable fan tightness with respect to dense subspaces is selectively a-star-Lindelöf.

Proof. Let X be a star-Lindelöf space of countable fan tightness with respect to dense subspaces. Let $\mathcal{U}$ be an open cover of X and let $(D_n : n \in \omega)$ be a sequence of dense subsets of X. By star-Lindelöfness of X, there is a countable subset $C \subset X$ such that $St(C,\mathcal{U}) = X$. Using countable fan tightness with respect to dense subspaces, we can take finite sets $F_n \subset D_n$ with $C \subset \overline{\bigcup\{F_n : n \in \omega\}}$. This implies $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = X$. □

Note that, having countable fan tightness with respect to dense subspaces and selective a-star-Lindelöfness are indipendent properties. Indeed, a space having countable fan tightness with respect to dense subspaces need not be selectively a-star-Lindelöf: consider, for example, any uncountable discrete space. Also, $C_p(\omega^\omega)$ is Lindelöf, hence selectively a-star-Lindelöf, but it has not countable fan tightness with respect to dense subspaces.

Relation with selective separability

A space X is said to be *selectively separable* [20] if for any sequence $(D_n : n \in \omega)$ of dense subsets of X, there are finite sets $F_n \subset D_n$ $(n \in \omega)$ such that $\bigcup_{n\in\omega} F_n$ is dense in X. S. Bhowmik also showed that any selectively separable space is selectively a-star-Lindelöf and the converse is not true (see [23, Theorem 2.7 and Example 2.9]). We want to observe that another example of this fact is given by [20, Example 2.10]: the space $C_p(\omega^\omega)$ is Lindelöf hence selectively a-star-Lindelöf, but it is not selectively separable. We can easily see that every space with a countable π-base is selectively separable and a space is selectively separable if and only if it is separable and of countable fan tightness with respect to dense subspaces.

Hence, we have the following result.

Corollary 1.1.1. A selectively separable space is selectively a-star-Lindelöf. In particular, a space with a countable π-base is selectively a-star-Lindelöf.

Any Lindelöf non-separable space is a selectively a-star-Lindelöf space which is not selectively separable.

A characterization

Proposition 1.1.2. Every a-star-Lindelöf space of cardinality less than $\mathfrak{d}$ is selectively a-star-Lindelöf.

Proof. Let X be an a-star-Lindelöf space of cardinality less than $\mathfrak{d}$. Let $\mathcal{U}$ be an open cover of X and let $(D_n : n \in \omega)$ be a sequence of dense subsets of X. Using a-star-Lindelöfness, we take a countable set $C_n = \{c_{n,m} : m \in \omega\} \subset D_n$ such that $St(C_n, \mathcal{U}) = X$. For each $x \in X$ and $n \in \omega$, there are $U(x,n) \in \mathcal{U}$ and $m(x,n) \in \omega$ such that $\{x, c_{n,m(x,n)}\} \subset U(x,n)$. For each $x \in X$, define a function $\varphi_x \in \omega^\omega$ by $\varphi_x(n) = m(x,n)$. Since $\{\varphi_x : x \in X\}$ is not cofinal in $(\omega^\omega, \leq)$, there is a $\psi \in \omega^\omega$ such that for each $x \in X$, there is an $n_x \in \omega$ with $\psi(n_x) > \varphi_x(n_x)$. Let $F_n = \{c_{n,m} : m \leq \psi(n)\}$ for each $n \in \omega$. Then we have $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = X$. Indeed, if $x \in X$, then $\psi(n_x) > \varphi_x(n_x)$. This implies $U(x, n_x) \cap F_{n_x} \neq \emptyset$. □

By Example 1.1.4 and Proposition 1.1.2, we have the following result.

Corollary 1.1.2. The following assertions are equivalent:

(1) $\omega_1 < \mathfrak{d}$ holds;

(2) Every a-star-Lindelöf space of cardinality ω_1 is selectively a-star-Lindelöf.

Behaviour with respect to subspaces

Now we are going to analyze the behaviour of selective a-star Lindelöfness with respect to particular subspaces.

Selective a-star-Lindelöfness is not closed hereditary. Indeed, consider the Isbell-Mrówka space $\Psi = \omega \cup \mathcal{R}$, where $\mathcal{R}$ is an infinite maximal almost disjoint family of infinite subsets of ω. Since Ψ has a countable π-base, it is selectively a-star-Lindelöf (see Corollary 1.1.1), but $\mathcal{R}$ is an uncountable closed discrete subspace of Ψ. The space $\omega_1 \times (\omega_1 + 1)$ can be a regular closed subset of an acc space [64, Example 4.4]. Therefore, selective a-star-Lindelöfness is not hereditary even for a regular closed subset.

It is easy to see that selective-a-star-Lindelöfness is not hereditary also with respect to dense subspaces, consider a compactification βX of a space X wich is not selectively a-star-Lindelöf. Indeed, βX is compact and hence is selectively a-star-Lindelöf, X is dense in βX and it is not selectively a-star-Lindelöf.

Continuous images

In [25], it is proved that the perfect image and preimage of an acc space need not be a-star-Lindelöf. This implies that the perfect image and preimage of a selectively a-star-Lindelöf space need not be selectively a-star-Lindelöf.

Definition 1.1.2. [64] A map $f : X \to Y$ is *varpseudo-open* if $\text{Int}_Y(f(O)) \neq \emptyset$ for any non-empty open set $O \subset X$.

Easily we can check that a continuous map $f : X \to Y$ is varpseudo-open if and only if for every dense subspace D of Y, $f^{-1}(D)$ is dense in X, and a closed irreducible map is varpseudo-open. In [25] and [64], it is proved that acc property and a-star-Lindelöfness, respectively, are preserved by varpseudo-open maps.

Proposition 1.1.3. Selective a-star-Lindelöfness is preserved by varpseudo-open maps.

Proof. Let $f : X \to Y$ be a varpseudo-open onto map from a selectively a-star-Lindelöf space X. Let $\mathcal{U}$ be an open cover of Y and let $(D_n : n \in \omega)$ be a sequence of dense subspaces of Y. Then, $\mathcal{U}_0 = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X and, since f is varpseudo-open, $(f^{-1}(D_n) : n \in \omega)$ is a sequence of dense subspaces of X. Then there are finite sets $F_n \subset f^{-1}(D_n)$ $(n \in \omega)$ such that $St(\bigcup_{n\in\omega} F_n, \mathcal{U}_0) = X$. Hence, we have $St(\bigcup_{n\in\omega} f(F_n), \mathcal{U}) = Y$. □

Corollary 1.1.3. [23] Selective a-star-Lindelöfness is preserved by open maps.

Corollary 1.1.4. Selective a-star-Lindelöfness is preserved by closed irreducible maps.

Sum

The union of two star-Lindelöf spaces is star-Lindelöf. However, the union of two acc spaces need not be a-star-Lindelöf. As described in Example 1.1.1, $\omega_1 \times (\omega_1 + 1)$ is not a-star-Lindelöf. Let $Y = \omega_1 \times \{\omega_1\}$ and let $Z = \omega_1 \times \omega_1$, then $\omega_1 \times (\omega_1 + 1) = Y \cup Z$, Y is a closed acc subspace of X and Z is an open dense acc subspace of X.

In this Section, we examine when the union of two selectively a-star-Lindelöf spaces is selectively a-star-Lindelöf.

Definition 1.1.3. A subspace Y of a space X is *selectively a-star-Lindelöf in* X if for any open cover $\mathcal{U}$ of X and any sequence $(D_n : n \in \omega)$ of dense subsets in X there are finite sets $F_n \subset D_n$ such that $Y \subset St(\bigcup_{n\in\omega} F_n, \mathcal{U})$.

Lemma 1.1.2. Let $Y \subset X$. If Y is Lindelöf, or regular closed in X and selectively a-star-Lindelöf, then Y is selectively a-star-Lindelöf in X.

Proof. Let $\mathcal{U}$ be an open cover of X and let $(D_n : n \in \omega)$ be a sequence of dense subsets of X. If Y is Lindelöf, take a countable family $\{U_n : n \in \omega\} \subset \mathcal{U}$ covering Y, and take points $x_n \in U_n \cap D_n$. Then, we have $Y \subset St(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$. If Y is regular closed in X and selectively a-star-Lindelöf, consider the open cover $\{Y \cap U : U \in \mathcal{U}\}$ of Y and dense subsets $(\text{Int}_X Y) \cap D_n$ in Y, and apply selective a-star-Lindelöfness of Y to them. □

Theorem 1.1.2. Let $X = Y \cup Z$. If Y is closed and selectively a-star-Lindelöf in X and Z is selectively a-star-Lindelöf, then X is selectively a-star-Lindelöf.

Proof. Let $\mathcal{U}$ be an open cover of X and let $(D_n : n \in \omega)$ be a sequence of dense subsets of X. Since Y is selectively a-star-Lindelöf in X, there are finite sets $F_n \subset D_n$ such that $Y \subset St(\bigcup_{n\in\omega} F_n, \mathcal{U})$. We put $V = St(\bigcup_{n\in\omega} F_n, \mathcal{U})$. For each point $x \in Z \setminus V$, take a member $U_x \in \mathcal{U}$ with $x \in U_x$. We put

$$\mathcal{C} = \{Z \cap V\} \cup \{U_x \setminus Y : x \in Z \setminus V\},$$

and

$$E_n = (D_n \setminus Y) \cup (Y \cap Z) \text{ for all } n \in \omega.$$

Since Y is closed in X, $\mathcal{C}$ is an open cover of Z and each E_n is a dense subset of Z. Hence, there are finite sets $G_n \subset E_n$ such that $Z = St(\bigcup_{n\in\omega} G_n, \mathcal{C})$. Then, we can see $X \setminus V \subset St(\bigcup_{n\in\omega}(G_n \setminus Y), \mathcal{U})$. Hence, if we put $H_n = F_n \cup (G_n \setminus Y)$, we have $H_n \subset D_n$ and $X = St(\bigcup_{n\in\omega} H_n, \mathcal{U})$. □

By Lemma 1.1.2 and Theorem 1.1.2, we have the following result.

Corollary 1.1.5. Let $X = Y \cup Z$ and let Z be selectively a-star-Lindelöf. If Y is closed and Lindelöf, or regular closed in X and selectively a-star-Lindelöf, then X is selectively a-star-Lindelöf.

Question 1.1.1. Let $X = Y \cup Z$. If both Y and Z are closed and selectively a-star-Lindelöf , then is X selectively a-star-Lindelöf?

Product

The product of a T_2 acc space with a first-countable compact space is acc [64, Theorem 2.3], and the product of a star-Lindelöf space with a separable compact space is star-Lindelöf [84, p.100]. In particular, if X is a T_2 acc (respectively, star-Lindelöf) space, then $X \times (\omega + 1)$ is also acc (respectively, star-Lindelöf).

As described in the first Section, ω_1 is acc (hence, selectively a-star-Lindelöf). However, $\omega_1 \times (\omega_1 + 1)$ is not a-star-Lindelöf: see Example 1.1.1. Therefore, the product of a selectively a-star-Lindelöf space with a compact space need not be a-star-Lindelöf, where this fact answers in the negative to Problem 2.14 in [23]. In fact, we may prove even more, as the following example shows.

Example 1.1.5. The product of a selectively a-star-Lindelöf space with a compact first-countable space which is not a star-Lindelöf space.

Proof. Recall the selectively a-star-Lindelöf space $\Psi = \omega \cup \mathcal{R}$ (*i.e.*, the Isbell-Mrówka space) described in the first Section. Let $|\mathcal{R}| = \mathfrak{c}$. Let $Y = \mathbb{I} \times \mathbb{D}$ be the Alexandroff duplicate [86] of the unit interval $\mathbb{I}$. Obviously, Y is compact and first-countable. Note that the quotient space obtained by identifying all points in $\mathbb{I} \times \{0\}$ is homeomorphic to the one-point compactification $A(\mathfrak{c})$ of the discrete space of cardinality $\mathfrak{c}$. It is known [84, Example 3.3.4] that $\Psi \times A(\mathfrak{c})$ is not star-Lindelöf. Since every continuous image of a star-Lindelöf space is star-Lindelöf, $\Psi \times Y$ is not star-Lindelöf. $\square$

For a (selectively) a-star-Lindelöf space X, we do not know if $X \times (\omega+1)$ is (selectively) a-star-Lindelöf. In this Section, we show that if a space X is countably metacompact and a-star-Lindelöf, $X \times (\omega + 1)$ is a-star-Lindelöf. Unfortunately we know nothing about selective a-star-Lindelöfness of $X \times (\omega + 1)$.

Recall that in [52], F. Ishikawa proved the following theorem.

Theorem 1.1.3. [52] A space X is countably paracompact if and only if for every decreasing sequence $\{F_i\}$ of non-empty closed sets F_i with vacuous intersection there exists a decreasing sequence G_i of open sets such that their closures $\overline{G_i}$ have vacuous intersection and $G_i \supset F_i$.

In [52] it was given, without proof, the following result. For sake of completeness, we give the proof of it.

Corollary 1.1.6. A space X is countably metacompact if and only if for every decreasing sequence $\{F_i\}$ of non-empty closed sets F_i with vacuous intersection there exists a decreasing sequence G_i of open sets with vacuous intersection such that $G_i \supset F_i$.

Proof. Let X be countably metacompact and let $\{F_i\}$ be a decreasing sequence of closed subsets of X, then $\{X \setminus F_i\}$ is a countable open cover of X, therefore it has a point finite refinement $\mathcal{B}$. For each open set W of $\mathcal{B}$ let $g(W)$ be the first $X \setminus F_i$ containing W, and let V_i be the union of all W for which $g(W) = X \setminus F_i$. Then V_i is open and $V_i \subset X \setminus F_i$, $\{V_i\}$ is a point finite cover of X. Put: $G_i = \bigcup_{n=i+1}^{\infty} V_n$. Then G_i is open, $G_i \supset X \setminus (V_1 + ... + V_i) \supset X \setminus (X \setminus F_i) = F_i$, hence $G_i \supset F_i$. For every point $x \in X$, x belongs only to a finite number of V_i. Then there exists an i such that $x \notin \bigcup_{n=i+1}^{\infty} V_n$, that is, $x \notin G_i$. Hence $\bigcap G_i = \emptyset$.

Conversely, let X be a space and let $\{U_i\}_{i\in\omega}$ a countable open cover of X. Put $F_i = X \setminus \bigcup_{n=1}^{i} U_n$. Then, $\{F_i\}_{i\in\omega}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Therefore, for the hypothesis, there exist open sets G_i such that $G_i \supset F_i$, $G_1 \supset G_2, \ldots$ and $\bigcap_{i\in\omega} G_i = \emptyset$.

Put $X \setminus G_i = E_i$. Then E_i is obviously closed and $E_i \subset \bigcup_{n=1}^{i} U_n$. Finally put: $V_i = U_i \setminus E_{i-1}$. Then V_i is clearly open and $V_i \subset U_i$. Moreover, since $V_i = U_i \setminus E_{i-1} \supset U_i \setminus \bigcup_{n=1}^{i-1} U_n$, we have $\bigcup_{i=1}^{\infty} V_i \supset \bigcup_{i=1}^{\infty}(U_i \setminus \bigcup_{n=1}^{i-1} U_n) = \bigcup_{i=1}^{\infty} U_i = X$, thus $\{V_i\}$ is a refinement of $\{U_i\}$. Finally, for every $x \in X$ let i be the first index such that $x \notin G_i$. Then $x \in X \setminus G_i = E_i$. But $x \in E_i$ and then $x \notin V_{i+j}$ for every $j = 1, 2, \ldots$. Therefore $\{V_i\}$ is point finite. □

The previous corollary is equivalent to the following condition.

Corollary 1.1.7. A space X is countably metacompact if and only if for any increasing open cover $\{U_n : n \in \omega\}$ of X, there is an increasing cover $\{A_n : n \in \omega\}$ of X consisting of closed subsets in X such that $A_n \subset U_n$ for all $n \in \omega$.

Theorem 1.1.4. For a space X, the following assertions are equivalent.

(1) $X \times (\omega + 1)$ is a-star-Lindelöf;

(2) For any open cover $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ of X with $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ and any sequence $(D_n : n \in \omega)$ of dense subsets in X, there are countable sets $C_n \subset D_n$ $(n \in \omega)$ such that $\bigcup\{St(C_n, \mathcal{U}_n) : n \in \omega\} = X$.

Proof. (1)→(2): Let $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ be an open cover of X with $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ and let $(D_n : n \in \omega)$ be a sequence of dense subsets in X. For each $n \in \omega$, we put

$$\mathcal{P}_n = \{U \times [n, \omega] : U \in \mathcal{U}_n\},$$

and

$$\mathcal{P} = \{X \times \omega\} \cup \bigcup\{\mathcal{P}_n : n \in \omega\}.$$

Then $\mathcal{P}$ is an open cover of $X \times (\omega+1)$. Applying a-star-Lindelöfness of $X \times (\omega+1)$ to the open cover $\mathcal{P}$ and the dense subset $D = \bigcup\{D_n \times \{n\} : n \in \omega\}$ in $X \times (\omega+1)$, we have countable sets $C_n \subset D_n$ such that $St(\bigcup_{n\in\omega}(C_n \times \{n\}), \mathcal{P}) = X \times (\omega + 1)$. We see $\bigcup\{St(C_n, \mathcal{U}_n) : n \in \omega\} = X$. Let $x \in X$. By $(x, \omega) \in St(\bigcup_{n\in\omega}(C_n \times \{n\}), \mathcal{P})$, there are some $n, m \in \omega$ and a member $U \in \mathcal{U}_n$ such that $(x, \omega) \in U \times [n, \omega] \in \mathcal{P}_n$ and $(U \times [n, \omega]) \cap (C_m \times \{m\}) \neq \emptyset$ (in particular, $n \leq m$). Hence, $x \in U \in \mathcal{U}_n \subset \mathcal{U}_m$ and $U \cap C_m \neq \emptyset$, so $x \in St(C_m, \mathcal{U}_m)$.

(2)→(1): Let $\mathcal{V}$ be an open cover of $X \times (\omega + 1)$ and let D be a dense subset in $X \times (\omega+1)$. Without loss of generality, we may assume $D \subset X \times \omega$, because $X \times \omega$ is open and dense in $X \times (\omega + 1)$. Let

$$D_n = \{x \in X : (x, n) \in D\},$$
$$\mathcal{U}_n = \{U : U \text{ is open in } X, U \times [n, \omega] \subset V \text{ for some } V \in \mathcal{V}\},$$

and

$$\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n.$$

Then each D_n is dense in X and $\mathcal{U}$ is an open cover of X with $\mathcal{U}_n \subset \mathcal{U}_{n+1}$. Hence, there are countable sets $C_n \subset D_n$ such that $\bigcup\{St(C_n, \mathcal{U}_n) : n \in \omega\} = X$. Let $C = \bigcup\{C_n \times \{n\} : n \in \omega\}$. We show $X \times \{\omega\} \subset St(C, \mathcal{V})$. Let $x \in X$. Then, there are some $n \in \omega$ and $U \in \mathcal{U}_n$ such that $x \in U$ and $U \cap C_n \neq \emptyset$. The condition $U \in \mathcal{U}_n$ means $U \times [n, \omega] \subset V$ for some $V \in \mathcal{V}$, and we have $V \cap C \supset (U \times [n, \omega]) \cap (C_n \times \{n\}) = (U \cap C_n) \times \{n\} \neq \emptyset$. Thus $(x, \omega) \in St(C, \mathcal{V})$. Note that our assumption (2) implies that X is a-star-Lindelöf, indeed consider the case that all $\mathcal{U}_n$ are identical and all D_n are identical. Hence, there are countable sets $C'_n \subset D_n$ such that $X \times \{n\} \subset St(C'_n \times \{n\}, \mathcal{V})$. Let $C' = \bigcup\{C'_n \times \{n\} : n \in \omega\}$. Finally $C \cup C'$ is a countable subset of D and we have $St(C \cup C', \mathcal{V}) = X \times (\omega + 1)$. □

Corollary 1.1.8. If a space X is countably metacompact and a-star-Lindelöf, then $X \times (\omega + 1)$ is a-star-Lindelöf.

Proof. We examine the condition (2) in Theorem 1.1.4. Let $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ be an open cover of X with $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ and let $(D_n : n \in \omega)$ be a sequence of dense subsets in X. Applying countable metacompactness of X to the increasing open cover $\{\bigcup \mathcal{U}_n : n \in \omega\}$ of X, we have an increasing cover $\{A_n : n \in \omega\}$ of X consisting of closed subsets in X such that $A_n \subset \bigcup \mathcal{U}_n$ for all $n \in \omega$. We define an open cover $\mathcal{V}_n$ of X as follows: $\mathcal{V}_n = \{X \setminus A_n\} \cup \mathcal{U}_n$. Applying a-star-Lindelöfness of X to $\mathcal{V}_n$ and D_n, we have a countable set $C_n \subset D_n$ such that $St(C_n, \mathcal{V}_n) = X$. We see $\bigcup\{St(C_n, \mathcal{U}_n) : n \in \omega\} = X$. Let $x \in X$. Then $x \in A_n$ for some $n \in \omega$, and for this n, $x \in St(C_n, \mathcal{V}_n)$. This means that there is some $U \in \mathcal{U}_n$ such that $x \in U$ and $U \cap C_n \neq \emptyset$. Thus we have $x \in St(C_n, \mathcal{U}_n)$. □

An acc space is, of course, countably metacompact. However not every selectively a-star-Lindelöf space is countably metacompact. Since a space with a countable π-base is selectively a-star-Lindelöf, we have only to find a space with a countable π-base which is not countably metacompact. The space $\mathbb{N}^{\omega_1}$ is nowhere dense and closed in $\mathbb{R}^{\omega_1}$, and it is not countably metacompact. Take a countable set $D \subset \mathbb{R}^{\omega_1} \setminus \mathbb{N}^{\omega_1}$ which is dense in $\mathbb{R}^{\omega_1}$. Consider the subspace $D \cup \mathbb{N}^{\omega_1} \subset \mathbb{R}^{\omega_1}$, and let X be the space obtained by isolating all points of D in $D \cup \mathbb{N}^{\omega_1}$. Then X is a Tychonoff space we need.

1.1.2 Selective strongly star Menger property

Recall that a space X is said to be *strongly star-Menger* [33, 57] (briefly *SSM*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there exists a sequence $(F_n : n \in \omega)$ of finite subsets of X such that $\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n) = X$.

We introduce the following property.

Definition 1.1.4. A space X is *selectively strongly star-Menger* (briefly *selSSM*) if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X and each sequence

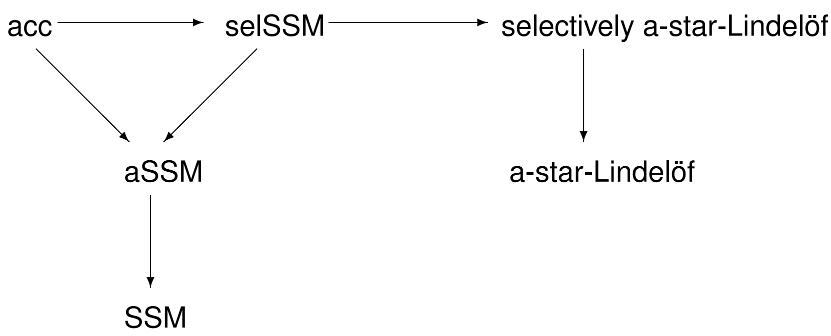
$(D_n : n \in \omega)$ of dense subspaces of X, there exists a sequence $(F_n : n \in \omega)$ of finite subsets $F_n \subset D_n,\ n \in \omega$, such that $\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n) = X$.

This property is between acc and selectively a-star-Lindelöfness and represents a selective version of *SSM* property.

Note that another selective version of *SSM* was given by A. Caserta, G. Di Maio and L. Kočinac.

Definition 1.1.5. [35] A space X is *absolutely strongly star-Menger* (briefly *aSSM*) if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X and each dense subspace D of X, there exists a sequence $(F_n : n \in \omega)$ of finite subsets of D such that $\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n) = X$.

The implications in the following diagram are easy to see:



Examples

In the diagram above, almost all the implications which involve *selSSM*, in general, are not invertible, as we will see with the following examples.

Example 1.1.6. A selectively a-star-Lindelöf space which is not *SSM* (hence not *aSSM* neither *selSSM*).

Proof. Let $X = \Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ be the Isbell-Mrwóka space, where $\mathcal{A}$ is the maximal almost disjoint family of infinite subsets of ω with $|\mathcal{A}| = \mathfrak{c}$. By Proposition 2 in [31] we have that X is not *SSM*. On the other hand, since X has a countable π-base, it is selectively a-star-Lindelöf ([29]). □

Example 1.1.7. A *selSSM* space which is not acc.

Proof. The countable discrete space is a *selSSM* not cc, hence not acc, space. □

The following problem remains open.

Main Problem Is there an *aSSM* not *selSSM* space?

Relations with Menger-type properties

The *selSSM* property is a weaker form of Menger property.

Property 1.1.1. Every Menger space is *selSSM*.

Proof. Let X be a Menger space and let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open cover of X and $(D_n : n \in \omega)$ be a sequence of dense subspaces of X. Then there are finite families $\mathcal{V}_n \subset \mathcal{U}_n,\ n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{V}_n$ is a cover for X. Put $\mathcal{V}_n = \{V_{n_1}, V_{n_2}, ..., V_{n_k}\}$ and $F_n = \{x_{n_i} : i = 1, 2, ..., k\} \subset D_n$, where $x_{n_i} \in V_{n_i} \cap D_n,\ \forall i = 1, 2, ..., k$. We want to show that $X = \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)$. Let $x \in X$ and $\mathcal{V}_n \subset \mathcal{U}_n$ such that $x \in \bigcup \mathcal{V}_n$. Since $\bigcup \mathcal{V}_n \cap F_n \neq \emptyset$, *a fortiori* there exists an $U_n \in \mathcal{U}_n$ such that $U_n \cap F_n \neq \emptyset$, and so $x \in St(F_n, \mathcal{U}_n)$. □

The converse of Property 1.1.1 is not true, as confirmed in the next example.

Example 1.1.8. A *selSSM* not Menger space.

Proof. $X = \omega_1$ is a cc space which has countable tightness. So, X is an acc space [64, Theorem 1.8], then *selSSM*. Moreover, it is not Menger since is not Lindelöf. □

Also recall that [57] a space X is *star-Menger* (briefly *SM*) provided for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \omega)$ such that for every $n \in \omega,\ \mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

We have that:

$$M \longrightarrow selSSM \longrightarrow aSSM \longrightarrow SSM \longrightarrow SM.$$

Since in the class of paracompact Hausdorff spaces, *M* and *SM* are equivalent properties (see [57]) we have that in the class of paracompact Hausdorff spaces all the previous Menger-type properties are equivalent.

Some easy results and questions

It can be proved the following property.

Property 1.1.2. Every *selSSM* space has countable fan tightness with respect to dense subspaces.

Proof. Let (X, τ) be a topological space, $(D_n : n \in \omega)$ be a sequence of dense subspaces of X and $x \in X$. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X. Put $\mathcal{U}_n = \tau$, $\forall n \in \omega$.

Since X is *selSSM*, there exist $(F_n : n \in \omega)$ where, for every $n \in \omega,\ F_n$ represents a finite subset of D_n such that

$$\bigcup_{n \in \omega} St(F_n, \mathcal{U}_n) = X \quad (*).$$

We want to prove that $x \in \overline{\bigcup_{n \in \omega} F_n}$, that is $U \cap (\bigcup_{n \in \omega} F_n) \neq \emptyset, \forall U \in \tau$.

From $(*)$ it is clear that there exists $n \in \omega$ such that $x \in St(F_n, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : U \cap F_n \neq \emptyset\}$. So we are done. □

Taking into account Proposition 1.1.1, it is natural to pose the following question.

Question 1.1.2. Is every *SSM* space having countable fan tightness with respect to dense subspaces a *selSSM* space?

Now we define the following technical notion which represents a selective version of property (a).

Definition 1.1.6. A space X has the *selective strong property (a)* provided for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X and every sequence $(D_n : n \in \omega)$ of dense subspaces of X there exist $F_n \subset D_n$ such that F_n are closed and discrete in X for every $n \in \omega$ and $\bigcup_{n \in \omega} St(F_n, \mathcal{U}_n) = X$.

It turns out that:

Proposition 1.1.4. Every cc space with the selective strong property (a) is *selSSM*.

Remark 1.1.1. In [35] the authors called a space X *selectively (a)* if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X and every dense subspace D of X there exist $F_n \subset D$ such that F_n are closed and discrete in X for every $n \in \omega$ and $\bigcup_{n \in \omega} St(F_n, \mathcal{U}_n) = X$. It is easy to see that selective strong property (a) implies selective property (a), but we know nothing about the converse.

Relation with selective separability

The next two examples show that selective separability and *selSSM* are indipendent properties.

Example 1.1.9. A *selSSM* space which is not selectively separable.

Proof. The space of Example 1.1.8 gives a *selSSM* not separable (hence not selectively separable) space. □

Example 1.1.10. A selectively separable space which is not *selSSM*.

Proof. The space X of Example 1.1.6 is a selectively separable space because it has a countable π-base (see [20, Proposition 2.3]). □

Remark 1.1.2. In order to give an answer to the main problem, we need a separable not selectively separable space (this will give a not *selSSM* space) which is not paracompact.

Behaviour with respect to subspaces

We can investigate the behaviour of *selSSM* with respect to particular subspaces. In [81] it was proved that the property *aSSM* is not hereditary with respect to G_δ subspaces and with respect to regular-closed subspaces. Since the space that the author considered is acc, we may conclude that *selSSM* is not a hereditary property with respect to G_δ subspaces and with respect to regular-closed subspaces.

Moreover, *selSSM* is not hereditary also with respect to dense subspaces. In fact, consider the one-point compactification $\alpha(X)$ where X is the space of Example 1.1.6.

Continuous images

In [81, Example 3.11] it was proved that the continuous image of an acc (therefore *selSSM*) space need not be *aSSM*. So we can say that the continuous image of a *selSSM* space need not be *selSSM*. This allows us to study the behaviour of *selSSM* respect to varpseudo-open maps.

Proposition 1.1.5. Let X and Y be topological spaces and $f: \ X \to Y$ be a varpseudo-open and surjective map. If X is a *selSSM* space then Y is a *selSSM* space.

Proof. Let $(\mathcal{U}_n : \ n \in \omega)$ a sequence of open covers of Y and $(D_n : \ n \in \omega)$ a sequence of dense subspaces of Y. For every $n \in \omega$, let $\mathcal{V}_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. So $\{\mathcal{V}_n : n \in \omega\}$ is a sequence of open covers of X. Being f varpseudo-open, $f^{-1}(D_n) = D_n^{'}$ is a dense subspace of X.

Since X is *selSSM*, there exist finite subsets E_n of $D_n^{'}$ $\forall n \in \omega$, such that $X = \bigcup_{n\in\omega} St(E_n, \mathcal{V}_n)$. Put $F_n = f(E_n)$, $\forall n \in \omega$ where F_n is a finite subset of D_n, $\forall n \in \omega$. We have that

$$Y = f(X) = f\left(\bigcup_{n\in\omega} St(E_n, \mathcal{V}_n)\right) = \bigcup_{n\in\omega} f\left(St(E_n, \mathcal{V}_n)\right) = \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n),$$

and so we are done.

□

As a consequence of the proposition above, we have the following result.

Proposition 1.1.6. Let X and Y be topological spaces. If $X \times Y$ is *selSSM* then X and Y are *selSSM*.

The converse of the previous proposition does not hold. In fact, in [25] it was showed that the product $\omega_1 \times (\omega_1 + 1)$ is not *a-star-Lindelöf*.

In [64], it was proved that the product of a Hausdorff acc space with a first countable compact space is acc. It is natural to pose the following question.

Question 1.1.3. Is the product of a *selSSM* space with a compact first countable space a *selSSM* space?

Sum

In this Section, we examine when the union of two *selSSM* spaces is *selSSM*. We will follow the same idea in Theorem 1.1.2.

Definition 1.1.7. A subspace Y of a space X is *selectively strongly star-Menger in* X if for every sequence of open cover $(\mathcal{U}_n : n \in \omega)$ of X and every sequence $(D_n : n \in \omega)$ of dense subsets in X, there are finite sets $F_n \subset D_n$ such that $Y \subset \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)$.

Theorem 1.1.5. Let $X = Y \cup Z$. If Y is closed and selectively strongly star-Menger in X and Z is *selSSM*, then X is *selSSM*.

Proof. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open cover of X and let $(D_n : n \in \omega)$ be a sequence of dense subsets of X. Since Y is *selectively strongly star-Menger* in X, there are finite sets $F_n \subset D_n$ such that $Y \subset \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)$. We put $V = \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)$. For each point $x \in Z \setminus V$, take a member $U_{n_x} \in \mathcal{U}_n$ with $x \in U_{n_x}$. We put

$$\mathcal{C}_n = \{Z \cap V\} \cup \{U_{n_x} \setminus Y : x \in Z \setminus V\},$$

and

$$E_n = (D_n \setminus Y) \cup (Y \cap Z) \text{ for all } n \in \omega.$$

Since Y is closed in X, $\mathcal{C}_n$ is an open cover of Z for every $n \in \omega$ and each E_n is a dense subset of Z. Hence, there are finite sets $G_n \subset E_n$ such that $Z = \bigcup_{n\in\omega} St(G_n, \mathcal{C}_n)$. Then, we can see $X \setminus V \subset \bigcup_{n\in\omega} St(G_n \setminus Y, \mathcal{U}_n)$. Hence, if we put $H_n = F_n \cup (G_n \setminus Y)$, we have $H_n \subset D_n$ and $X = \bigcup_{n\in\omega} St(H_n, \mathcal{U}_n)$. $\square$

1.2 Monotone normality and related properties

A characterization of monotone normality in terms of funcions

Recall that, by Urysohn's Lemma, X is normal if and only if for every pair of disjoint non-empty closed sets F and H there is an $f \in C(X, I)$ such that $f(x) = 0$ for every $x \in F$ and $f(x) = 1$ for every $x \in H$.

The following theorem gives a characterization of monotone normality in terms of monotone functions.

Theorem 1.2.1. A space X is monotonically normal if and only if one can assign to every pair of disjoint non-empty closed sets F and H a function $f_{F,H} \in C(X, I)$ so that

(1) $f_{F,H}(x) = 0$ for every $x \in F$ and $f_{F,H}(x) = 1$ for every $x \in H$;

(2) $f_{F_2,H_2} \leq f_{F_1,H_1}$ whenever $F_1 \subset F_2$ and $H_1 \supset H_2$.

Proof. Suppose that the function $f_{F,H}$ such as in the theorem is given. Define

$$r(F, H) = f^{-1}_{F,H}([0, 1/2)).$$

Assume that we have two pairs of disjoint closed sets F_1, H_1 and F_2, H_2 such that $F_1 \subset F_2$ and $H_2 \subset H_1$. Then if $p \in r(F_1, H_1)$ we have $f_{F_1,H_1}(p) < 1/2$. It means that $f_{F_2,H_2}(p) \leq f_{F_1,H_1}(p) < 1/2$, *i.e.*, $p \in r(F_2, H_2)$.

For the other one, fix an enumeration $\{q_n : n \in \mathbb{N}\}$ of $Q = \mathbb{Q} \cap [0, 1]$; we assume that all q_n are different and moreover that $q_1 = 0$ and $q_2 = 1$. Let F and H be arbitrary disjoint closed subsets of X. As in the standard proof of Urysohn's Lemma, we will construct for every $n \in \mathbb{N}$ an open subset V_{q_n} of X such that

1. $\overline{V}_{q_n} \subset V_{q_m}$ whenever $q_n < q_m$,

2. $F \subset V_{q_1}$ and $H \subset X \setminus V_{q_1}$.

We put $V_{q_1} = r(F, H)$ and $V_{q_2} = X \setminus H$. Then (1) and (2) are clearly satisfied. Let $n > 2$ and assume that V_{q_i} are defined for all $i < n$. Put

$$r = \max\{q_i : i < n, q_i < q_n\} \text{ and } s = \min\{q_i : i < n, q_n < q_i\}.$$

Then, by our inductive hypothesis we have $\overline{V}_r \subseteq V_s$. Put $V_{q_n} = r(\overline{V}_r, X \setminus V_s)$. Then, our inductive hypotheses are clearly satisfied. We now define, as in the proof of Urysohn's Lemma, the function $f_{F,H}$ by the formula:

$$f_{F,H}(x) = \begin{cases} \inf\{q_n \in Q : x \in V_{q_n}\} & \text{if } x \in V_{q_2}, \\ 1 & \text{if } x \notin V_{q_2}. \end{cases}$$

Furthermore, $f_{F,H}$ is continuous.

To prove that this assignment of functions is 'monotone', consider two pairs of disjoint closed sets F_1, H_1 and F_2, H_2 such that $F_1 \subset F_2$ and $H_2 \subset H_1$. For F_1, H_1 we use the above notation V_{q_n}, and for F_2, H_2 we use the notation W_{q_n}. Observe that $V_{q_1} = r(F_1, H_1)$ and $V_{q_2} = X \setminus H_1$. Moreover, $W_{q_1} = r(F_2, H_2)$ and $W_{q_2} = X \setminus H_2$. Observe that

$$V_{q_1} \subset W_{q_1} \text{ and } V_{q_2} \subset W_{q_2}.$$

Claim. For every $n \in \mathbb{N}$, $V_{q_n} \subset W_{q_n}$.

For $n = 1, 2$ there is nothing to prove. Let $n > 2$ and assume that we have what we want for all $i < n$. In the above inductive construction, we put

$$r = \max\{q_i : i < n, q_i < q_n\} \text{ and } s = \min\{q_i : i < n, q_n < q_i\},$$

and $V_{q_n} = r(\overline{V}_r, X \setminus V_s)$ and $W_{q_n} = r(\overline{W}_r, X \setminus W_s)$. By our inductive hypothesis we have

$$V_r \subset W_r \text{ and } V_s \subset W_s,$$

hence

$$\overline{V}_r \subset \overline{W}_r \text{ and } X \setminus W_s \subset X \setminus V_r,$$

and so

$$V_{q_n} = r(\overline{V}_r, X \setminus V_s) \subset r(\overline{W}_r, X \setminus W_s) = W_{q_n}.$$

This completes the proof of the claim.

Now assume that $x \in X$. We want to prove that $f_{F_2,H_2}(x) \leq f_{F_1,H_1}(x)$. Assume first that $f_{F_1,H_1}(x) < 1$. If $x \notin V_{q_2}$, then $f_{F_1,H_1}(x) = 1$, which is impossible. Hence $x \in V_{q_2}$ and so $x \in W_{q_2}$. For every q_n such that $x \in V_{q_n}$ we also have that $x \in W_{q_n}$. From this we see that

$$\{q_n \in Q : x \in V_{q_n}\} \subset \{q_n \in Q : x \in W_{q_n}\},$$

and so

$$f_{F_2,H_2}(x) = \inf\{q_n \in Q : x \in W_{q_n}\} \leq \inf\{q_n \in Q : x \in V_{q_n}\} = f_{F_1,H_1}(x).$$

If $f_{F_1,H_1}(x) = 1$, then there is nothing to prove and so we are done. □

Monotone star-normality

As we already noted, there are a lot of properties that can be defined or characterized in terms of stars. The following theorem gives a star characterization of normality.

Theorem 1.2.2. [39] The following conditions are equivalent for a topological space X

(1) X is normal;

(2) Every two-element open cover of X has an open star-refinement;

(3) Every finite open cover of X has a finite open star-refinement.

In order to give a monotone version of the previous theorem, we define monotone versions of (2) and (3) in Theorem 1.2.2:

Definition 1.2.1. A space X is *2-monotonically star-normal* if there exists an operator that assigns to every two-element open cover $\mathcal{U}$ an open star refinement $r(\mathcal{U})$ so that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. The function r is called *2-monotone star-normality operator* for X.

Definition 1.2.2. A space X is *finitely-monotonically star-normal* if there exists an operator that assigns to every finite open cover $\mathcal{U}$ a finite open star refinement $r(\mathcal{U})$ so that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. The function r is called *finite-monotone star-normality operator* for X.

Note that both the previous definitions are weak form of monotone paracompactness.

For a cover $\mathcal{A}$ of a set X, let $\mathcal{A}^b = \{St(x, \mathcal{A}) : x \in X\}$. Introduce the following useful definition.

Definition 1.2.3. A space X satisfies property $(*)$ if for each binary open cover $\mathcal{U}$ of X there is an open cover $r(\mathcal{U})$ of X such that $r(\mathcal{U})^b$ refines $\mathcal{U}$ (*i.e.*, $r(\mathcal{U})$ is a barycentric open refinement of $\mathcal{U}$) and $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$.

The following fact easily follows from Definition 1.2.3.

Proposition 1.2.1. A 2-monotonically star-normal space has property $(*)$.

Proposition 1.2.2. A space with property $(*)$ is monotonically normal.

Proof. For a point $x \in X$ and an open neighborhood U of x, consider the binary open cover $\mathcal{U}(x, U) = \{U, X \setminus \{x\}\}$. Let $H(x, U) = St(x, r(\mathcal{U}(x, U)))$. Obviously $x \in H(x, U) \subset U$. Let U be an open neighborhood of $x \in X$ and let V be an open neighborhood of $y \in X$. Assume $y \notin U$ and $x \notin V$. We show $H(x, U) \cap H(y, V) = \emptyset$. Consider the binary open cover $\mathcal{W} = \{X \setminus \{x\}, X \setminus \{y\}\}$. Since both $\mathcal{U}(x, U)$ and $\mathcal{U}(y, V)$ are refinements of $\mathcal{W}$, both $r(\mathcal{U}(x, U))$ and $r(\mathcal{U}(y, V))$ are refinements of $r(\mathcal{W})$. Hence, we have $H(x, U) \cap H(y, V) \subset St(x, r(\mathcal{W})) \cap St(y, r(\mathcal{W}))$. Assume that there is a point $z \in St(x, r(\mathcal{W})) \cap St(y, r(\mathcal{W}))$. Then there are some $W_0, W_1 \in r(\mathcal{W})$ such that $\{x, z\} \subset W_0$ and $\{y, z\} \subset W_1$. Since $r(\mathcal{W})$ is a barycentric refinement of $\mathcal{W}$, $St(z, r(\mathcal{W}))$ is contained in $X \setminus \{x\}$, or $X \setminus \{y\}$. This is a contradiction, because $\{x, y\} \subset St(z, r(\mathcal{W}))$. □

Corollary 1.2.1. A 2-monotonically star-normal space is monotonically normal.

Question 1.2.1. Does monotone normality imply property $(*)$?

By propositions 1.2.1 and 1.2.2, a negative answer to Question 1.2.1 permits to prove that monotone normality and 2-monotone star-normality are not equivalent conditions.

Now introduce the following useful definition.

Definition 1.2.4. A space X has *property* $(**)$ if for each finite open cover $\mathcal{U}$ of X, there is an open cover $r(\mathcal{U})$ of X such that $r(\mathcal{U})^b$ refines $\mathcal{U}$, and $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$.

Also in this case, the following fact easily follows from Definition 1.2.4.

Proposition 1.2.3. A finitely-monotonically star-normal space has property $(**)$.

In [68], the authors introduced the concept of acyclic monotonically normal operator.

Definition 1.2.5. [68] A space X is *acyclically monotonically normal* if there is an operator r which assigns to each x and open set U containing x an open set $r(x, U)$ containing x which satisfies

(1) $r(x, U) \subseteq r(x, U')$ whenever $U \subseteq U'$,

(2) $r(x, X \setminus \{y\}) \cap r(y, X \setminus \{x\}) = \emptyset$ if $x \neq y$,

(3) $\bigcap_{t=0}^{n-1} r(x_t, X \setminus \{x_{t+1}\}) = \emptyset$ whenever $n \geq 2, x_0, \ldots, x_{n-1}$ are distinct and $x_n = x_0$.

The operator r is called *acyclic monotonically operator* for X.

Observe that (3) implies (2) and that (1) and (2) are precisely the conditions for a space to be monotonically normal.

Using a quite similar argument of Proposition 1.2.2, we have the following result.

Proposition 1.2.4. A space with property $(**)$ is acyclically monotonically normal.

Since M. E. Rudin [73] constructed a monotonically normal space which is not acyclically monotonically normal, monotone normality does not imply property $(**)$ (hence finite monotone star-normality).

Then, monotone normality and finite-monotone star-normality are not equivalent conditions.

The following question is still open.

Question 1.2.2. Are 2-monotone star-normality and finite-monotone star-normality equivalent conditions?

Monotone versions of property (a) and related spaces

It is easy to see that like normality, property (a) follows from paracompactness. In fact, property (a) is rather close to normality (it is very difficult to distinguish between property (a) and normality in the class of countably compact space). M. V. Matveev asked if monotonically normal spaces have property (a). M. E. Rudin, I. Stars and J. Vaughan in [75] answered in the affirmative to this question.

Theorem 1.2.3. [75] Monotonically normal spaces satisfy property (a).

Motived by the previous result it is natural to pose the following question.

Question 1.2.3. Is it possible to define a monotone version of property (a) in order to prove that monotone normality implies such a property?

Clearly, depending on the way in which the monotone operator works, we obtain different monotone version of property (a). We want to weight up all the options. The following results leave out two possibilities.

Proposition 1.2.5. Let X be a space. If there exists a function r that assigns to every open cover $\mathcal{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathcal{U}, D) \subset D$ such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U}, D) \subseteq r(\mathcal{V}, D)$. Then X is discrete.

Proof. Let X be a space, D a dense subset of X and r be the same as in the hypothesis. Let $\mathcal{U} = \{X\}$ be the trivial cover of X and $F = r(\mathcal{U}, D)$ be the closed in X and discrete such that $St(F, \mathcal{U}) = X$.

Claim: $X = F$.

Assume the contrary and fix $a \in X \setminus F$. Put $V = X \setminus \{a\}$ and $U = X \setminus F$. Clearly, the sets U and V are open in X and $\mathcal{C} = \{U, V\}$ covers X. Since $\mathcal{C}$ refines $\mathcal{U}$, from the hypothesis, we have $E \subset F$, where $E = r(\mathcal{C}, D)$. Therefore, $St(E, \mathcal{C}) \neq X$, since $St(E, \mathcal{C})$ does not contain the point a. Then $St(F, \mathcal{U}) \neq X$, a contradiction. □

Corollary 1.2.2. Let X be a space. If there exists a function r that assigns to every open cover $\mathcal{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathcal{U}, D) \subset D$ such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$ and $D \subseteq E$ then $r(\mathcal{U}, D) \subseteq r(\mathcal{V}, E)$. Then X is discrete.

There are still two options.

Definition 1.2.6. A space X is:

- $\mathbf{sm(a)}$ or has *strongly monotone property (a)* if there exists a function r that assigns to every open cover $\mathcal{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathcal{U}, D) \subset D$ such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and if $\mathcal{U}$

refines $\mathcal{V}$ and $D \subseteq E$ then $r(\mathcal{U}, D) \supseteq r(\mathcal{V}, E)$. The function r is called $\mathbf{sm}(\mathbf{a})$ *operator* for X.

- $\mathbf{m}(\mathbf{a})$ or has *monotone property (a)* if there exists a function r that assigns to every open cover $\mathcal{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathcal{U}, D) \subset D$ such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U}, D) \supseteq r(\mathcal{V}, D)$. The function r is called $\mathbf{m}(\mathbf{a})$ *operator* for X. A space having property $\mathbf{m}(\mathbf{a})$ is called $\mathbf{m}(\mathbf{a})$ *space*.

Obviously every monotone version of property (a) implies property (a). The converse is not true: consider ω_1 (see Proposition 1.2.5 and Example 1.2.2). Clearly, $\mathbf{sm}(\mathbf{a}) \Rightarrow \mathbf{m}(\mathbf{a})$.

We will show that none of the previous definitions gives a positive answer to Question 1.2.3, in particular by the next examples 1.2.1 and 1.2.2, we answer in the negative Question 1.2.3.

Example 1.2.1. A monotonically normal space which is not **sm(a)**.

Consider the real line, $E = \mathbb{R}$. Suppose E be a $\mathbf{sm}(\mathbf{a})$ space, and r be a **sm(a)** operator for E. Let $\mathcal{V} = \{\mathbb{R}\}$, then there exists a closed in $\mathbb{R}$ and discrete $r(\mathcal{V}, E) \subset E$. Let $\mathcal{U} = \{\mathbb{R}\}$ and let $D = \mathbb{R} \setminus r(\mathcal{V}, E)$. Clearly D is dense in $\mathbb{R}$ and then there exists a closed in $\mathbb{R}$ and discrete $r(\mathcal{U}, D) \subset D$. Therefore, since $D \subseteq E$, by hypothesis we have $r(\mathcal{U}, D) \supset r(\mathcal{V}, E)$; a contradiction since $r(\mathcal{U}, D) \subset D = \mathbb{R} \setminus r(\mathcal{V}, E)$.

Among monotone versions of covering properties, there is the definition of monotonically star closed-and-discrete that was given by S. G. Popvasillev and J. E. Porter.

Definition 1.2.7. [70] A space X is *monotonically star closed-and-discrete* if there exists an operator r which assigns to each open cover $\mathcal{U}$ a subspace $r(\mathcal{U}) \subseteq X$ such that $r(\mathcal{U})$ is closed and discrete in X, $St(r(\mathcal{U}), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$, then $r(\mathcal{U}) \supseteq r(\mathcal{V})$.

In [65], M. V. Matveev observed that every space is star closed-and-discrete (*i.e.* for every open cover $\mathcal{U}$ there is a closed and discrete subset $F \subseteq X$ such that $St(F, \mathcal{U}) = X$). However, monotone version of star closed-and-discrete turns out to be interesting since protometrizable spaces (recall that a *protometrizable* space is a paracompact space with an orthobase. In [41], P. M. Gartside and P. J. Moody showed that a space is protometrizable if and only if it is monotone paracompact) are monotonically star closed-and-discrete and the following propositions hold.

Proposition 1.2.6. [70] Monotonically star closed-and-discrete GO spaces are paracompact.

Proposition 1.2.7. Every $\mathbf{m}(\mathbf{a})-$space is monotonically star closed-and-discrete.

Example 1.2.2. A monotonically normal space which is not a $\mathbf{m}(\mathbf{a})$.

Since ω_1 is not paracompact, by propositions 1.2.7 and 1.2.6 it is a GO-space (hence a monotonically normal space) which is not $\mathbf{m}(\mathbf{a})$.

Recall that (see [65]) in the class of Hausdorff spaces countable compactness is equivalent to star-compactness.

The following definition of monotonically star-compact space was introduced in [70] and called monotonically star-finite space.

Definition 1.2.8. [70] A space X is *monotonically star-compact* (briefly msc) if there exists a function r that assigns to every open cover $\mathcal{U}$ of X a finite subset of X $r(\mathcal{U})$ such that $St(r(\mathcal{U}), \mathcal{U}) = X$ and such that if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U}) \supseteq r(\mathcal{V})$.

It is natural to pose the following question.

Question 1.2.4. Is monotone countable compactness equivalent to monotone star-compactness in the class of Hausdorff spaces?

Note that ω_1 is not monotonically star-compact (by Proposition 1.2.6) neither monotonically countably compact [69].

A countable compact space X which is not monotonically star-compact is given in [70, Example 21]. If such a space X is mcc then this example permits to give a negative answer to Question 1.2.4.

For sake of completeness, note that another possible monotone version of star-compactness could be given requiring that:

($\square$) there exists a function r, called $\square$-*operator*, that assigns to every open cover $\mathcal{U}$ of X a finite subset $r(\mathcal{U})$ of X such that $St(r(\mathcal{U}), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U}) \subseteq r(\mathcal{V})$.

However the following result proves the absurdness of the previous definition.

Theorem 1.2.4. Every space X having property ($\square$) is finite.

Proof. Let X be a space having property ($\square$) and r be the $\square$-operator. Let $\mathcal{U} = \{X\}$ be the trivial cover of X. Put $F = r(\mathcal{U})$.

Claim: $X = F$.

Assume the contrary, and fix $a \in X \setminus F$. Put $V = X \setminus \{a\}$ and $U = X \setminus F$. Clearly, the sets U and V are open in X and $\mathcal{C} = \{U, V\}$ covers X. Since X has property $\square$ and $\mathcal{C}$ refines $\mathcal{U}$, we have $E \subset F$, where $E = r(\mathcal{C})$. Therefore, $St(E, \mathcal{C}) \neq X$, since $St(E, \mathcal{C})$ does not contain the point a. Then, $St(F, \mathcal{U}) \neq X$, a contradiction. $\square$

Recall the following theorem.

Theorem 1.2.5. In the class of Hausdorff spaces, property acc is equivalent to property (a) plus countable compactness.

In order to give a monotone version of the previous theorem, we introduce the following monotone version of acc property.

Definition 1.2.9. A space X has *property monotone acc* if there exists a function r that assigns to every open cover $\mathcal{U}$ of X and every dense $D \subset X$ a finite subset $r(\mathcal{U}, D)$ of D such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and such that if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U}, D) \supseteq r(\mathcal{V}, D)$. A space having property macc is called *macc space*.

The following proposition gives a monotone version of Theorem 1.2.5.

Proposition 1.2.8. In the class of Hausdorff spaces, property macc is equivalent to property $\mathbf{m}(\mathbf{a})$ plus monotone countable compactness.

Proof. $\Rightarrow$) By hypothesis, there exists an operator r that assigns to every open cover $\mathcal{U}$ of X and every dense subset $D \subset X$, a finite set $r(\mathcal{U}, D) \subset D$ such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$, then $r(\mathcal{U}, D) \supseteq r(\mathcal{V}, D)$. By Hausdorfness, $r(\mathcal{U}, D)$ is closed and discrete and then X is a $\mathbf{m}(\mathbf{a})$ space.

$\Leftarrow$) Let X be a countable compact $\mathbf{m}(\mathbf{a})$ space. Then there exists an operator r that assigns to every open cover $\mathcal{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathcal{U}, D) \subset D$ such that $St(r(\mathcal{U}, D), \mathcal{U}) = X$ and if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U}, D) \supseteq r(\mathcal{V}, D)$. Since X is mcc, hence countable compact, every closed and discrete subset is finite and hence it is macc. $\square$

Obviously every macc space is an acc space. The converse is not true, as the following example shows.

Example 1.2.3. ω_1 is acc but not a mcc [69], hence not macc.

2 Some results on the cardinality of a topological space

The aim of this Chapter is to obtain bounds on the cardinality of topological spaces in terms of cardinal functions. We focus our attention on particular classes of topological spaces: n-Hausdorff and n-Urysohn spaces and topological groups. In [80], the author investigated the inequality of A. Hajnal and I. Juhasz $|X| \leq 2^{c(X)\chi(X)}$ where X is Hausdorff [46], for Urysohn spaces replacing the bound on cellularity $c(X) \leq \kappa$ with a bound on Urysohn cellularity $Uc(X) \leq \kappa$, which is a weaker condition, because $Uc(X) \leq c(X)$. In the first Section, we introduce the n-Urysohn cellularity n-$Uc(X)$ and prove that the previous inequality is true in the class of n-Urysohn spaces replacing the cellularity with the n-Urysohn cellularity. In this connection we also show that $|X| \leq 2^{Uc(X)\pi\chi(X)}$ if X is a power homogeneous Urysohn space.

In the second Section, we introduce the Hausdorff point separating weight $Hpw(X)$, and prove that

(1) for a Hausdorff space X, $|X| \leq Hpsw(X)^{aL_c(X)\chi(X)}$,

(2) and for a a Hausdorff space with a π-base consisting of compact sets with non-empty interior, $|X| \leq Hpsw(X)^{wL_c(X)\psi(X)}$..

The inequality (1) is a Hausdorff version of A. Charlesworth's inequality $|X| \leq psw(X)^{L(X)\chi(X)}$ [36]. These results are contained in the submitted paper [32].

In [14], it is proved that the character of a non-locally compact topological group with a first countable remainder does not exceed ω_1 and, as a consequence, that if G is a non-locally compact topological group with a first countable remainder, then $|G| \leq 2^{\omega_1}$. Moreover, a non-locally compact topological group of character ω_1 having a compactification whose reminder is first countable is given. In the third Section, we generalize these results in the general case of an arbitrary infinite cardinal κ. The obtained results are contained in [28].

We recall below several important cardinal functions.

The following properties represent weaker forms of the Lindelöf degree $L(X) = min\{\kappa : \forall$ open cover $\mathcal{U}$ of $X, \exists \ \ \mathcal{U}_0 \subset \mathcal{U} : |\mathcal{U}_0| \leq \kappa\}$. The *weak Lindelöf degree of* X, denoted $wL(X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{V}$ of X, there is a subcollection $\mathcal{V}_0$ of $\mathcal{V}$ such that $|\mathcal{V}_0| \leq \kappa$ and $\overline{\bigcup \mathcal{V}_0}$ covers X. The *almost Lindelöf degree of* X, denoted $aL(X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{V}$ of X, there is a subcollection $\mathcal{V}_0$ of $\mathcal{V}$ such that $|\mathcal{V}_0| \leq \kappa$ and $\{\overline{V} : V \in \mathcal{V}_0\}$ covers X. For every regular space X, $aL(X) = L(X)$. The *almost Lindelöf degree of* X *with respect to closed sets*, denoted $aL_c(X)$, is the smallest infinite cardinal κ such that for every closed subset H of X and every collection $\mathcal{V}$ of open sets in X that covers H, there is a subcollection $\mathcal{V}'$ of $\mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $\{\overline{V} : V \in \mathcal{V}'\}$ covers H. The *weak Lindelöf degree of* X *with respect to closed sets*, denoted $wL_c(X)$, is the smallest infinite cardinal κ such that for every closed subset H of X and every collection $\mathcal{V}$ of open sets in X that covers H, there is a subcollection $\mathcal{V}'$ of $\mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $H \subseteq \overline{\bigcup \mathcal{V}'}$.

The relationships between these cardinal functions are:

$$aL(X) \leq aL_c(X) \leq L(X) \text{ and equality holds for regular spaces}$$

and

$$wL_c(X) \leq aL_c(X) \leq L(X).$$

The *closed pseudo-character* of a space X, denoted $\psi_c(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\alpha<\kappa} \overline{V(\alpha, x)} = \{x\}$. The *Hausdorff pseudo-character of* X, denoted $H\psi(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x \neq y$, then there exist $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \emptyset$. These two cardinal functions are defined only for Hausdorff spaces. The *Urysohn pseudo-character of* X, denoted $U\psi(X)$, is similar to $H\psi(X)$ except that we require that $\overline{V(\alpha, x)} \cap \overline{V(\beta, y)} = \emptyset$. This cardinal function is defined only for Urysohn spaces. The following hold.

$$\psi(X) \leq \psi_c(X) \leq H\psi(X) \leq U\psi(X) \leq \chi(X).$$

An important role is played by the point separating weight.

A *point separating open cover* $\mathcal{S}$ for a space X is an open cover of X having the property that for each pair of distinct points x and y in X there is S in $\mathcal{S}$ such that x is in S but y is not in S. The *point separating weight* of a T_1 space X is the cardinal

$$psw(X) = min\{\tau : X \text{ has a point separating cover } \mathcal{S} \text{ such that each point of } X \text{ is contained in at most } \tau \text{ elements of } \mathcal{S}\}.$$

If $psw(X) = \omega$, one says that X has a *point-countable separating open cover*.

A set $S = \{p_\alpha : \alpha \in \nu\} \subset X$ is a *free sequence* of lenght ν in X if for each $\alpha \in \nu$ we have

$$\overline{\{p_\beta \in \alpha\}} \cap \overline{\{p_\beta : \beta \in \nu \setminus \alpha\}} = \emptyset.$$

We put $F(X) = sup\{\kappa :$ there exists a free sequence of lenght κ in $X\}$.

2.1 Variations of the Hajnal and Juhasz inequality $|X| \leq 2^{c(X)\chi(X)}$ for Hausdorff spaces

The inequality of A. Hajnal and I. Juhasz $|X| \leq 2^{c(X)\chi(X)}$ for Hausdorff spaces [46] is well known in literature. Several generalizations of the previous inequality were given. In [25], the author defined the *Hausdorff number* of X, denoted $H(X)$, as the smallest cardinal τ such that for every subset $A \subset X, |A| \geq \tau$, there exist neighborhoods $U_a, a \in A$, such that $\bigcap_{a\in A} U_a = \emptyset$. A space X is said to be *n-Hausdorff* if $H(X) = n$. For every n-Hausdorff space X, the *n-Hausdorff pseudocharacter* of X, denoted n-$H\psi(X)$, is the smallest κ such that for each point x there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x_1, x_2, \ldots, x_n$ are distinct points from X, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n < \kappa$ such that $\bigcap_{i=1}^{n} V(\alpha_i, x_i) = \emptyset$. Then, using these definitions, it was proved that the inequality of Hajnal and Juhasz holds replacing the character with the Hausdorff pseudocharacter, and that for every 3- Hausdorff space the inequality $|X| \leq 2^{2^{c(X)3\text{-}H\psi(X)}}$ holds. Moreover, I. Gotchev proved that the last inequality is true for every space X having finite Hausdorff number, [43]. In [80], the author investigated the inequality of Hajnal and Juhasz for Urysohn spaces replacing the bound on cellularity $c(X) \leq \kappa$ with a bound on Urysohn cellularity $Uc(X) \leq \kappa$, which is a weaker condition, because $Uc(X) \leq c(X)$ (*i.e.* J. Schröder proved that for a Urysohn space X, $|X| \leq 2^{Uc(X)\chi(X)}$). It seems natural to ask if the previous inequality could be restated for n-Urysohn spaces. We prove that this is possible provided the Urysohn cellularity is replaced by the n-Urysohn cellularity (Theorem 2.1.1 below).

An analogue of the inequality of Hajnal and Juhasz in the setting of homogeneous spaces was established in [34]. In [34, Theorem 2.3] N. Carlson and G. J. Ridderbos use the Erdös-Rado's theorem to show that if X is a power homogeneous Hausdorff space then $|X| \leq 2^{c(X)\pi\chi(X)}$. This result can be modified in the setting of Urysohn spaces to give the homogeneous analogue of Schröder's result. We prove that if X is an Urysohn power homogeneous space then $|X| \leq 2^{Uc(X)\pi\chi(X)}$.

Recall that, a pairwise disjoint collection of non-empty open sets in X is called a *cellular family*. The *cellularity* of X is the supremum of the cardinality of cellular families in X.

In [80], J. Schröder gives the following definition.

Definition 2.1.1. [80] Let X be a topological space. A collection $\mathcal{V}$ of open subsets of X is called *Urysohn-cellular*, if O_1, O_2 in $\mathcal{V}$ and $O_1 \neq O_2$ implies $\overline{O_1} \cap \overline{O_2} = \emptyset$. The *Urysohn-cellularity* of X, $Uc(X)$, is defined by

$$Uc(X) = \sup\{|\mathcal{V}| : \mathcal{V} \text{ is Urysohn-cellular }\} + \omega.$$

Moreover, he gave the following variation of the Hajnal and Juhasz inequality in the class of Urysohn spaces: $|X| \leq 2^{Uc(X)\chi(X)}$.

We generalize the definition of Uryshon cellularity in order to generalize the previous inequality.

Recall that, in [26], the authors define the *Urysohn number* $U(X)$ of a space X as the smallest cardinal τ such that for every subset $A \subset X, |A| \geq \tau$, there exist neighborhoods $U_a, a \in A$, such that $\bigcap_{a\in A} \overline{U_a} = \emptyset$. A space X is said to be n*-Urysohn* if $U(X) = n$.

We introduce the following definition.

Definition 2.1.2. Let X be a topological space. A collection $\mathcal{V}$ of open subsets of X is called n*-Urysohn-cellular* , where $n \in \omega$, if $O_1, O_2, \ldots, O_n$ in $\mathcal{V}$ and $O_1 \neq O_2 \neq \ldots \neq O_n$ implies $\overline{O_1} \cap \overline{O_2} \cap \ldots \cap \overline{O_n} = \emptyset$. The n*-Urysohn-cellularity* of X, n-$Uc(X)$, is defined by

$$n\text{ - }Uc(X) = \sup\{|\mathcal{V}| : \mathcal{V} \text{ is } n\text{- Urysohn-cellular }\} + \omega.$$

Clearly, if $\mathcal{V}$ is a Urysohn cellular collection of open subsets, then $\mathcal{V}$ is n-Urysohn cellular for every $n \in \omega$. Also if $Uc(X) \leq \kappa$, then n-$Uc(X) \leq \kappa$ for every $n \in \omega$.

Question 2.1.1. Is there a space X such that $(n + 1)$-$Uc(X) = \kappa$ and n-$Uc(X) \neq \kappa$?

Recall that the *θ-closure* of a set A in the space X is the set $cl_\theta(A) = \{x \in X :$ for every neighborhood $U \ni x, \overline{U} \cap A \neq \emptyset\}$ [85].

Proposition 2.1.1. Let $\{A_\alpha\}_{\alpha\in A}$ be a collection of subsets of X, then

$$\bigcup_{\alpha\in A} cl_\theta(A_\alpha) \subseteq cl_\theta\left(\bigcup_{\alpha\in A} A_\alpha\right).$$

Proof. If $x \in \bigcup_{\alpha\in A} cl_\theta(A_\alpha)$, then there exists $\alpha \in A$ such that $x \in cl_\theta(A_\alpha)$. Therefore for every neighborhood U_x we have $\overline{U}_x \cap A_\alpha \neq \emptyset$, then $\overline{U}_x \cap (\bigcup_{\alpha\in A} A_\alpha) \neq \emptyset$. This implies $x \in cl_\theta(\bigcup_{\alpha\in A} A_\alpha)$. □

The next lemma represents a modification of Lemma 7 in [80].

Lemma 2.1.1. Let X be a topological space and $\mu = n\text{-}Uc(X)$. Let $(U_\alpha)_{\alpha\in A}$ be a collection of open sets. Then there are $B_1, B_2, \ldots, B_{n-1} \subseteq A$ with

$$|B_i| \leq \mu \quad \forall i = 1, 2, \ldots, n-1$$

and

$$\bigcup_{\alpha\in A} U_\alpha \subseteq cl_\theta \left(\bigcup_{\alpha\in B_1\cup B_2\cup\ldots\cup B_{n-1}} \overline{U_\alpha} \right).$$

Proof. Let $\mathcal{V} = \{V \subset X : V$ is open and $\exists\alpha \in A$ such that $V \subseteq U_\alpha\}$. By Zorn's Lemma, take a maximal n-Urysohn-cellular family $\mathcal{W} \subseteq \mathcal{V}$ and $|\mathcal{W}| \leq \mu$.

For every $W \in \mathcal{W}$ take $U_\beta \in \{U_\alpha : \alpha \in A\}$ such that $W \subseteq U_\beta$. We may assume $\beta \in B = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_{n-1}, B_i \subseteq A$ and $|B_i| \leq \mu, \forall i = 1, 2, \ldots, n-1$.

We want to prove that

$$\bigcup_{\alpha\in A} U_\alpha \subseteq cl_\theta \left(\left(\bigcup_{\alpha\in B_1} \overline{U_\alpha} \right) \cup \ldots \cup \left(\bigcup_{\alpha\in B_{n-1}} \overline{U_\alpha} \right) \right).$$

Assume the contrary, then there exists $x \in \bigcup_{\alpha\in A} U_\alpha$ and

$$x \notin cl_\theta \left(\left(\bigcup_{\alpha\in B_1} \overline{U_\alpha} \right) \cup \ldots \cup \left(\bigcup_{\alpha\in B_{n-1}} \overline{U_\alpha} \right) \right).$$

Then we can find $\alpha_0 \in A$ and a neighborhood U_x of x such that $x \in U_{\alpha_0}$ and

$$\overline{U_x} \cap \left(\left(\bigcup_{\alpha\in B_1} \overline{U_\alpha} \right) \cup \ldots \cup \left(\bigcup_{\alpha\in B_{n-1}} \overline{U_\alpha} \right) \right) = \emptyset.$$

Then

$$\overline{(U_{\alpha_0} \cap U_x)} \subseteq \overline{U_x}$$

and

$$(U_{\alpha_0} \cap U_x) \cup \mathcal{W}$$

is a n-Urysohn cellular family containing $\mathcal{W}$, a contradiction. □

Corollary 2.1.1. [80] Let X be a topological space and $\mu = Uc(X)$. Let $(U_\alpha)_{\alpha\in A}$ be a collection of open sets. Then there is $B \subseteq A$ with $|B| \leq \mu$ and $\bigcup_{\alpha\in A} U_\alpha \subseteq cl_\theta \bigcup_{\alpha\in B} \overline{U_\alpha}$.

We can restate Lemma 2.1.1 in the following way.

Lemma 2.1.2. Let X be a topological space and $\mu = n\text{-}Uc(X)$. Let $\mathcal{F}$ be a collection of open sets. Then there are $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \subseteq \mathcal{F}$ with

$$|\mathcal{G}_i| \leq \mu \quad \forall i = 1, 2, \ldots, n$$

and

$$\bigcup \mathcal{F} \subseteq cl_\theta \left(\bigcup_{i=1}^{n} \left(\bigcup \overline{\mathcal{G}_i} \right) \right).$$

Theorem 2.1.1. Let X be a n-Urysohn space. Then

$$|X| \leq 2^{n-Uc(X)\chi(X)}.$$

Proof. Set $\mu = n\text{-}Uc(X)\chi(X)$. For every $x \in X$ let $\mathcal{B}(x)$ denote an open neighbourhood base of x with $|\mathcal{B}(x)| \leq \mu$. Construct an increasing sequence $\{C_\alpha : \alpha < \mu^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha : \alpha < \mu^+\}$ of open collections of open subsets of X such that:

1. $|C_\alpha| \leq 2^\mu$ for all $\alpha < \mu^+$;
2. $\mathcal{V}_\alpha = \bigcup\{\mathcal{B}(c) : c \in \bigcup_{\tau<\alpha} C_\tau\}, \alpha < \mu^+$;
3. If $\{G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} : (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) \subseteq \mu\}$ is a collection of subsets of X and each $G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}$ is the union of closures of $\leq \mu$ many elements of $\mathcal{V}_\alpha$ and
$$\bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \neq X$$
then
$$C_\alpha \setminus \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \neq \emptyset.$$

The construction is by transfinite induction. Let x_0 be a point of X and put $C_0 = \{x_0\}$. Let $0 < \alpha < \mu^+$ and assume that C_β has been constructed for each $\beta < \alpha$. Note that $\mathcal{V}_\alpha$ is defined by 2. and $\mathcal{V}_\alpha \leq 2^\mu$. For each collection $\{G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} : (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) \subseteq \mu\}$ of subsets of X where each $G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}$ is the union of closures of $\leq \mu$ many elements of $\mathcal{V}_\alpha$ and

$$\bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \neq X,$$

choose a point of $X \setminus \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}$. Let H_α be the set of points choosen in this way, (clearly, $|H_\alpha| \leq 2^\mu$) and let $C_\alpha = H_\alpha \cup (\bigcup_{\beta<\alpha} C_\beta)$. It is clear that the family $\{C_\alpha : 0 < \alpha < \mu^+\}$ constructed in this way satisfies condition 1., 2. and 3..

Let $C = \bigcup_{\alpha<\mu^+} C_\alpha$. We shall show that $C = X$. Assume there is $y \in X \setminus C$. For every $B_{\gamma_1}, B_{\gamma_2}, \ldots, B_{\gamma_{n-1}} \in \mathcal{B}(y)$, with $|B_{\gamma_i}| > 1$ $\forall i = 1, 2, \ldots, n-1$

and $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \subseteq \mu$ define

$$\mathcal{F}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} = \{V_c : c \in C, V_c \in \mathcal{B}(c), \overline{V_c} \cap \overline{B_{\gamma_1}} \cap \overline{B_{\gamma_2}} \cap \ldots \cap \overline{B_{\gamma_{n-1}}} = \emptyset\}.$$

Since X is n-Urysohn, we have

$$C \subseteq \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} \bigcup \mathcal{F}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}.$$

By Lemma 2.1.2, we find for every $\{\gamma_1, \gamma_2, \ldots, \gamma_{n-1}\} \subseteq \mu$ subcollections

$$\mathcal{G}_{\gamma_1}, \mathcal{G}_{\gamma_2}, \ldots, \mathcal{G}_{\gamma_{n-1}} \subseteq \mathcal{F}_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}}, |\mathcal{G}_{\gamma_i}| \leq \mu \ \forall i = 1, 2, \ldots, n-1$$

such that

$$\bigcup \mathcal{F}_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}} \subseteq cl_\theta \bigcup_{i=1}^{n-1} \left(\bigcup \overline{\mathcal{G}_{\gamma_i}}\right).$$

Note $y \notin cl_\theta \bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}})$. Indeed, since

$$\left(\bigcup_{i=1}^{n-1} \left(\bigcup \overline{\mathcal{G}_{\gamma_i}}\right)\right) \cap \overline{B_{\gamma_1}} \cap \overline{B_{\gamma_2}} \cap \ldots \cap \overline{B_{\gamma_{n-1}}} = \emptyset,$$

and then

$$\left(\bigcup_{i=1}^{n-1} \left(\bigcup \overline{\mathcal{G}_{\gamma_i}}\right)\right) \cap \overline{(B_{\gamma_1} \cap B_{\gamma_2} \cap \ldots \cap B_{\gamma_{n-1}})} = \emptyset.$$

Find $\alpha < \mu^+$ such that $\bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu}(\mathcal{G}_{\gamma_1} \cup \mathcal{G}_{\gamma_2} \cup \ldots \cup \mathcal{G}_{\gamma_{n-1}}) \subseteq \mathcal{V}_\alpha$. Then

$$y \notin \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta \bigcup_{i=1}^{n-1} \left(\bigcup \overline{\mathcal{G}_{\gamma_i}}\right)$$

but

$$C_\alpha \subseteq C \subseteq \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} \bigcup \mathcal{F}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \subseteq \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta \bigcup_{i=1}^{n-1} \left(\bigcup \overline{\mathcal{G}_{\gamma_i}}\right).$$

Put $G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} = \bigcup_{i=1}^{n-1} \left(\bigcup \overline{\mathcal{G}_{\gamma_i}}\right)$. This contradicts 3. □

The Schröder's result can be now obtained as a corollary of the previous theorem.

Corollary 2.1.2. [80] Let X be a Urysohn space. Then

$$|X| \leq 2^{Uc(X)\chi(X)}.$$

Quite naturally we also raised the question concerning what happens in the n-Hausdorff case. It turns out that the inequality of Hajnal and Juhasz can be restated in the class of n-Hausdorff spaces provided cellularity is replaced by *n-cellularity*.

Definition 2.1.3. Let X be a topological space, $\mathcal{C}$ a collection of open subsets of X and $n \in \omega$. We say that $\mathcal{C}$ is a *n-cellular family* if for every $O_1, O_2, \ldots, O_n \in \mathcal{C}$ we have that $O_1 \cap O_2 \cap \ldots \cap O_n = \emptyset$.

We define the *n-cellularity of X* as:

$$n\text{ - }c(X) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is an } n\text{-cellular family of } X\} + \omega.$$

Clearly, if $\mathcal{C}$ is a cellular family, then $\mathcal{C}$ is a n-cellular family for every $n \in \omega$.

Lemma 2.1.3. Let X be a topological space and n-$c(X) = \kappa$, where $n \in \omega$. Let $(U_\alpha)_{\alpha\in A}$ be a collection of subsets of X. Then there exist $B_1, B_2, \ldots, B_{n-1} \subseteq A$ such that $|B_i| \le \kappa$ for every $i = 1, 2, \ldots, n-1$ and

$$\bigcup_{\alpha\in A} U_\alpha \subseteq \overline{\bigcup_{i=1}^{n-1} \bigcup_{\alpha\in B_i} U_\alpha}.$$

Proof. Let $\mathcal{G}$ be the collection of all non-empty open sets in X which are subsets of some element of $\mathcal{V}$. Use Zorn's Lemma to obtain a maximal n-cellular family $\mathcal{G}' \subseteq \mathcal{G}$. Then, $|\mathcal{G}'| \le n$-$c(X) = \kappa$. For every $G \in \mathcal{G}'$ take $U_\beta \in \{U_\alpha : \alpha \in A\}$ such that $G \subseteq U_\beta$. We may assume

$$\beta \in B = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_{n-1}, \qquad B_i \subseteq A$$

and $|B_i| \le \kappa$, for every $i = 1, 2, \ldots, n-1$. We want to prove that

$$\bigcup_{\alpha\in A} U_\alpha \subseteq \overline{\bigcup_{i=1}^{n-1} \bigcup_{\alpha\in B_i} U_\alpha}.$$

Assume the contrary, then there exists $x \in \bigcup_{\alpha\in A}$ and $x \notin \overline{\bigcup_{i=1}^{n-1} \bigcup_{\alpha\in B_i} U_\alpha}$. Then we can find $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$ and a neighborhood U_x of x such that

$$U_x \cap \bigcup_{i=1}^{n-1} \bigcup_{\alpha\in B_i} U_\alpha = \emptyset.$$

Then $(U_{\alpha_0} \cap U_x) \cup \mathcal{G}'$ is a n-cellular family containing $\mathcal{G}'$, a contradiction. □

Theorem 2.1.2. Let X be a n-Hausdorff topological space, then

$$|X| \le 2^{n-c(X)\chi(X)}.$$

Proof. Let $n - c(X)\chi(X) = \kappa$, and for each $p \in X$ let $\mathcal{V}_p$ be a local base for p with $\mathcal{V}_p \leq \kappa$. Construct a sequence $\{A_\alpha : 0 \leq \alpha < \kappa'\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa'\}$ of open collections in X such that:

1. $A_\alpha \leq 2^\kappa, 0 \leq \alpha < \kappa'$;
2. $\mathcal{V}_\alpha = \{V : V \in \mathcal{V}_p, p \in \bigcup_{\beta<\alpha} A_\beta\}, 0 < \alpha < \kappa'$;
3. if $\{G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} : \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \subseteq \kappa\}$ is a collection of $\leq \kappa$ open sets of X, each of which is the union of $\leq \kappa$ elements of $\mathcal{V}_\alpha$ and $\bigcup_{\gamma_1,\gamma_2,..,\gamma_{n-1}\subseteq\kappa} \overline{G_{\gamma_1,\gamma_2,..,\gamma_{n-1}}} \neq X$, then $A_\alpha \setminus (\bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \overline{G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}})$ $\neq \emptyset$.

Let $A = \bigcup_{\alpha<\kappa'} A_\alpha$. The proof is complete if $A = X$. Suppose not, let $q \in X \setminus A$, and let $\{B_\gamma : 0 \leq \gamma < \kappa\}$ be a local base at q. For every $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \subseteq \kappa$ let

$$\mathcal{W}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} = \{V \in \mathcal{V}_p, p \in A,\ V \cap B_{\gamma_1} \cap B_{\gamma_2} \cap \ldots \cap B_{\gamma_{n-1}} = \emptyset\}.$$

Note that for each $p \in A$, there exist $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ such that

$$p \in \bigcup \mathcal{W}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}.$$

By the previous Lemma, there are $\mathcal{G}_{\gamma_1}, \mathcal{G}_{\gamma_2}, \ldots, \mathcal{G}_{\gamma_{n-1}} \subseteq \mathcal{W}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}$, with $|\mathcal{G}_{\gamma_i}| \leq \kappa$ for every $i = 1, 2, \ldots n-1$, such that $\bigcup \mathcal{W}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \subseteq \overline{\bigcup_{i=1}^{n-1} \bigcup \mathcal{G}_{\gamma_i}}$. Let $G_{\gamma_1,\gamma_2,\ldots\gamma_{n-1}} = \bigcup_{i=1}^{n-1} \bigcup \mathcal{G}_{\gamma_i}$ and note that

$$A \subseteq \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\subseteq\kappa} \overline{\bigcup_{i=1}^{n-1} \bigcup \mathcal{G}_{\gamma_i}}$$

and

$$q \notin \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\subseteq\kappa} \overline{\bigcup_{i=1}^{n-1} \bigcup \mathcal{G}_{\gamma_i}}.$$

Choose $\alpha < \kappa'$ such that $\mathcal{G}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \subseteq \mathcal{V}_\alpha$ for all $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \subseteq \kappa$. By 3., $A_\alpha \setminus (\bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\subseteq\kappa} \overline{G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}}) \neq \emptyset$. This contradicts

$$A \subseteq \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\subseteq\kappa} \overline{G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}}.$$

□

Recall that a topological space X is said to be *quasiregular* provided for every open set V, there is a non-empty open set U such that the closure of U is contained in V. We observe the following properties.

Lemma 2.1.4. If X is a quasiregular space, then for every cellular family $\mathcal{U}$ such that $|\mathcal{U}| = \kappa$ there exists an Urysohn cellular family $\mathcal{U}'$ such that $|\mathcal{U}'| = \kappa$.

Proof. Let $\mathcal{U}$ be a cellular family with $|\mathcal{U}| = \kappa$. For every $U \in \mathcal{U}$ there exists an open set $V_U \subset U$ such that $\overline{V_U} \subset U$. Clearly, if U_1 and U_2 are distinct elements of $\mathcal{U}$ such that $U_1 \cap U_2 = \emptyset$, we have $\overline{V_{U_1}} \cap \overline{V_{U_2}} = \emptyset$. Hence $\mathcal{U}' = \{V_U : U \in \mathcal{U}\}$ is an Urysohn cellular family for X such that $|\mathcal{U}'| = \kappa$. □

Property 2.1.1. If X is a quasiregular space, $c(X) = Uc(X)$.

Proof. Clearly, $Uc(X) \leq c(X)$. Let $Uc(X) = \kappa$ and suppose that $c(X) > \kappa$. Then by Lemma 2.1.4 there exists an Urysohn cellular family $\mathcal{U}$ such that $|\mathcal{U}| > \kappa$; a contradiction. □

Recall the following theorem.

Theorem 2.1.3. [43, Corollary 3.2] Let X be a space with $H(X)$ finite. Then $|X| \leq 2^{c(X)\chi(X)}$.

The previous result together with Property 2.1.1 gives the following corollary.

Corollary 2.1.3. If X is a quasiregular n-Hausdorff space, $|X| \leq 2^{\chi(X)Uc(X)}$.

We end this Section with a new cardinality bound for power homogeneous Urysohn spaces involving the Urysohn cellularity $Uc(X)$. It is well established that cardinality bounds on a topological space can be improved if the space possesses homogeneous-like properties. The first result in this direction was obtained by E. van Douwen in 1978. He proved that the cardinality of any power homogeneous space is at most $2^{\pi w(X)}$. Afterwards, J. van Mill proved that for every compact, power homogeneous space X, the inequality $|X| \leq 2^{c(X)\pi\chi(X)}$ holds. In 2006, R. De la Vega answered a long-standing question of A. V. Arhangel'skii by proving that the cardinality of a compact homogeneous space is at most $2^{t(X)}$. A. V. Arhangel'skii, J. van Mill, and G. J. Ridderbos improved this result by showing that the same bound holds for compact power homogeneous spaces. In 2008, N. Carlson and G. J. Ridderbos [34] have shown that if X is power homogeneous then $|X| \leq 2^{c(X)\pi\chi(X)}$. This result represents both an improvement of van Douwen's theorem and an analogue of the Hajnal and Juhasz's inequality $|X| \leq 2^{c(X)\chi(X)}$ (where X is Hausdorff), for power homogeneous spaces. By modifying this result, we show below that an analogous result holds for Urysohn power homogeneous spaces when $Uc(X)$ is used in place of $c(X)$.

We recall the following Carlson and Ridderbos' theorem.

Theorem 2.1.4. [34] If X is a homogeneous space, then

$$|X| \leq 2^{\pi\chi(X)c(X)}.$$

We restate the previous result in the case of an Urysohn or quasiregular space.

Theorem 2.1.5. If X is a homogeneous Hausdorff space that is Urysohn or quasiregular then

$$|X| \leq 2^{Uc(X)\pi\chi(X)}.$$

Proof. Let X be a homogeneous Hausdorff space. If X is quasiregular, then $Uc(X) = c(X)$ and the result follows from Theorem 2.1.4. Now we assume X is Urysohn. Let $\kappa = Uc(X)\pi\chi(X)$, fix $p \in X$, and let $\mathcal{B}$ be a local π-base at p such that $|\mathcal{B}| \leq \kappa$. As X is homogeneous, for all $x \in X$ there exists a homeomorphism $h_x : X \to X$ such that $h_x(p) = x$.

As X is Urysohn, for all $x \neq y \in X$ there exist open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. Then $p \in h_x^{-1}[U] \cap h_y^{-1}[V]$, an open set. As $\mathcal{B}$ is a local π-base at p, there exists $B(x,y) \in \mathcal{B}$ such that $B(x,y) \subseteq h_x^{-1}[U] \cap h_y^{-1}[V]$. Thus, $h_x[B(x,y)] \subseteq U$, $h_y[B(x,y)] \subseteq V$, and

$$\overline{(h_x[B(x,y)])} \cap \overline{(h_y[B(x,y)])} = \emptyset.$$

The existence of $B(x,y)$ for each $x \neq y \in X$ defines a function $B : [X]^2 \to \mathcal{B}$.

Suppose by way of contradiction that $|X| > 2^\kappa$. As $|\mathcal{B}| \leq \kappa$, we can apply the Erdös-Rado Theorem to the function B. Thus, there exists $Y \in [X]^{\kappa^+}$ and $A \in \mathcal{B}$ such that $B(x,y) = A$ for all $x, y \in Y$.

Observe that for every $x \neq y \in Y$, we have

$$\overline{(h_x[A])} \cap \overline{(h_y[A])} = \overline{(h_x[B(x,y)])} \cap \overline{(h_y[B(x,y)])} = \emptyset.$$

This shows $\{h_x[A] : x \in Y\}$ is a Urysohn cellular family. However,

$$|\{h_x[A] : x \in Y\}| = |Y| = \kappa^+ > Uc(X),$$

which is a contradiction. Thus, $|X| \leq 2^\kappa = 2^{Uc(X)\pi\chi(X)}$. □

Moreover, N. Carlson and G. J. Ridderbos proved in [34], that Theorem 2.1.4 is also valid for power homogeneous spaces.

Theorem 2.1.6. [34] If X is power homogeneous, then

$$|X| \leq 2^{c(X)\pi\chi(X)}.$$

To establish this more general theorem in the Urysohn or quasiregular cases, we adapt the proof Theorem 2.3 in [34].

Whenever μ is a cardinal number and $A \subseteq \mu$, then by π_A we denote the projection of X^μ onto X^A. If $\alpha \in \mu$, then we write π_α for $\pi_{\{\alpha\}}$, which is the projection on the α-*th* coordinate. This notation is ambiguous because α is also a subset of μ. As a rule, we will always use π_α and π_β as projections on the respective coordinates and for $\kappa \subseteq \mu$, we will use π_κ for the projection onto X^κ. Finally, if $x \in X^\mu$, then we write x_A instead of $\pi_A(x)$, and π is always the projection onto the first coordinate, *i.e.* $\pi = \pi_0$. By $\Delta(X, \kappa)$, we denote the diagonal in X^κ which is given by

$$\big\{x \in X^\kappa : \forall \alpha, \beta \in \kappa (x_\alpha = x_\beta)\big\}.$$

We will call a space X^μ Δ-homogeneous if for all points $x, z \in \Delta(X, \mu)$ there is a homeomorphism of X^μ mapping x onto z. A space X is power homogeneous if and only if there is a cardinal μ such that X^μ is Δ-homogeneous (G. J. Ridderbos, [71]). Let X be a power homogeneous space and let μ be a cardinal number such that X^μ is Δ-homogeneous. Let κ be a cardinal number such that $\pi\chi(X) \leq \kappa$. Without loss of generality, we may assume that $\kappa \leq \mu$. Fix $p \in \Delta(X, \mu)$ and a local π-base $\mathcal{U}$ at $\pi(p)$ in X. For $B \subseteq A \subseteq \mu$, let $\pi_{A \to B}$ be the projection of X^A onto X^B. For $A \subseteq \mu$, define $\mathcal{U}(A)$ by

$$\big\{\pi^{-1}_{A \to B}\big[\Pi_{b \in B} U_b\big] : B \in [A]^{<\omega}, \forall b \in B(U_b \in \mathcal{U})\big\}.$$

Note that $\mathcal{U}(A)$ is a local π-base at p_A in X^A. We also need the following Lemma from [34].

Lemma 2.1.5. [34] For every $x \in \Delta(X, \mu)$ there is a homeomorphism $h_x : X^\mu \to X^\mu$ such that $h_x(p) = \{x\}$ and the following conditions are satisfied.

(1) for all $z \in X^\mu$, if $z_\kappa = p_\kappa$, then $\pi(h_x(z)) = \pi(x)$,

(2) for all $U \in \mathcal{U}(\kappa)$, there is a point $q(U)$ (depending on x) $\in \pi_\kappa^{-1}[U]$ and a basic open neighborhood U_x of $h_x(q(U))_\kappa$ in X^κ such that

 (a) $q(U)_\alpha = p_\alpha$ for all $\alpha \in \mu \setminus \kappa$ and

 (b) $\pi_\kappa^{-1}[U_x] \subseteq h_x[\pi_\kappa^{-1}[U]]$.

Theorem 2.1.7. If X is a power homogeneous Hausdorff space that is Urysohn or quasiregular then

$$|X| \leq 2^{Uc(X)\pi\chi(X)}.$$

Proof. Let X be a power homogeneous Hausdorff space. If X is quasiregular then again $Uc(X) = c(X)$ and the proof follows directly from Theorem 2.1.6. So we assume X is Urysohn.

Let $\kappa = Uc(X)\pi\chi(X)$ and fix $\mu > \kappa$ such that X^μ is homogeneous. For every $x \in \Delta(X, \mu)$, we fix an homeomorphism $h_x : X^\mu \to X^\mu$ as in Lemma 2.1.5. For $x \in \Delta(X, \mu)$ and $U \in \mathcal{U}(\kappa)$, the open set U_x is a basic open subset of X^κ, so we may fix a collection $\{U_{x,\alpha} : \alpha \in \kappa\}$ of open subsets of X such that

$$U_x = \bigcap_{\alpha < \kappa} \pi_\alpha^{-1}[U_{x,\alpha}].$$

For every $\alpha \in \kappa$, we also fix a local π-base $\{V(x,U,\alpha,\beta) : \beta < \kappa\}$ of the point $h_x(q(U))_\alpha$ in X. We first observe the following claim.
Claim 1. Whenever $x \neq y \in \Delta(X,\mu)$, there is $U \in \mathcal{U}(\kappa)$ and $\alpha, \beta < \kappa$ such that

$$\overline{V(x,U,\alpha,\beta)} \subseteq \overline{U_{x,\alpha}} \text{ and } \overline{V(x,U,\alpha,\beta)} \cap \overline{U_{y,\alpha}} = \emptyset.$$

Since $\pi(x) \neq \pi(y)$ we have that $h_y^{-1}(x)_\kappa \neq p_\kappa$. As X^κ is Urysohn, fix an open neighbourhood W of p_κ in X^κ such that $h_y^{-1}(x)_\kappa \notin cl_\theta(\overline{W})$ and let

$$\mathcal{W} = \{\mathcal{U} \in \mathcal{U}(\kappa) : \mathcal{U} \subseteq \mathcal{W}\}.$$

Note that $\mathcal{W}$ is a local π-base at p_κ in X^κ and $h_y^{-1}(x)_\kappa \notin cl_\theta(\overline{\bigcup \mathcal{W}})$. So we have the following, as in the proof of Theorem 2.3 in [34]:

$$x \in \overline{\{h_x(q(U)) : U \in \mathcal{W}\}}. \tag{2.1}$$

As $h_y^{-1}(x)_\kappa \notin cl_\theta(\overline{\bigcup \mathcal{W}})$, there exists a basic open set S in X^κ containing $\pi_\kappa h_y^{-1}(y)$ such that $\overline{S} \cap \overline{\bigcup \mathcal{W}} = \emptyset$. As $\pi_\kappa h_y^{-1}(y) \in S$, we have

$$x \in h_y \pi_\kappa^{-1}[S] \subseteq h_y \pi_\kappa^{-1}[\overline{S}] = \overline{h_y \pi_\kappa^{-1}[S]}.$$

Now suppose $\overline{h_y \pi_\kappa^{-1}[S]} \cap \pi_\kappa^{-1}[\overline{U_y}] \neq \emptyset$ for some $U \in \mathcal{W}$. Then, as $\pi_\kappa^{-1}[U_y] \subseteq h_y \pi_\kappa^{-1}[U]$, we have

$$h_y \pi_\kappa^{-1}[\overline{S} \cap \overline{U}] = h_y \pi_\kappa^{-1}[\overline{S}] \cap h_y \pi_\kappa^{-1}[\overline{U}] \neq \emptyset.$$

But this is a contradiction as $\overline{S} \cap \overline{U} \subseteq \overline{S} \cap (\overline{\bigcup \mathcal{W}}) = \emptyset$. Thus, $(\overline{h_y \pi_\kappa^{-1}[S]}) \cap \pi_\kappa^{-1}[\overline{U_y}] = \emptyset$ for all $U \in \mathcal{W}$, and

$$x \notin cl_\theta \bigcup \{\pi_\kappa^{-1}[\overline{U_y}] : U \in \mathcal{W}\} = cl_\theta \left(\pi_\kappa^{-1}\left[\bigcup\{\overline{U_y} : U \in \mathcal{W}\}\right]\right).$$

Choose a basic open set T in X^μ such that $x \in T$ and

$$\overline{T} \cap \pi_\kappa^{-1}\left[\bigcup\{\overline{U_y} : U \in \mathcal{W}\}\right] = \emptyset. \tag{2.2}$$

Now, by (2.1) above there exists $U \in \mathcal{W}$ such that $h_x(q(U)) \in T$, where $q(U)$ is as in Lemma 2.2 in [34]. As T is basic-open, it follows from (2.2) that $\pi_\kappa[\overline{T}] \cap [\overline{U_y}] = \emptyset$. As $\pi_\kappa h_x(q(U)) \in \pi_\kappa[T]$, we see that

$$\pi_\kappa[h_x(q(U))] \notin cl_\theta(\overline{U_y}).$$

Thus there exists a basic-open set Z in X^κ containing $\pi_\kappa[h_x(q(U)]$ such that $\overline{Z} \cap \overline{U_y} = \emptyset$. By the definition of U_y and the fact that both Z and U_y are basic-open, there exists an $\alpha < \kappa$ such that $\pi_\alpha[\overline{Z}] \cap \overline{U_{y,\alpha}} = \emptyset$. Thus, $\pi_\alpha[h_x(q(U)] \notin cl_\theta(\overline{U_{y,\alpha}})$.

Therefore, $\pi_\alpha[h_x(q(U))] \in U_{x,\alpha} \setminus cl_\theta(\overline{U_{y,\alpha}})$. There exists an open set A in X containing $\pi_\alpha[h_x(q(U))]$ such that $\overline{A} \cap \overline{U_{y,\alpha}} = \emptyset$. Thus $A \cap U_{x,\alpha}$ is an open set containing $\pi_\alpha[h_x(q(U))]$. Since $\{V(x, U, \alpha, \beta) : \beta < \kappa\}$ is a local π-base at $h_x(q(U))_\alpha$ in X, there exists $\beta < \kappa$ such that

$$V(x, U, \alpha, \beta) \subseteq A \cap U_{x,\alpha}.$$

Thus, $\overline{V(x, U, \alpha, \beta)} \subseteq \overline{A}$ and $\overline{V(x, U, \alpha, \beta)} \cap \overline{U_{y,\alpha}} = \emptyset$. As $\overline{V(x, U, \alpha, \beta)} \subseteq \overline{U_{x,\alpha}}$, this completes the proof of the claim.

Assume that $|X| > 2^\kappa$. We fix a well-ordering $\prec$ on X and define a map $G : X^2 \to \mathcal{U}(\kappa) \times \kappa \times \kappa$ as follows: let $\{x, y\} \in [X]^2$ and assume that $x \prec y$. Applying the previous claim, we may let $G(\{x, y\}) = \langle U, \alpha, \beta\rangle$ be such that

$$\overline{V(x, U, \alpha, \beta)} \subseteq \overline{U_{x,\alpha}} \text{ and } \overline{V(x, U, \alpha, \beta)} \cap \overline{U_{y,\alpha}} = \emptyset.$$

Here we have identified $\Delta(X, \mu)$ with X. Note that $|\mathcal{U}(\kappa) \times \kappa \times \kappa| = \kappa$. Since $|X| > 2^\kappa$, we apply the Erdös-Rado Theorem to find $Y \subseteq X$ and $\langle U, \alpha, \beta\rangle \in \mathcal{U}(\kappa) \times \kappa \times \kappa$ such that $|Y| = \kappa^+$ and for all $\{x, y\} \in [Y]^2, G(\{x, y\}) = \langle U, \alpha, \beta\rangle$. By possibly removing the $\prec$-largest element from Y, we may assume that for all $y \in Y$, $V(y, U, \alpha, \beta) \subseteq U_{y,\alpha}$. Consider the collection $\mathcal{C} = \{V(x, U, \alpha, \beta) : x \in Y\}$ of open subsets of X^κ. If $x, y \in Y$ are different with $x \prec y$, then we have $\overline{V(x, U, \alpha, \beta)} \cap \overline{U_{y,\alpha}} = \emptyset$ and $\overline{V(y, U, \alpha, \beta)} \subseteq \overline{U_{y,\alpha}}$, and therefore $\{V(x, U, \alpha, \beta) : x \in Y\}$ is a Urysohn cellular family. However,

$$|\mathcal{C}| = |Y| = \kappa^+ > Uc(X),$$

which, a contradiction. Thus $|X| \leq 2^\kappa = 2^{Uc(X)\pi\chi(X)}$. □

The above result shows that Schröder's cardinality bound $2^{Uc(X)\chi(X)}$ for Urysohn spaces can be improved in the power homogeneous setting.

Question 2.1.2. If X is power homogeneous and n-Urysohn, is

$$|X| \leq 2^{n-Uc(X)\pi\chi(X)}?$$

2.2 Bounds on cardinality of a space involving the Hausdorff point separating weight

In this Section, we give a partial solution to Arhangel'skii's problem [5, Problem 5.2] concerning whether the continuum is an upper bound for the cardinality of a Hausdorff Lindelöf space having countable pseudocharacter. In [36], the author gave a partial solution to this question proving that $|X| \leq psw(X)^{L(X)\psi(X)}$ for every T_1 space X, where $psw(X)$ is the mininum infinite cardinal κ such that X has an open cover $\mathcal{S}$ (called *separating open cover*) having the property that for each pair of distinct points x and y in X there is an $S \in \mathcal{S}$ such that $x \in S$ and $y \notin S$ and such that each point of X is in at most κ elements of $\mathcal{S}$. In [32], we gave the analogous of $psw(X)$ for Hausdorff space, denoted $Hpsw(X)$, and prove that for Hausdorff spaces, $|X| \leq Hpsw(X)^{aL_c(X)\chi(X)}$. Also it is proved that for a Hausdorff space with a π-base consisting of compact sets with non-empty interior, $|X| \leq Hpsw(X)^{wL_c(X)\psi(X)}$..

Definition 2.2.1. A *Hausdorff point separating open cover* $\mathcal{S}$ for a space X is an open cover of X having the property that for each pair of distinct points x and y in X there is S in $\mathcal{S}$ such that x is in S but y is not in $\overline{S}$. The *Hausdorff point separating weight* of a Hausdorff space X is the cardinal

$$Hpsw(X) = min\{\tau : X \text{ has a Hausdorff point separating cover } \mathcal{S} \text{ such that each point of } X \text{ is contained in at most } \tau \text{ elements of } \mathcal{S}\} + \omega.$$

We say that X has a *Hausdorff point-continuum separating open cover* if and only if $Hpsw(X) \leq \mathfrak{c}$. Clearly, if X is a Hausdorff space, then $psw(X) \leq Hpsw(X)$. In [36], A. Charlesworth proved the following theorem.

Theorem 2.2.1. [36, Theorem 2.1] If X is T_1, then $nw(X) \leq psw(X)^{L(X)}$.

As a consequence of the previous result, he proved that

$$|X| \leq psw(X)^{L(X)\psi(X)}.$$

We prove the following theorem.

Theorem 2.2.2. If X is a Hausdorff space, then $nw(X) \leq Hpsw(X)^{aL_c(X)}$.

Proof. Let $aL_c(X) = \kappa$ and let $\mathcal{S}$ be a Hausdorff separating open cover for X such that for each $x \in X$ we have $|\mathcal{S}_x| \leq \lambda$, where $\mathcal{S}_x$ denotes the collection of members of $\mathcal{S}$ containing x and such that if x and y are different points of X then there exists $U \in \mathcal{S}$ such that $x \in U$ and $y \notin \overline{U}$. We first show that $d(X) \leq \lambda^\kappa$. For each $\alpha < \kappa^+$ construct a subset D_α of X such that

1. $D_\alpha \leq \lambda^\kappa$;
2. If $\mathcal{U}$ is a subcollection of $\bigcup\{\mathcal{S}_x : x \in \bigcup_{\beta<\alpha} D_\beta\}$ such that $|\mathcal{U}| \leq \kappa$ and $X \setminus \bigcup\overline{\mathcal{U}} \neq \emptyset$, then $D_\alpha \setminus \bigcup\overline{\mathcal{U}} \neq \emptyset$.

Such a D_α can be constructed since the number of possible $\mathcal{U}$'s at the α-th stage of construction is $\leq (\lambda^\kappa \cdot \kappa \cdot \lambda)^\kappa = \lambda^\kappa$. Let $D = \bigcup_{\alpha<\kappa^+} D_\alpha$. Clearly

$|D| \leq \lambda^\kappa$. Furthermore, D is a dense subset of X. In fact, if there is a point $p \in X \setminus \overline{D}$, since $Hpsw(X) \leq \lambda$, for every $x \in \overline{D}$ there exists an open set $V_x \in \mathcal{S}_x$ such that $x \in V_x$ and $p \notin \overline{V_x}$. Moreover, since $x \in \overline{D}$, we have $V_x \cap D \neq \emptyset$, then there exists $y \in V_x \cap D$ and then $V_x \in \bigcup\{\mathcal{S}_y : y \in D\}$. Put $\mathcal{W} = \{V_x : x \in \overline{D}\} \subseteq \bigcup\{\mathcal{S}_y : y \in D\}$. Clearly, $\mathcal{W}$ is an open cover of $\overline{D}$. Using $aL_c(X) \leq \kappa$ we can select a subcollection $\mathcal{W}' \subseteq \mathcal{W}, |\mathcal{W}'| \leq \kappa$ such that $\overline{D} \subseteq \bigcup\{\overline{V} : V \in \mathcal{W}'\}$ and $p \notin \bigcup\{\overline{V} : V \in \mathcal{W}'\}$. This contradicts 2. Since $d(X) \leq \lambda^\kappa$ we have that $|\mathcal{S}| \leq \lambda^\kappa$. Let $\mathcal{N} = \{X \setminus S | S$ is the union of at most κ members of $\mathcal{S}\}$. Then $|\mathcal{N}| \leq \lambda^\kappa$ and $\mathcal{N}$ is a network for X.

□

Theorem 2.2.3. If X is a Hausdorff space, then $|X| \leq Hpsw(X)^{aL_c(X)\psi(X)}$.

Proof. It is known that if X is a T_1 space, $|X| \leq nw(X)^{\psi(X)}$. Then by Theorem 2.2.2, we have $|X| \leq Hpsw(X)^{aL_c(X)\psi(X)}$. □

Corollary 2.2.1. If X is a Hausdorff space with $L(X) = \omega, \psi(X) = \omega$ and $Hpsw(X) \leq \mathfrak{c}$, then $|X| \leq \mathfrak{c}$.

The previous corollary gives a partial solution to Arhangel'skii's problem [5, Problem 5.2] concerning whether the continuum is an upper bound for the cardinality of a Hausdorff Lindelöf space having countable pseudocharacter.

Remark 2.2.1. Using Remark 2.5 in [36] we note that countable pseudocharacter is essential in Corollary 2.2.1: if X is the product of 2^ω copies of the two point discrete space, then X is Hausdorff, Lindelöf and $\psi(X) > \omega$ but $|X| > 2^\omega$.

The following theorem, under additional hypothesis, gives a result similar to Theorem 2.2.2 in which the weakly Lindelöf degree with respect to closed sets takes the place of the almost Lindelöf degree with respect to closed sets.

Theorem 2.2.4. If X is a Hausdorff space with a π-base consisting of compact sets with non-empty interior, then $nw(X) \leq Hpsw(X)^{wL_c(X)}$.

Proof. Let $wL_c(X) = \kappa$ and let $\mathcal{S}$ be a Hausdorff point separating open cover for X such that for each $x \in X$ we have $|\mathcal{S}_x| \leq \lambda$, where $\mathcal{S}_x$ denotes the collection of members of $\mathcal{S}$ containing x. Without loss of generality, we can suppose that the family $\mathcal{S}_x$ is closed under finite intersection. We first show that $d(X) \leq \lambda^\kappa$. For each $\alpha < \kappa^+$ construct a subset D_α of X such that:

1. $D_\alpha \leq \lambda^\kappa$.

2. If $\mathcal{U}$ is a subcollection of $\bigcup\{\mathcal{S}_x : x \in \bigcup_{\beta<\alpha} D_\beta\}$ such that $|\mathcal{U}| \leq \kappa$ and $X \setminus \overline{\bigcup\mathcal{U}} \neq \emptyset$, then $D_\alpha \setminus \overline{\bigcup\mathcal{U}} \neq \emptyset$.

Such a D_α can be constructed since the number of possible $\mathcal{U}$'s at the αth stage of construction is $(\leq \lambda^\kappa \cdot \kappa \cdot \lambda)^\kappa = \lambda^\kappa$. Let $D = \bigcup_{\alpha<\kappa^+} D_\alpha$. Clearly $|D| \leq \lambda^\kappa$. Furthermore D is a dense subset of X. Indeed if $\overline{D} \neq X$, $X \setminus \overline{D}$ is a non-empty open set. Since X has a π-base consisting of compact sets with non-empty interior, we can find a non empty open subset $W \subseteq X$ such that $\overline{W}$ is compact and $\overline{W} \subset X \setminus \overline{D}$, hence $\overline{W} \cap \overline{D} = \emptyset$. Fix $x \in \overline{D}$. For every $p \in \overline{W}$ there exists an open subset $V_p \in \mathcal{S}_x$ such that $p \notin \overline{V_p}$. Then, we can find a family $\{V_p : p \in \overline{W}\}$ of open subsets of X such that $\bigcap\{\overline{V_p} : p \in \overline{W}\} \cap \overline{W} = \emptyset$. So, for the compactness of $\overline{W}$ the family $\{\overline{V_p} \cap \overline{W} : p \in \overline{W}\}$ can not have the finite intersection property. So put $F_x = \overline{V_{p_1}} \cap ... \cap \overline{V_{p_k}}$, where $p_1, ..., p_\kappa \in \overline{W}$ are such that $F_x \cap \overline{W} = \emptyset$. Put $G_x = V_{p_1} \cap ... \cap V_{p_k}$. Since $\mathcal{S}_x$ is closed under finite intersection, $G_x \in \mathcal{S}_x$ and $G_x \cap \overline{W} = \emptyset$. Since $G_x \in \mathcal{S}_x$ then $G_x \in \mathcal{S}_y$ for some $y \in D$. Clearly, $\mathcal{V} = \{G_x : x \in \overline{D}\}$ is an open cover of $\overline{D}$. Using $wL_c(X) \leq \kappa$ we can select a subcollection $\mathcal{V}' \subseteq \mathcal{V}, |\mathcal{V}'| \leq \kappa$ such that $\overline{D} \subseteq \overline{\bigcup\{V : V \in \mathcal{V}'\}}$. For every $U \in \bigcup\mathcal{V}'$, $U \cap \overline{W} = \emptyset$, hence $\bigcup\mathcal{V}' \cap \overline{W} = \emptyset$. Since W is a nonempty open set, $\overline{\bigcup\mathcal{V}'} \cap W = \emptyset$ and then $X \setminus \overline{\bigcup\mathcal{V}'} \neq \emptyset$. This contradicts 2. Since $d(X) \leq \lambda^\kappa$ we have that $|\mathcal{S}| \leq \lambda^\kappa$. Let $\mathcal{N} = \{X \setminus S | S$ is the union of at most κ members of $\mathcal{S}\}$. Then $|\mathcal{N}| \leq \lambda^\kappa$ and $\mathcal{N}$ is a network for X. □

Then we have the following result.

Corollary 2.2.2. If X is a Hausdorff space with a π-base consisting of compact sets with non-empty interior, then $|X| \leq Hpsw(X)^{wL_c(X)\psi(X)}$.

2.3 On the cardinality of a topological group

In this Section, by *space*, we mean a Tychonoff space.

Topological groups represent an area of Topological Algebra that is well developed and has a long tradition, see for example [13, 82].

A group $(G, \cdot)$ provided with a topology τ is a topological group if the multiplication mapping $(x, y) \to x \cdot y$ and the inverse $x \to x^{-1}$ are continuous with respect to τ. This is equivalent to saying that the mapping of $G \times G$ to G, where $G \times G$ carries the usual product topology, defined by $(x, y) \to x \cdot y^{-1}$ is continuous.

Every topological group is homogeneous. If the topological space (G, τ) is compact, one says briefly *compact group*.

For $A, B \subset G$ and $p \in G$ we write

$$AB = \{a \cdot b : a \in A, b \in B\},$$
$$A^2 = AA, \qquad A^{n+1} = AA^n (2 \leq n < \omega),$$
$$Ap = A\{p\}, \qquad pA = \{p\}A.$$

We note that the relation $A^n = \{a^n : a \in A\}$ is false. The *identity* or *neutral element* of a group G is denoted by e. For a topological group G and $p \in G$, we denote by $\mathcal{N}_G(p)$ or $\mathcal{N}(p)$ the set of open neighborhoods of p in G. In particular, $\mathcal{N}(e)$ is the open neighborhood system at e, and we have

$$\mathcal{N}(p) = \{pU : U \in \mathcal{N}(e)\} = \{Up : U \in \mathcal{N}(e)\} \quad \forall p \in G.$$

The presence of an algebraic structure nicely related to a topology changes dramatically the relationship between topological invariants. Important classical results in this direction are well known. For example, it is important to mention Birkhoff-Kakutani's theorem that first countability is equivalent to metrizability in topological groups, Pontryagin's theorem that every topological group which satisfies the T_0 separation property is a Tychonoff space, and Bourbaki's theorem that every locally compact topological group is paracompact.

Theorem 2.3.1. [82] Let G be a topological group.

(i) If $\alpha \geq \omega$ and there is $S \subset G$ such that $int\overline{S} \neq \emptyset$ and $\chi(S) = \alpha$, then $\chi(G) = \alpha$;

(ii) if G has a dense, first countable subspace, then G is first countable.

Theorem 2.3.2. [82] Every locally compact topological group G with $|G| < 2^\omega$ is discrete.

Cardinal functions behave much better in topological groups than in topological spaces. Another particular phenomenon is the coincidence of some cardinal functions in the class of topological groups while they are different even for compact spaces, for example, the following theorem can not be extend to compact spaces.

Theorem 2.3.3. [7] Every topological group G satisfies the inequalities:

(1) $w(G) \leq d(G) \cdot \chi(G)$;

(2) $w(G) \leq L(G) \cdot \chi(G)$.

The following theorem from [7] shows that the differences between several cardinal functions disappear in the realm of topological groups.

Proposition 2.3.1. [7] Let G be a topological group. Then

(1) $\chi(G) = \pi\chi(G)$;

(2) $w(G) = \pi w(G)$.

For a locally compact Hausdorff group G we always have that $t(G) = w(G)$. In particular, a locally compact Hausdorff group of countable tightness is metrizable.

Some relations between cardinal invariants of topological groups disappear in the general case of Tychonoff spaces. These relations play an important role in the study of topological groups. For example, the cardinality of a topological group G can be estimated by the cellularity and the pseudocharacter as well as by the Lindelöf number and the pseudocharacter: $|G| \leq 2^{L(G)\psi(G)}$ and $|G| \leq 2^{c(G)\psi(G)}$. For a Tychonoff spaces these inequalities are not always satisfied (for any cardinal τ there exists a Tychonoff space X whose cardinality is larger than τ while its pseudocharacter and cellularity are countable). It is interesting and important to know,

how properties of a space X are related to the properties of some or all remainders of X. In particular, when does a space X have a compactification with a remainder belonging to a given class of spaces? A famous classical result in this direction is the following theorem of M. Henriksen and J. Isbell [48].

Theorem 2.3.4. [48] A space X is of countable type if and only if the remainder in any, or in some, compactification of X is Lindelöf.

It follows from the theorem of Henriksen and Isbell that every remainder of a metrizable space is Lindelöf and hence, paracompact. Arhangel'skii studied in deep the previous question, in particular those spaces whose remainders are close, in some sense, to being metrizable, and obtained a lot of results in this direction, see [1, 2, 3, 8, 9, 12, 14, 15, 16]. It turns out to be much easier to answer this question for topological groups than in the general case. For example, every remainder of a Lindelöf p-space is a Lindelöf p-space. However this statment does not generalize to paracompact p-spaces: the remainders of such spaces need not be paracompact p-spaces. However, Arhangel'skii established that if a topological group G has a remainder that is a paracompact p-space, then G is a paracompact p-space.

The following represents a Dichotomy theorem.

Theorem 2.3.5. [12] For any topological group G, any remainder of G in a compactification bG of G is either pseudocompact or Lindelöf.

Using this theorem, Arhangel'skii noted that no Dowker space can be a remainder of a topological group, and that normality is equivalent to collectionwise normality in remainders of topological groups. Perfect normality of a remainder Y of a topological group is shown to be equivalent to hereditary Lindelöfness of Y. Moreover, Arhangel'skii proved that a non-locally compact topological group G is separable and metrizable if (and only if) some remainder Y of G has locally a G_δ-diagonal.

Arhangel'skii's results show that the remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. Of course, there is an important exception to this rule: the case of locally compact topological groups. Indeed, every locally compact non-compact topological group has a remainder consisting of exactly one point. Thus, we will be interested only in the case of non-locally compact topological groups.

The basic problem is: *when does a topological group have a first-countable remainder?* The early results suggested the following conjecture: *Does a non-locally compact topological group G have a first-countable remainder if and only if G is metrizable?*

The early results by Arhangel'skii suggested a positive answer.

Theorem 2.3.6 (Arhangel'skii [2])**.** Let G be a non-locally compact topological group such that G^ω has a first-countable remainder. Then G is metrizable.

Theorem 2.3.7. [Arhangel'skii [2] $(MA\neg CH)$] Suppose that G is a σ-compact topological group with a remainder of countable tightness. Then either G is locally compact, or G is metrizable.

The last result is very surprising. Under a very strong set theoretical assumption, the basic problem can be solved for the class of all σ-compact groups. In particular, for the class of all countable topological groups.

Taking into account the Birkhoff-Kakutani's theorem, the basic problem can be formulated as follows. If G is a non-locally compact topological group, are the statements

1. $\chi(G) = \omega$ *(G is metrizable)*,
2. G has a compactification bG such that $\chi(bG \setminus G) = \omega$ *($bG \setminus G$ is first-countable)*

equivalent? If true, this would have been a very elegant result.

In a series of papers, Arhangel'skii and van Mill [14, 15, 16] solved the basic problem. Their first result gives information on topological groups with a first-countable remainder, and their second result gives a negative answer to the basic problem.

In particular, in [14], Arhangel'skii and van Mill, answer in the negative to the following problem.

Problem 2.3.1. [14, Problem 1.1] Suppose that G is a non-locally compact topological group with a first countable remainder. Is G metrizable?

Also, the following necessary condition for a non-locally compact topological group to have a first countable remainder is established.

Theorem 2.3.8. [14, Theorem 2.1] Suppose that G is a non-locally compact topological group with a first countable remainder. Then the character of the space G does not exceed ω_1.

As a consequence of the previous result, the following theorem holds.

Theorem 2.3.9. [14, Theorem 2.4] If G is a non-locally compact topological group with a first countable remainder, then $|G| \leq 2^{\omega_1}$.

In [14], it is proved that Theorem 2.3.8 is the best possible giving the following example.

Example 2.3.1. [14, Section 3] A non-locally compact topological group G of character ω_1 which has a compactification bG such that $bG \setminus G$ is first countable.

In this Section, we show that the methods used by Arhangel'skii and van Mill permit to generalize Theorem 2.3.8 and Example 2.3.1 in the case of an arbitrary infinite cardinal κ.

We show that Arhangel'skii and van Mill's proof of [14, Theorem 2.1], works in the general case of an arbitrary infinite cardinal κ.

Theorem 2.3.10. Let κ be an infinite cardinal and let G be a non-locally compact topological group. Assume that G has a compactification such that its remainder $bG \setminus G$ has character κ. Then the character of the space G does not exceed κ^+.

To prove Theorem 2.3.10, we need the following propositions 2.3.2 and 2.3.3. In particular, Proposition 2.3.2 is known (see for example [10], also note that the concept of free sequence was introduced in [6]). We include the proof of Proposition 2.3.2 for completeness of the exposition.

Proposition 2.3.2. Suppose that Y is a space with tightness $t(Y) = \kappa$ satisfying the following condition:

(s) *for any subset A of Y such that $|A| \leq \kappa^+$, the closure of A in Y is compact.*

Then Y is compact.

Proof. Striving for a contradiction, assume that Y is not compact and let X be a compactification of Y. Pick an arbitrary point $x \in X \setminus Y$. Then:
Fact 1. Every non-empty G_κ-subset P of X that contains x meets Y.

Indeed, let $P = \bigcap\{V_\alpha : \alpha < \kappa\}$, where each V_α is open. For each α take an open set U_α in X such that $x \in \overline{U}_\alpha \subseteq V_\alpha$. Put $\{U_\alpha : \alpha < \kappa\} = \mathcal{U}$. We may assume without any loss of generality that $\mathcal{U}$ is closed under finite intersections. For any $U \in \mathcal{U}$ pick a point $y_U \in U \cap Y$ and let $A = \{y_U : U \in \mathcal{U}\}$. By condition (s), the set $S = \overline{A}^Y$ is compact. As the family $\mathcal{F} = \{\overline{U} \cap S : U \in \mathcal{U}\}$ has the finite intersection property, we must have $\bigcap \mathcal{F} \neq \emptyset$. Since $\bigcap \mathcal{F} \subseteq P \cap Y$, we are done.

Using Fact 1, we define a sequence (P_ξ, y_ξ) for every $\xi < \kappa^+$ such that $(P_\xi)_{\xi \in \kappa^+}$ is a decreasing sequence and $y_\xi \in P_\xi$, as follows. Let y_0 be any element of Y, and put $P_0 = X$. Now assume that $\xi < \kappa^+$, and that the points $y_\beta \in Y$ and the closed G_κ-subsets P_β of X have been defined for every $\beta < \xi$. Denote by F_ξ the closure of the set $\{y_\beta : \beta < \xi\}$ in X. Then, by condition (s), $F_\xi \subseteq Y$ and $x \notin F_\xi$. Since F_ξ is closed in X and X is Tychonoff, it follows that there exists a closed G_δ-subset V of x in X such that $x \in V$ and $V \cap F_\xi = \emptyset$. Put $P_\xi = V \cap \bigcap_{\beta<\xi} P_\beta$. Clearly, $x \in P_\xi$, and P_ξ is a closed G_κ-subset of X. By Fact 1, have $P_\xi \cap Y \neq \emptyset$. This completes the transfinite construction.

Obviously, the following statements hold for any $\xi < \kappa^+$ (Fact 4 follows directly from facts 2 and 3).

Fact 2. $\overline{\{y_\beta : \beta < \xi\}} \cap P_\xi = \emptyset$.

Fact 3. $\overline{\{y_\beta : \xi \leq \beta < \kappa^+\}} \subseteq P_\xi$.

Fact 4. $\overline{\{y_\beta : \beta < \xi\}} \cap \overline{\{y_\beta : \xi \leq \beta < \kappa^+\}} = \emptyset$.

Fact 4 implies that $\eta = \{y_\xi : \xi < \kappa^+\}$ is a free sequence in X. Its closure is compact and is contained in Y. Hence this contradicts the fact that the tightness of Y is at most κ (For every compact and Hausdorff space X, $t(X) = F(X)$, [53]). □

Following the argument from [14, Proposition 2.3], and using Proposition 2.3.2 instead of [14, Proposition 2.2] we obtain the following result.

Proposition 2.3.3. Suppose that X is a nowhere locally compact space with remainder Y such that $\chi(Y) = \kappa$, where κ is an infinite cardinal. Then the π-character of the space X does not exceed κ^+ at some point of X.

Proof of Theorem 2.3.10. It follows from Proposition 2.3.3 that there exists a π-base $\mathcal{P}$ of G at the neutral element e of G such that $|\mathcal{P}| \leq \kappa^+$. Then, clearly, the family $\mu = \{UU^{-1} : U \in \mathcal{P}\}$ is a base of G at e such that $|\mu| \leq \kappa^+$. □

Theorem 2.3.11. If G is a non-locally compact topological group with remainder Y such that $\chi(Y) = \kappa$, then $|G| \leq 2^{\kappa^+}$.

Proof. Let bG be a compactification of the space G such that the remainder $Y = bG \setminus G$ has character κ. By Theorem 2.3.10, the character of the space G does not exceed κ^+. Since $\chi(Y) = \kappa$ and Y and G are both dense in bG, we conclude that $\chi(bG) \leq \kappa^+$. Since bG is compact, it follows that $|bG| \leq 2^{\kappa^+}$. Hence, $|G| \leq 2^{\kappa^+}$. □

Following the method in [14, Section 3] we construct the following example.

Example 2.3.2. A non-locally compact topological group G of character κ^+ which has a compactification bG such that $bG \setminus G$ has character κ.

Let X be a space with a dense subset D and consider the subspace

$$X(D) = (X \times \{0\}) \cup (D \times \{1\})$$

of the Alexandroff duplicate of X.
Observe that $X(D)$ is compact if X is compact.

The idea used by authors in [14, Section 3] is to consider the space $X(D, Y)$, obtained by replacing every isolated point of the form $(d, 1)$ in $X(D)$ by a copy of a fixed non-empty space Y. Also they note that if both X and Y are compact, then so is $X(D, Y)$ and that the function $\pi : X(D, Y) \to X \times \{0\}$ that collapses each set of the form $\{d\} \times Y \times \{1\}$ to $(d, 0)$ is a retraction.

Let $\kappa \geq \omega$ and let $K = 2^\kappa(2^\kappa)$, *i.e.*, the Alexandroff duplicate of the Cantor cube 2^κ. Following the idea used in [14, Section 3] and using this building block repeatedly, we will construct an inverse sequence of compact spaces $X_\alpha, \alpha < \kappa^+$.

In particular following step by step [14, Section 3] and defining $X_0 = 2^\kappa$ instead of 2^ω, we construct all X_α, where $\alpha < \omega_1$ and $X_{\omega_1} = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$. Let $\pi^{\omega_1}_\alpha : X_{\omega_1} \to X_\alpha$ denotes the projection for all $\alpha < \omega_1$.

Also the points $p \in X_{\omega_1}$, for which $\pi^{\omega_1}_\alpha(p)$ is isolated for every successor ordinal number $\alpha < \omega_1$, form a dense subspace H in X_{ω_1}.

Now put $X_{\omega_1+1} = X_{\omega_1}(H)$, and let $\pi^{\omega_1+1}_{\omega_1}$ be the standard retraction. We continue as before, replacing each isolated point by a copy of K, etc. Let $X_{\omega_1+\omega_1}$ be the inverse limit of spaces $X_{\omega_1+\beta}, \beta < \omega_1$. Continuing in this way for all $\alpha < \omega_2$, we get an inverse sequence $\{X_\alpha, \pi^\alpha_\beta\}$ of compact spaces having character equal to κ. Let $X_{\omega_2} = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$ with retractions $\pi^{\omega_2}_\alpha : X_{\omega_2} \to X_\alpha$ for all $\alpha < \omega_2$.

Continuing in this way for all $\alpha < \kappa^+$, we get an inverse sequence $\{X_\alpha, \pi^\alpha_\beta\}$ of compact spaces having character equal to κ.

Let $X_{\kappa^+} = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$ with retractions $\pi^{\kappa^+}_\alpha : X_{\kappa^+} \to X_\alpha$ for all $\alpha < \kappa^+$.

The following fact holds.

If $p \in X_{\kappa^+}$ and there exists a successor ordinal number $\alpha < \kappa^+$ such that $\pi^{\kappa^+}_\alpha(p)$ is not isolated, then

$$(\pi^{\kappa^+}_\alpha)^{-1}(\{\pi^{\kappa^+}_\alpha(p)\}) = \{p\}.$$

Hence, X_{κ^+} has character equal to κ at p.

The points $p \in X_{\kappa^+}$, for which $\pi^{\kappa^+}_\alpha(p)$ is isolated for every successor ordinal number $\alpha < \kappa^+$, form a dense subspace G in X_{κ^+}. The space G is easily seen to be homeomorphic to the space $(2^\kappa)^{\kappa^+}$ with the G_κ-topology (where the topology on 2^κ is the standard product topology). The reason that we get the G_κ-topology is clear: because if $p \in G$, then, for every $\alpha < \kappa^+$, we have that $\pi^{\kappa^+}_{\alpha+1}(p)$ is isolated. Hence, G is a topological group, and so we are done.

It seems natural to pose the following question.

Question 2.3.1. Is it possible to generalize Theorem 2.3.10 and Example 2.3.2 in other class of topological groups?

In particular, since the theory of paratopological groups is quite different from the theory of topological groups, in fact, many concepts in the theory of paratopological groups have no analogues in topological groups at all, it seems to be interesting to study the previous question in the case of paratopological group. (Recall that a group G provided with a topology τ is a *paratopological* group if multiplication in G is continuous as a mapping of $G \times G$ to G, where $G \times G$ carries the usual product topology. It is clear from the definitions that every topological group is a paratopological group.)

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