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Integral operators and partial differential equations

in

Morrey–type spaces

Doctoral Thesis

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Introduction

Morrey spaces were introduced by C. Morrey in 1938. They appeared to be quite useful in the study of local behaviour of the solutions of elliptic partial differential equations. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter $0 < \operatorname{diam}\Omega < \infty$. For $1 \le p < \infty$ and $\lambda \ge 0$, the Morrey space $L^{p,\lambda}(\Omega)$ is the subspace of $L^p(\Omega)$ defined via

$$L^{p,\lambda}(\Omega) = \left\{ u \in L^p(\Omega) : \|u\|_{L^{p,\lambda}(\Omega)} < \infty \right\},$$

where

$$\|u\|_{L^{p,\lambda}(\Omega)} = \left(\sup_{\substack{x\in\Omega \\ 0<\rho\le \operatorname{diam}\Omega}} \rho^{-\lambda} \int\limits_{\Omega\cap B(x,r)} |u(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}.$$

The theory of boundedness of classical operators of Real Analysis, such as maximal operator, fractional maximal operator, Riesz potential, singular integral operator etc, is by now well studied. These results can be applied fruitful in the theory of partial differential equations. It should be noted that in the theory of partial differential equations, in the last years, general Morrey-type spaces play an important role.

In the nineties of the XX century an extensive study of general Morrey type spaces, characterized by a functional parameter, started. In particular, V.S. Guliyev in his doctoral thesis (1994) introduced local and complementary local Morrey-type spaces and studied the boundedness in these spaces of fractional integral operators and singular integral operators defined on homogeneous Lie groups. A number of results on boundedness of classical operators in general Morrey type spaces were obtained by several authors. However in all these results only sufficient conditions on the functional parameters, characterizing general Morrey-type spaces, ensuring boundedness, were obtained.

At the beginning of the XXI century there were new deep developments in this research area. In particular, V.S. Guliyev, jointly with V.I. Burenkov, has developed a new perspective trend in harmonic analysis, related to the study of classical operators in general spaces of Morrey type. The significance of the developed methods lies in the fact that they allow to obtain necessary and sufficient conditions for the boundedness of classes of singular type operators with the subsequent application to regularity theory for solutions to elliptic and parabolic partial differential equations. As a result, for a certain range of the numerical parameters necessary and sufficient conditions were obtained on the functional parameters ensuring boundedness of classical operators of Real Analysis (maximal operator, fractional maximal operator, Riesz potential, genuine singular integrals) from one general local Morrey-type space to another one. Results of such type are very important for the development of contemporary Real Analysis and its applications, first of all, to Partial Differential Equations.

In this thesis, after a brief introduction of the classical operators of Real Analysis, the author treats the boundedness of some integral operators on generalized local Morrey spaces, on mixed Morrey spaces and on generalized local modified Morrey spaces and investigates some regularity properties of solutions to partial differential equations.

Precisely, the work is organized as follows.

Chapter 1. For the sake of completeness, we summarize without proofs the relevant material on some classical integral operators and functional spaces. In particular, we briefly recall some important results from the standard L^p-theory and many of them will be obtained again in the next chapters in the framework of various Morrey-type spaces.

Chapter 2. This chapter is devoted to the study of the boundedness of Hardy-Littlewood maximal operator in terms of the sharp maximal function and, as a consequence, the boundedness of commutators generated by a Calderón-Zygmund singular integral operator and a bounded mean oscillation function is obtained.

Chapter 3. This chapter provides a detailed exposition of a new Morrey-type space, defined by M.A. Ragusa and A. Scapellato. In particular, this chapter deals with mixed

Morrey spaces and, at first, we show some embedding results; later, in this new functional class, we discuss the boundedness of several classical integral operators and, as a consequence, we obtain a regularity result for solutions to parabolic equations.

Chapter 4. Aim of this chapter is to show boundedness results for a particular integral operator that occurs in the study of regularity of the solutions to elliptic partial differential equations in divergence form. The functional spaces under consideration are recently defined, represent a refinement of the generalized local Morrey spaces and are called *modified local generalized Morrey spaces*. We emphasize that the results proved in this chapter are new and unpublished.

All the results, except the ones contained in Chapter 4 that are completely new, have been present during conferences and workshops and some questions are still topic of research and further improvements.

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CHAPTER 1

Classical results on integral operators and Morrey spaces

In this chapter we collect some classical definitions and results dealing with various kind of integral operators and useful functional classes.
Namely, as integral operators, we introduce the Hardy-Littlewood maximal function and some its variants, Calderón-Zygmund singular integral operators and the Riesz potential. Furthermore, in the framework of BMO class, we discuss about the commutators of singular and fractional integral operators and introduce the Sarason class VMO. Last but not least, we introduce the classical Morrey spaces and some embedding results.

1. Hardy-Littlewood maximal function

Let us give the definition of Hardy-Littlewood maximal function which plays a very important role in harmonic analysis. There are a lot of definitions of the Hardy-Littlewood maximal function. We now state the definition that we will use in the sequel of the work.

Definition 1.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The Hardy-Littlewood maximal function $Mf(x)$ of f is defined by*

$$M(f) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)| \, \mathrm{d}y,$$

where $B(x,r)$ denotes the ball having center at x and with radius r.

We point out that some authors define the Hardy-Littlewood maximal function as

$$M'f(x) = \sup_{B \ni x} \frac{1}{|B|} \int\limits_B |f(y)| \, \mathrm{d}y$$

where the supremum is taken over all balls B containing x.

However, it can be shown that, although these definition looks different at first, they are actually pointwise comparable. Moreover, in literature the definitions above are also formulated considering cubes instead of balls.

It is worth pointing out that the mapping $M : f \mapsto Mf$ is not linear, but it satisfies the sub-additive property, that is $M(f_1 + f_2) \leq Mf_1 + Mf_2$. Furthermore, the Hardy-Littlewood maximal operator M is not a bounded operator from $L^1(\mathbb{R}^n)$ to itself. Let us show this fact only in the case $n = 1$. Take $f(x) = \chi_{[0,1]}(x)$ then, for any $x \geq 1$, we have

$$Mf(x) \geq \frac{1}{2x} \int\limits_0^{2x} |f(y)| \, \mathrm{d}y = \frac{1}{2x}.$$

Hence

$$\int\limits_{\mathbb{R}} Mf(x) \, \mathrm{d}x \geq \int\limits_1^{\infty} Mf(x) \, \mathrm{d}x \geq \int\limits_0^{2x} \frac{1}{2x} \, \mathrm{d}x = \infty.$$

Although M is not a bounded operator on $L^1(\mathbb{R})$, it is possible to show that M is a bounded operator from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, i.e., the weak $L^1(\mathbb{R}^n)$ space. This remark motivates the following definitions.

Definition 1.2. *Let $1 \leq p < \infty$ and let f be a measurable function on $\mathbb{R}^n$. The function f is said to belong to the weak L^p space on $\mathbb{R}^n$, if there is a constant $C > 0$ such that*

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} \leq C < \infty.$$

Equivalently, the weak $L^p(\mathbb{R}^n)$ is defined by

$$L^{p,\infty}(\mathbb{R}^n) : \{f : \|f\|_{p,\infty} < \infty\},$$

where

$$\|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}}$$

denotes the seminorm of f in the weak $L^p(\mathbb{R}^n)$.

It is easy to verify that for $1 \leq p < \infty$, $L^p(\mathbb{R}^n) \subsetneq L^{p,\infty}(\mathbb{R}^n)$.

Definition 1.3. *Let T be a sublinear operator and $1 \leq p, q \leq \infty$.*

- *T is said to be of weak type (p,q) if T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. That is, there exists a constant $C>0$ such that for any $\lambda>0$ and $f\in L^p(\mathbb{R}^n)$,*
$$|\{x\in\mathbb{R}^n : |Tf(x)|>\lambda\}| \le \left(\frac{C}{\lambda}\|f\|_p\right)^q; \tag{1.1}$$
- *T is said to be of (strong) type (p,q) if T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. That is, there exists a constant $C>0$ such that for any $f\in L^p(\mathbb{R}^n)$*
$$\|Tf\|_q \le C\|f\|_p. \tag{1.2}$$

It is easy to see that an operator of type (p,q) is also of weak type (p,q), but its reverse doesn't in general hold.

The following theorem summarize some important properties of the Hardy - Littlewood maximal function. Precisely, the theorem states that the Hardy-Littlewood maximal operator M is of weak type $(1,1)$ and type (p,p) for $1<p\le\infty$, respectively.

Theorem 1.4 ([84]). *Let f be a function defined on $\mathbb{R}^n$.*

(1) *If $f\in L^p(\mathbb{R}^n)$, $1\le p\le\infty$ then, $M(f)$ is finite almost everywhere.*

(2) *If $f\in L^1(\mathbb{R}^n)$ then, for every $\lambda>0$, there exists a constant $C=C(n)>0$ such that*
$$|\{x\in\mathbb{R}^n : Mf(x)>\lambda\}| \le \frac{C}{\lambda}\int\limits_{\mathbb{R}^n}|f(y)|\,\mathrm{d}y.$$

(3) *If $f\in L^p(\mathbb{R}^n)$, $1<p\le\infty$ then, $Mf\in L^p(\mathbb{R}^n)$ and there exists a constant $C=C(n,p)>0$ such that*
$$\|Mf\|_p \le A_p\|f\|_p.$$

We emphasize that, by weak $(1,1)$ boundedness of the Hardy-Littlewood maximal operator M, the celebrated Lebesgue differentiation theorem follows.

Theorem 1.5 (Lebesgue differentiation theorem)**.** *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*
$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} f(y)\,\mathrm{d}y, \qquad \text{a.e. in } \mathbb{R}^n$$
where $B(x,r)$ denotes the ball with center x and radius r.

1.1. Sharp maximal function.

Definition 1.6. *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$. The sharp maximal function $M^\sharp f(x)$ of f is defined by*
$$M^\sharp(f) = \sup_{B(x,r)}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-f_{B(x,r)}|\,\mathrm{d}y,$$

where $B(x,r)$ denotes the ball having center at x and with radius r.

As for the Hardy-Littlewood maximal function, the supremum in the definition above can be taken also over all balls B in $\mathbb{R}^n$ which contain x.

There exist a lot of properties dealing with the Hardy-Littlewood maximal function and the sharp maximal function but we restrict the attention to the celebrated Fefferman and Stein inequality that, as we show in the next chapters, was generalized in the framework of generalized and mixed Morrey spaces.

Theorem 1.7 (Fefferman-Stein, **[31]**). *Let $1 \leq p_0 < \infty$. Then for any $p_0 < p < \infty$, there exists a constant C, independent of f, such that*

$$\|Mf\|_p \leq C\|M^\sharp f\|_p$$

for any function f such that $Mf \in L^{p_0}(\mathbb{R}^n)$.

2. Singular integral operators

Singular integrals, as they are currently understood, are a natural outgrowth of the Hilbert transform. This last operator has a long tradition in complex and harmonic analysis. The generalization to n dimensions, due to Calderón and Zygmund in 1952, revolutionized the subject. This made a whole new body of techniques available for higher-dimensional analysis. Cauchy problems, commutators of operators, boundary value problems, and many other natural contexts for analysis can be analyzed in connection with some boundedness of singular integral operators.

The next simple example gives an elementary motivation to study singular integral operators. It is well known that, when $n \geq 3$, the fundamental solution of the Laplacian operator Δ is

$$\Gamma(x) = \frac{1}{(2-n)\omega_{n-1}}\frac{1}{|x|^{n-2}}.$$

Thus, when f has good properties, for example $f \in \mathscr{S}(\mathbb{R}^n)$, the convolution $\Gamma * f$ is a solution of the Poisson equation $\Delta u = f$, that is

$$u(x) = \Gamma * f(x) = C_n \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\,\mathrm{d}y.$$

Formally, by taking partial derivatives of second order of u, we obtain that

$$\frac{\partial^2 u(x)}{\partial x_j^2} = \int\limits_{\mathbb{R}^n} \frac{\Omega_j(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y := \lim_{\epsilon\to 0^+} \int\limits_{|x-y|>\epsilon} \frac{\Omega_j(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y,$$

where $\Omega_j(y) = C_n(1 - n|y|^{-2}y_j^2)$. If we denote by Σ_{n-1} the unit sphere , it is easy to prove that Ω_j satisfies the following properties:

(1) $\Omega_j(\lambda y) = \Omega_j(y)$, for all $\lambda > 0$;

(2) $\int\limits_{\Sigma_{n-1}} \Omega_j(y')\,\mathrm{d}\sigma(y') = 0$;

(3) $\Omega_j \in L^1(\Sigma_{n-1})$.

If we set

$$T_j f(x) = \lim_{\epsilon\to 0^+} \int\limits_{|x-y|>\epsilon} \frac{\Omega_j(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y,$$

then, L^p regularity of solution of equation $\Delta u = f$ is converted on the L^p boundedness of the operator T_j.

2.1. Calderón-Zygmund singular integral operators.

Definition 2.1. *Let us assume that $K(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ and satisfies the following conditions:*

(1) *$|K(x)| \le B|x|^{-n}$, for all $x \neq 0$;*

(2) *$\int\limits_{r\le|x|\le R} K(x)\,\mathrm{d}x = 0$ for $0 < r < R < \infty$;*

(3) *$\int\limits_{|x|\ge 2|y|} |K(x-y) - K(x)|\,\mathrm{d}x \le B$ for all $y \neq 0$.*

Then K is said to be a Calderón-Zygmund kernel, where B is a constant independent of x and y. Condition (3) is called Hörmander's condition.

Theorem 2.2 (**[50]**). *Let us assume that K is a Calderón-Zygmund kernel. For $\epsilon > 0$ and $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, let*

$$T_\epsilon f(x) = \int\limits_{|y|\ge\epsilon} f(x-y)K(y)\,\mathrm{d}y.$$

Then, the following statements hold:

- *$\|T_\epsilon f\|_p \le A_p\|f\|_p$, where A_p is independent of ϵ and f.*
- *For any $f \in L^p(\mathbb{R}^n)$, $\lim\limits_{\epsilon\to 0^+} T_\epsilon f$ exists in the sense of L^p norm. That is, there exists a linear operator T such that*

$$Tf(x) = P.V.\int\limits_{\mathbb{R}^n} f(x-y)K(y)\,\mathrm{d}y. \tag{2.3}$$

- *$\|Tf\|_p \le A_p\|f\|_p$.*

The linear operator T defined by Theorem 2.2 is called the Calderón-Zygmund singular integral operator. T_ϵ is also called the truncated operator of T.

Let us now consider a dilation δ_ϵ in $\mathbb{R}^n$. Define $\delta_\epsilon f(x) = f(\epsilon x)$ for $\epsilon > 0$ and $x \in \mathbb{R}^n$. Suppose that $Tf = K * f$, and T commute with the dilation, i.e., $T\delta_\epsilon = \delta_\epsilon T$. Then, the kernel $K(x)$ of T satisfies

$$K(\epsilon x) = \epsilon^{-n} K(x). \tag{2.4}$$

Formula (2.4) shows that K is homogeneous of degree $-n$. Thus, we can rewrite $K(x)$ as $\frac{\Omega(x)}{|x|^n}$, where Ω satisfies the homogeneous condition of degree zero, i.e., $\Omega(\lambda x) = \Omega(x)$, for every $\lambda > 0$ and $x \neq 0$. In this case, due to conditions (1) and (3), $\Omega(x')$ in Definition 2.1, $\Omega(x')$ should satisfy:

- $|K(x)| \leq \frac{B}{|x|^n} \quad \Leftrightarrow \quad |\Omega(x')| \leq B$, for every $x' \in \Sigma_{n-1}$;
- $\displaystyle\int\limits_{r \leq |x| \leq R} K(x)\,\mathrm{d}x = 0 \quad \Leftrightarrow \quad \int\limits_{\Sigma_{n-1}} \Omega(x')\,\mathrm{d}\sigma(x') = 0.$
- condition (3) will be changed to a stronger $L^\infty-$Dini's condition

$$\int_0^1 \frac{\omega_\infty(\delta)}{\delta}\,\mathrm{d}\delta < \infty,$$

where

$$\omega_\infty(\delta) = \sup_{\substack{x',y' \in \Sigma_{n-1} \\ |x'-y'|<\delta}} |\Omega(x') - \Omega(y')|.$$

Note that the condition

$$\int\limits_{\Sigma_{n-1}} \Omega(x')\,\mathrm{d}\sigma(x') = 0$$

is a consequence of the following equality:

$$\int\limits_{r \leq |x| \leq R} K(x)\,\mathrm{d}x = \int\limits_{r \leq |x| \leq R} \frac{\Omega(x)}{|x|^n}\,\mathrm{d}x = \int_r^R \int\limits_{\Sigma_{n-1}} \Omega(x')\,\mathrm{d}\sigma(x')\frac{\mathrm{d}r}{r} = \ln\frac{R}{r} \int\limits_{\Sigma_{n-1}} \Omega(x')\,\mathrm{d}\sigma(x').$$

Theorem 2.3 (**[50]**)**.** *Let us assume that $\Omega(x)$ is a homogeneous bounded function of degree 0 on $\mathbb{R}^n$ such that*

$$\int\limits_{\Sigma_{n-1}} \Omega(x')\,\mathrm{d}\sigma(x') = 0 \tag{2.5}$$

and

$$\int_0^1 \frac{\omega_\infty(\delta)}{\delta}\,\mathrm{d}\delta < \infty. \tag{2.6}$$

Let

$$T_\epsilon f(x) = \int\limits_{|y|\geq\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\,\mathrm{d}y$$

for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. *Then, the following three statements hold:*

- $\|T_\epsilon f\|_p \leq A_p \|f\|_p$, *where* A_p *is independent of* ϵ *and* f.
- *There exists a linear operator* T *such that* $\lim\limits_{\epsilon\to 0^+} T_\epsilon f(x) = Tf(x)$ *in* L^p *norm.*
- $\|Tf\|_p \leq A_p \|f\|_p$.

If $K(x) = \frac{\Omega(x)}{|x|^n}$, where Ω is homogeneous of degree zero, then T_Ω defined by

$$T_\Omega f(x) = P.V. \int\limits_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)\,\mathrm{d}y \tag{2.7}$$

is also called *singular integral operator with homogeneous kernel*.

Both Theorem 2.2 and Theorem 2.3 show that the L^p-norm limit of Calderón-Zygmund singular integral operator and singular integral operator with homogeneous kernel exist while considering them as the truncated operator family, and they are both operators of type (p, p), $1 < p < \infty$.

A natural question is whether the limit of $\{T_\epsilon f(x)\}$ in pointwise sense exists for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. It is possible to give an affirmative answer to the above question introducing the maximal operator of singular integral operator. Precisely, suppose that T_Ω is the singular integral operator defined by (2.7) and let Ω be a homogeneous function of degree 0 on $\mathbb{R}^n$ and satisfy (2.5) and (2.6). For $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, let us define the maximal singular integral operator as

$$T^*_\Omega f(x) = \sup_{\epsilon>0} |T_{\Omega,\epsilon} f(x)|,$$

where $T_{\Omega,\epsilon}$ is the truncated operator of T_Ω defined by

$$T_{\Omega,\epsilon} f(x) = \int\limits_{|y|\geq\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\,\mathrm{d}y, \qquad \epsilon > 0.$$

Furthermore, it is possible to show the weak $(1, 1)$ boundedness of T_Ω.

For more details we refer the reader to **[50]**.

3. Fractional integral operators

In the last decades there is a significant interest in the study of an important class of convolution operators known as *fractional integral operators*. The behavior of these operators on functions in the L^p spaces is of particular interest. In addition, a number of closely related topics dealing with how far a function is from its integral average are treated. The classes of Hölder continuous functions as well as the class of functions having bounded mean oscillation arise naturally in this context.

Let f be a real-valued measurable function on $\mathbb{R}^n$, $n \geq 1$, and let $0 < \alpha < n$. The *fractional integral* or *Riesz potential of f of order α* is defined as follows:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y, \qquad x \in \mathbb{R}^n$$

provided the integral above exists.

By allowing f to vary, the mapping defined by

$$I_\alpha : f \to I_\alpha f,$$

that is, the convolution operator with kernel $|x|^{\alpha-n}$, is called the *fractional integral operator of order α*.

The case $\alpha = 1$ play an important role, although the theory for general α, with $0 < \alpha < n$, was extensively studied by a lot of authors and it is very interesting nowadays.

Following **[84]**, as a motivation for studying fractional integrals, we begin by deriving a subrepresentation formula for any sufficiently smooth function f in terms of the Riesz potential of order $\alpha = 1$ of the first partial derivatives of f.

In $\mathbb{R}$ the situation is very simple. It is well known that an absolutely continuous function f defined on an interval $[a,b] \subset \mathbb{R}$ satisfies

$$f(x) - f(y) = \int_y^x f'(t) \, \mathrm{d}t, \qquad x, y \in [a,b].$$

In particular, taking the absolute value of both sides and integrating in y from a to b, we obtain the inequality

$$\frac{1}{b-a} \int_a^b |f(x) - f(y)| \, \mathrm{d}y \leq \int_a^b |f'|, \qquad x \in [a,b].$$

Moreover, since

$$f(x) - \frac{1}{b-a} \int_a^b f(y) \, \mathrm{d}y = \frac{1}{b-a} \int_a^b [f(x) - f(y)] \, \mathrm{d}y,$$

we also obtain the pointwise estimate

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(y)\,\mathrm{d}y \right| \le \int_a^b |f'|, \qquad x \in [a,b].$$

In order to derive analogues of these inequalities in $\mathbb{R}^n$, we initially assume that f is a function defined in an open ball $B \subset \mathbb{R}^n$ and that f belongs to the class $C^1(B)$ of functions with continuous first partial derivatives in B. It is worth pointing out that the C^1 restriction can be weakened (see Theorem 15.16 in [**84**]).

The gradient vector of such an f will be denoted, as usual, by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

and its magnitude is

$$|\nabla f| = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2}.$$

We emphasize that since B is open, if $f \in C^1(B)$, neither f nor $|\nabla f|$ may belong to $L^1(B)$.

We have the following result.

Theorem 3.1 (Subrepresentation Formula, [**84**]). *Let B be an open ball in $\mathbb{R}^n$ and $f \in C^1(B)$ Then*

$$\frac{1}{|B|} \int_B |f(x) - f(y)|\,\mathrm{d}y \le c_n \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y, \qquad x \in B, \tag{3.8}$$

where c_n is a constant that depends only on n. If in addition $f \in L^1(B)$, then

$$|f(x) - f_B| \le c_n \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y, \qquad x \in B, \tag{3.9}$$

where $f_B = \frac{1}{|B|} \int_B f(y)\,\mathrm{d}y$ is the integral average of f on B.

We remark that the integrals on the right sides of (3.8) and (3.9), except that their domain of integration is B and not $\mathbb{R}^n$, are $I_1(|\nabla f|)(x)$. Indeed, f is assumed to be defined only on B. If g is a function defined on B, but not necessary outside B, it is possible to extend g to $\mathbb{R}^n$ defining it to be 0 outside B. It is usual to denote this extension by $g\chi_B$:

$$(g\chi_B)(x) = \begin{cases} g(x) & \text{if } x \in B \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B \end{cases}.$$

Taking into account the extension above, the integrals on the right sides of (3.8) and (3.9) are simply $I_1(|\nabla f|\chi_B)(x)$.

Now, we state a corollary of Theorem 3.1 that gives analogues of the subrepresentation formula (3.9) without the integral average f_B on the left side.

Corollary 3.2 ([**84**]). *Let us assume that B is an open ball in $\mathbb{R}^n$ and $f \in C^1(B)$.*

(1) *If $f = 0$ in a measurable set $E \subset B$ satisfying $|E| \geq \gamma|B|$ for some constant $\gamma > 0$, then*

$$|f(x)| \leq \frac{c_n}{\gamma} \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}} \, \mathrm{d}y, \qquad x \in B,$$

where c_n is a constant that depends only on n.

(2) *If f has compact support in B, then*

$$|f(x)| \leq c_n \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}} \, \mathrm{d}y, \qquad x \in B,$$

where c_n is a constant that depends only on n.

The significance of the subrepresentation formulas in Theorem 3.1 and Corollary 3.2 can be understood considering, for instance, the behaviour of L^q norm of $I_\alpha f$ when $f \in L^p$. For example, it is possible to bound $L^q(B)$ norms of $f - f_B$ by $L^p(B)$ norms of $|\nabla f|$ for suitable values of p and q. These inequalities are very famous and are called Poincaré-Sobolev estimates. We refer the reader to Chapter 15 of [**84**] for further details on Poincaré-Sobolev estimates under less restrictive smoothness assumptions on f than continuous differentiability.

3.1. L^p−estimates for I_α. The next well known theorem states that I_α is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. As usual, we use the notation $\|f\|_p$ for the $L^p(\mathbb{R}^n)$ norm of f, $1 \leq p \leq \infty$.

Theorem 3.3 (Hardy-Littlewood, Sobolev, [**84**]). *Let*

$$0 < \alpha < n, \quad 1 \leq p < \frac{n}{\alpha} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then, for every $f \in L^p(\mathbb{R}^n)$, I_α exists a.e. and is measurable in $\mathbb{R}^n$. Moreover,

(1) *if $1 < p < \frac{n}{\alpha}$, then*

$$\|I_\alpha f\|_q \leq c\|f\|_p$$

for a constant c that depends only on α, n and p.

(2) *if* $p = 1$, *then*

$$|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| \leq \left(\frac{c}{\lambda}\|f\|_1\right)^q, \qquad \left(q = \frac{n}{n-\alpha}\right),$$

for a constant c *that depends only on* α *and* n.

Hardy and Littlewood considered the case $n = 1$ and Sobolev the case $n > 1$. When $p > 1$, Thorin obtained estimates and the case $p = 1$ was studied by Zygmund.

It is possible to obtain some estimates in the case $p = \frac{n}{\alpha}$. Precisely, in **[84]** are shown some variants of Theorem 3.3 for the case $p = \frac{n}{\alpha}$ either by restricting I_α to the subspace of compactly supported $f \in L^{n\alpha}(\mathbb{R}^n)$ or modifying the definition of I_α for general $f \in L^{n\alpha}(\mathbb{R}^n)$. These results have been extensively studied and are often called *Trudinger estimates* or *Moser-Trudinger type estimates*.

However, the norm inequality

$$\|I_\alpha f\|_q \leq c\|f\|_p, \qquad \forall f \in L^p(\mathbb{R}^n) \tag{3.10}$$

for some constant c independent of f, holds only for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. We refer the reader to **[84]** for some comments and examples that explain why the restriction on p and q mentioned above are necessary for the validity of (3.10).

3.2. Fractional maximal operator. Let us introduce the fractional maximal operator M_α. For $0 < \alpha < n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define M_α by

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|y|\leq r} |f(x-y)|\, \mathrm{d}y.$$

An equivalent definition of M_α is

$$M_\alpha(f)(x) = \sup_{B(x,r)} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int\limits_{B(x,r)} |f(y)|\, \mathrm{d}y,$$

where the supremum is taken over all balls $B(x,r)$ in $\mathbb{R}^n$ with center x and radius r.

The fractional maximal operator M_α will be dominate by I_α in some sense. Precisely, for $0 < \alpha < n$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$M_\alpha f(x) \leq I_\alpha(|f|)(x).$$

In fact, for fixed $x \in \mathbb{R}^n$ and $r > 0$, we have that

$$I_\alpha(|f|)(x) = \int\limits_{\mathbb{R}^n} \frac{|f(x-y)|}{|y|^{n-\alpha}}\, \mathrm{d}y \geq \int\limits_{\mathbb{R}^n} \frac{|f(x-y)|}{|y|^{n-\alpha}}\, \mathrm{d}y \geq \frac{1}{r^{n-\alpha}} \int\limits_{|y|\leq r} |f(x-y)|\, \mathrm{d}y.$$

The desired assertion follows from taking the supremum for $r > 0$ of the inequality above.

The reverse inequality does not hold in general, but the two integral operators are comparable in norm.

Theorem 3.4 ([55, 1, 2]). *For $1 < p < \infty$ and $0 < \alpha < n$, there exists a constant $C_{a,p}$ such that*

$$\|I_\alpha f\|_q \leq C_{\alpha,p}\|M_\alpha f\|_p.$$

The next theorem shows that M_α is of type (p, q) and of weak type $(1, \frac{n}{n-\alpha})$.

Theorem 3.5 ([50]). *Let us assume that $0 < \alpha < n$, $1 \leq p \leq \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

(1) *If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \frac{n}{\alpha}$, then*

$$\|M_\alpha f\|_q \leq C\|f\|_p.$$

(2) *If $f \in L^1(\mathbb{R}^n)$, then for any $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}| \leq \left(\frac{C}{\lambda}\|f\|_1\right)^{\frac{n}{n-\alpha}}.$$

The above constant C only depends on α, n, p.

4. Bounded and Vanishing Mean Oscillation

4.1. BMO class. This section concerns with a class of functions that satisfy particular mean oscillation inequalities. Precisely, we list some well known properties of functions f such that

$$\frac{1}{|B|}\int_B |f - f_B|\,\mathrm{d}x \leq c, \qquad B \subset \mathbb{R}^n, \tag{4.11}$$

where B range in the class of the balls of $\mathbb{R}^n$.

Such f are said to belong to the class $BMO(\mathbb{R}^n)$ of functions having (uniformly) *bounded mean oscillation on* $\mathbb{R}^n$. They can be characterized in terms of the size of the distribution function of $|f - f_B|$ on B, rather than in terms of a pointwise condition.

Equivalently, if we denote

$$\|f\|_* = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f - f_B|\,\mathrm{d}x, \tag{4.12}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, then $f \in BMO(\mathbb{R}^n)$ means that f is locally integrable and $\|f\|_* < \infty$.

Let us observe that by the definition of BMO and the sharp maximal function, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$f \in BMO(\mathbb{R}^n) \quad \Leftrightarrow \quad M^\sharp f \in L^\infty(\mathbb{R}^n).$$

Note that if f and g are two generic locally integrable functions, then $\|f+g\|_* \leq \|f\|_* + \|g\|_*$ and $\|cf\|_* = |c|\,\|f\|_*$ for any constant c. However, $\|\cdot\|_*$ is not a norm in the usual sense since $\|f\|_* = 0$ if and only if f is constant a.e. on $\mathbb{R}^n$. In order to turn BMO into a complete normed space it is enough to quotient out these constant functions and it is precisely this quotient space that in the sequel is understood by BMO.

The space BMO was introduced by F. John and L. Nirenberg in **[45]** motivated by earlier works of John on uniqueness of solutions to some nonlinear PDEs arising from the theory of elasticity. It has since found a central position in analysis, particularly in light of the discovery of C. Fefferman that identifies BMO as the dual of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.

It easy to see that the following facts hold.

- If $f \in BMO(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, then $f(\cdot - h)$, the translation of f, satisfies $f(\cdot - h) \in BMO(\mathbb{R}^n)$ and
$$\|f(\cdot - h)\|_* = \|f\|_*.$$
- If $f \in BMO(\mathbb{R}^n)$ and $\lambda > 0$, then $f(\lambda\cdot) \in BMO(\mathbb{R}^n)$ and
$$\|f(\lambda\cdot)\|_* = \|f\|_*.$$

The next proposition shows that if f is measurable and (4.11) holds with the integral average f_B replaced by a different constant depending on B, then f belongs to $BMO(\mathbb{R}^n)$.

Proposition 4.1. *Let f be a measurable function on $\mathbb{R}^n$. If there is a constant C such that*

$$\frac{1}{|B|}\int_B |f(x) - c(f,B)|\,\mathrm{d}x \leq C$$

for every ball $B \subset \mathbb{R}^n$ and for some constant $c(f,B)$ depending on f and B, then $f \in BMO(\mathbb{R}^n)$. Moreover, $\|f\|_* \leq 2C$.

From Proposition 4.1, it follows that balls can be replaced by cubes in the definition on $BMO(\mathbb{R}^n)$. Precisely, let us assume that f is a locally integrable function and satisfies the analogue of condition (4.11) for cubes,

$$\|f\|_{**} := \sup \frac{1}{|Q|} \int_{Q} |f(x) - f_Q| \, \mathrm{d}x < \infty,$$

where the supremum is taken over all cubes Q with edges parallel to the coordinate axes and, as usual, $f_Q = \frac{1}{|Q|} \int_Q f$. Then, given any ball B, by enclosing B in a cube Q with $|Q| \le c_n |B|$, we obtain

$$\int_B |f - f_Q| \, \mathrm{d}x \le \int_Q |f - f_Q| \, \mathrm{d}x \le \|f\|_{**} |Q| \le c_n \|f\|_{**} |B|.$$

It follows from Proposition 4.1 with $c(f, B) = f_Q$ that $f \in BMO(\mathbb{R}^n)$ and $\|f\|_* \le 2c_n \|f\|_{**}$. The converse is also true, that is, the definition of $BMO(\mathbb{R}^n)$ using balls implies the analogous definition using cubes and $\|f\|_{**} \le c\|f\|_*$ for some constant c that depends only on n.

It is easy to see that $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. In fact, if $f \in L^\infty(\mathbb{R}^n)$, then f is locally integrable and, for any ball B, we have

$$\frac{1}{|B|} \int_B |f(x) - f_B| \, \mathrm{d}x \le \|f - f_B\|_\infty \le \|f\|_\infty + |f_B| \le 2\|f\|_\infty.$$

Hence, $f \in BMO(\mathbb{R}^n)$ and $\|f\|_* \le 2\|f\|_\infty$.

However, the inclusion $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ is proper. For example, the essentially unbounded function $\ln|x|$ is of bounded mean oscillation on $\mathbb{R}^n$. Furthermore, if $\lambda \in \mathbb{R} \setminus \{0\}$, then $|x|^\lambda \notin BMO(\mathbb{R}^n)$ and there are also functions having compact support that belong to $L^p(\mathbb{R}^n)$ for all p, $0 < p < \infty$, but do not belong to $BMO(\mathbb{R}^n)$.

We can now formulate a classical result dealing with functions with bounded mean oscillation.

Theorem 4.2 (John-Nirenberg, **[45]**). *There exist positive constants c_1 and c_2 depending only on n such that if $f \in BMO(\mathbb{R}^n)$, B is a ball in $\mathbb{R}^n$ and $\lambda > 0$, then*

$$|\{x \in B : |f(x) - f_B| > \lambda\}| \le c_1 \left[\exp\left(-\frac{c_2 \lambda}{\|f\|_*} \right) \right] |B|. \tag{4.13}$$

In the Theorem 4.2 we have assumed that $\|f\|_* \neq 0$; otherwise, f is constant a.e. in $\mathbb{R}^n$ and the left side of (4.13) is zero for all B and all $\lambda > 0$.

Thus, from Theorem 4.2, it follows that when f lies in $BMO(\mathbb{R}^n)$, the distribution function of $f - f_B$ decays exponentially as $\lambda \to \infty$. This fact implies the L^p-integrability of f and more, as the following corollaries show.

Corollary 4.3 ([84]). *Let $f \in BMO(\mathbb{R}^n)$ and $1 \leq p < \infty$. There exists a positive constant c depending only on n and p such that for every ball $B \subset \mathbb{R}^n$,*

$$\left(\frac{1}{|B|}\int_B |f - f_B|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq c\|f\|_*.$$

In particular, $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, and for every ball B,

$$\left(\frac{1}{|B|}\int_B |f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq c\|f\|_* + |f_B|.$$

Corollary 4.4 (Exponential integrability, **[84]**)**.** *Let c_1 and c_2 be as in Theorem 4.2. If $f \in BMO(\mathbb{R}^n)$ and c_0 is a positive constant such that $c_0\|f\|_* < c_2$, then*

$$\frac{1}{|B|}\int_B \exp\left(c_0|f - f_B|\right)\,\mathrm{d}x \leq 1 + \frac{c_0 c_1}{c_2 - c_0\|f\|_*}$$

for every ball $B \subset \mathbb{R}^n$. In particular,

$$\int_B \exp(c_0|f|)\,\mathrm{d}x < \infty$$

for every ball $B \subset \mathbb{R}^n$.

4.2. Commutators of singular integral operators. In 1965, Calderón defined the Calderón commutator in studying the boundedness of the Cauchy integral on Lipschitz curves, and its definition is

$$C_{h,\varphi}(f)(x) = P.V.\int_{-\infty}^{\infty} h\left(\frac{\varphi(x) - \varphi(y)}{x - y}\right)\frac{f(y)}{x - y}\,\mathrm{d}y,$$

where $h \in C^\infty(\mathbb{R})$ and φ is a Lipschitz function on $\mathbb{R}$.

It is clear that, if $h(t) = (1 + it)^{-1}$, then $C_{h,\varphi}(f)$ is the Cauchy integral along the curve $y = \varphi(x)$; if $h = 1$, then $C_{h,\varphi}(f)$ is the Hilbert transform; if $h(t) = t^k$, with $k \in \mathbb{N}$, then $C_{h,\varphi}(f)$ is the commutator of degree k of the Hilbert transform about φ.

In 1976, Coifman, Rochberg and Weiss studied the L^p boundedness, $1 < p < \infty$, of the commutator $[b, T_\Omega]$ generated by the Calderón-Zygmund singular integral operator T_Ω and a function b, where $[b, T_\Omega]$ is defined by

$$[b, T_\Omega](f)(x) = P.V.\int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n}[b(x) - b(y)]f(y)\,\mathrm{d}y, \tag{4.14}$$

where Ω satisfies the condition of homogeneity of degree zero

$$Q(\lambda x) = Q(x), \qquad \forall\lambda > 0,\ \forall x \in \mathbb{R}^n, \tag{4.15}$$

the vanishing condition on Σ_{n-1}

$$\int\limits_{\Sigma_{n-1}} \Omega(x')\,\mathrm{d}\sigma(x') = 0, \tag{4.16}$$

moreover, $\Omega \in \mathrm{Lip}1(\Sigma_{n-1})$ and $b \in BMO(\mathbb{R}^n)$. Using the L^p boundedness of the commutator $[b, T_\Omega]$, Coifman, Rochberg and Weiss successfully gave a decomposition of Hardy space $H^1(\mathbb{R}^n)$.

The commutator defined by (4.14) is often called CRW-type commutator and CRW-commutator plays an important role in the study of the regularity of solutions of elliptic partial differential equations of second order.

In this section, as an application of the properties of BMO class, we briefly state an important result on L^p boundedness, $1 < p < \infty$, of CRW-type commutator. Precisely, the following theorem shows that L^p boundedness of the commutator of singular integral operator T_Ω, where Ω is a Lipschitz function, can be used to characterize BMO functions.

Theorem 4.5 ([81, 50]). *Let us assume that $\Omega \in \mathrm{Lip}1(\Sigma_{n-1})$ satisfies (4.15) and (4.16) and let T_Ω be a singular integral operator with kernel Ω. Then the following two statements hold.*

(1) *If $b \in BMO(\mathbb{R}^n)$, then $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*
(2) *Suppose $1 < p_0 < \infty$ and $b \in \bigcup_{q>1} L^q_{\mathrm{loc}}(\mathbb{R}^n)$. If $[b, T_\Omega]$ is bounded on $L^{p_0}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.*

4.3. Commutators of Riesz potential. In this section we show some $(L^p, L^q)-$boundedness results of commutators of Riesz potential I_α. We also emphasize that boundedness of commutators of the integral operator I_α can characterize $BMO(\mathbb{R}^n)$ space.

First we give the definition of the commutator $[b, I_\alpha]$.

Let us assume that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The commutator generated by b and the Riesz potential I_α is defined by

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x) = \int\limits_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} f(y)\,\mathrm{d}y.$$

The following result holds.

Theorem 4.6 ([50, 26]). *Let us assume that $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, Then $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

The necessity of Theorem 4.6 with $n-\alpha$ being an even and the sufficiency of the theorem were first proved by Chanillo ([**14**]) in 1982.

For the commutator $[b, M_\alpha]$ of fractional maximal operator M_α, there exists an analogue of Theorem 4.6. Precisely, $[b, M_\alpha]$ is defined by

$$[b, M_\alpha](f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|y-x|\le r} |b(x)-b(y)||f(y)|\,\mathrm{d}y.$$

Theorem 4.7 ([**50**]). *Let us assume that* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Then,* $[b, M_\alpha]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *if and only if* $b\in BMO(\mathbb{R}^n)$.

4.4. VMO class. Now we introduce a proper subspace of BMO. Precisely we define the VMO class of functions having vanishing mean oscillation. This space was defined firstly by Sarason in [**70**].

Definition 4.8 ([**70**]). *Given a function* $f\in BMO(\mathbb{R}^n)$, *let us set*

$$\eta_f(r) = \sup_{\rho\le r} \frac{1}{|B_\rho|} \int\limits_{B_\rho} |f(x)-f_{B_\rho}|\,\mathrm{d}x, \tag{4.17}$$

where B_ρ *varies in the class of balls of radius* ρ. *We say that* $f\in VMO(\mathbb{R}^n)$ *if*

$$\lim_{r\to 0}\eta_f(r)=0$$

and refer to $\eta_f(r)$ *as* $VMO-$*modulus of the function* f.

In a similar manner, we can set the spaces $BMO(\Omega)$ and $VMO(\Omega)$ of functions defined on a domain $\Omega\subset\mathbb{R}^n$, replacing B in (4.11), (4.12) and B_ρ in (4.17) by the intersections of the respective balls with Ω.

We emphasize that, given a function $f\in VMO(\Omega)$, it is possible to extend it to the whole $\mathbb{R}^n$ preserving the $VMO-$modulus, if the boundary $\partial\Omega$ is $C^{1,1}-$smooth (see [**46**], [**80**]).

It is worth to point reader's attention to some embeddings of VMO and BMO into well known functional spaces. First of all, the space of bounded and uniformly continuous functions belong to VMO. In fact, it suffices to take as VMO modulus the modulus of continuity. Moreover, using the Poincaré inequality, we easily obtain that $W^{1,n}(\mathbb{R}^n)\subset VMO(\mathbb{R}^n)$.

In fact,

$$\frac{1}{|B|}\int\limits_B |f(x)-f_B|\,\mathrm{d}x \le C(n)\left(\int\limits_B |Df(x)|^n\,\mathrm{d}x\right)^{\frac{1}{n}}$$

and the term on the right-hand side of the inequality above, tends to zero as $|B| \to 0$ by the absolute continuity of the integral. The space $W^{1,n}$ is a proper subset of VMO as shows the function $f_\alpha(x) = |\ln|x||^\alpha$ for $\alpha \in (0,1)$. Standard calculations yield that $f_\alpha \in VMO$ for all $\alpha \in (0,1)$, $f_\alpha \in W^{1,n}$ for $\alpha \in (0, 1-\frac{1}{n})$, while $f_\alpha \notin W^{1,n}$ for $\alpha \in [1-\frac{1}{n}, 1)$.

Further on, $W^{\theta,n/\theta}(\mathbb{R}^n) \subset VMO$ for $0 < \theta < 1$. In fact,

$$\begin{aligned}
\frac{1}{|B|}\int_B |f(x) - f_B|\mathrm{d}x &= \left(\frac{1}{|B|}\int_B |f(x) - f_B|^{n/\theta}\mathrm{d}x\right)^{\theta/n} \\
&= \left(\frac{1}{|B|}\int_B \left|\frac{1}{|B|}\int_B (f(x) - f(y))\,\mathrm{d}y\right|^{n/\theta} \mathrm{d}x\right)^{\theta/n} \\
&\le \left(\frac{1}{|B|}\int_B \frac{1}{|B|}\int_B |f(x) - f(y)|^{n/\theta}\,\mathrm{d}x\mathrm{d}y\right)^{\theta/n} \\
&\le C(n)\left(\int_B\int_B \frac{|f(x) - f(y)|^{n/\theta}}{|x-y|^{2n}}\,\mathrm{d}x\mathrm{d}y\right)^{\theta/n}.
\end{aligned}$$

Since $f \in W^{\theta,n/\theta}(\mathbb{R}^n)$ implies

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^{n/\theta}}{|x-y|^{2n}}\,\mathrm{d}x\mathrm{d}y\right)^{\theta/n} < +\infty,$$

by the absolute continuity of the integral, we infer $f \in VMO(\mathbb{R}^n)$.

It is interesting to state a simple and useful criterion, due to Bramanti, to show that a function belong to VMO.

Proposition 4.9. *Let $f(|x|)$ be a radially symmetric function, $f(r) \in C^1(0,R)$ and let*

$$\lim_{r\to 0^+} rf'(r) = 0.$$

Then, $f(|x|) \in VMO$.

The next characterization of VMO belongs to Sarason.

Theorem 4.10 ([70]). *For a function $f \in BMO$, the following conditions are equivalent:*

(1) *$f \in VMO$;*

(2) *f belongs to the BMO closure of the space of bounded and uniformly continuous functions;*

(3) $\lim_{h\to 0} \|f(\cdot - h) - f(\cdot)\|_* = 0.$

Let us observe that condition (3) implies the good behaviour of the mollifiers of VMO functions. Precisely, for a given $f \in VMO$ with modulus $\eta_f(r)$ we can find a sequence $\{f_k\}_{k\in\mathbb{N}} \in C^\infty(\mathbb{R}^n)$ of functions with $\eta_{f_k}(r)$ moduli, such that $f_k \to f$ in BMO as $k \to \infty$ and $\eta_{f_k}(r) \leq \eta_f(r)$ for all integers k.

5. Morrey spaces

Morrey spaces were introduced by Morrey in 1938 (**[54]**) in his work on systems of second order elliptic partial differential equations and together with the now well-studied Sobolev spaces, constitute a very useful family of spaces useful for proving regularity results for solutions to various partial differential equations.

Definition 5.1 (**[54]**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter $0 < \operatorname{diam}\Omega < \infty$. For $1 \leq p < \infty$ and $\lambda \geq 0$, the Morrey space $L^{p,\lambda}(\Omega)$ is the subspace of $L^p(\Omega)$ defined via*

$$L^{p,\lambda}(\Omega) = \left\{ u \in L^p(\Omega) : \|u\|_{L^{p,\lambda}(\Omega)} < \infty \right\},$$

where

$$\|u\|_{L^{p,\lambda}(\Omega)} = \left(\sup_{\substack{x\in\Omega \\ 0<\rho\leq \operatorname{diam}\Omega}} \rho^{-\lambda} \int_{\Omega\cap B(x,r)} |u(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}}. \tag{5.18}$$

Using standard arguments it is easy to see that the quantity defined by (5.18) defines a norm on $L^{p,\lambda}$ and that the resulting normed space is complete, that is, it is a Banach space.

In order to present some embedding results, it is convenient to fix the notation used to indicate the embeddings. In general, if X and Y are two normed linear spaces and there exists a continuous embedding from X into Y, as usual, we write $X \hookrightarrow Y$. If simultaneously $X \hookrightarrow Y$ and $Y \hookrightarrow X$, then we shall write $X \rightleftarrows Y$.

Theorem 5.2 (**[61]**). *Let Ω be a bounded domain of $\mathbb{R}^n$.*

(1) *Let $p \in]1, +\infty[$. Then*

$$L^{p,0}(\Omega) \rightleftarrows L^p(\Omega).$$

(2) *Let $p \in]1, +\infty[$. Then*

$$L^{p,n}(\Omega) \rightleftarrows L^\infty(\Omega).$$

(3) *Let* $1 \leq p \leq q < +\infty$, *let* λ *and* ν *be nonnegative numbers. If*

$$\frac{\lambda - n}{p} \leq \frac{\nu - n}{q},$$

then

$$L^{q,\nu}(\Omega) \hookrightarrow L^{p,\lambda}(\Omega).$$

Remark 5.3. *It is easy to see that* $L^{p,\lambda}(\Omega) = \{0\}$ *for* $\lambda > n$. *Further, it follows from Theorem 5.2 that the collection* $\{L^{p,\lambda}(\Omega)\}_{\lambda \in [0,n]}$ *for fixed* $p \in [1, +\infty[$ *generates a "scale of spaces" between* $L^p(\Omega)$ *and* $L^\infty(\Omega)$.

A lot of authors extended the results on singular and fractional integral operator and their commutators in the framework of classical Morrey spaces. For instance, Di Fazio and Ragusa in **[23]** generalized the classical Fefferman and Stein inequality, studied the boundedness of the fractional maximal operator and, as a consequence, obtained the boundedness of the commutator generated by a Calderón-Zygmund singular integral operator and a function having bounded mean oscillation. Furthermore, in **[23]** the authors gave a strong condition that ensure the boundedness of a commutators generated by a fractional integral operator of order α, I_α, and a function with bounded mean oscillation. Di Fazio and Ragusa in **[23]** obtained necessary and sufficient conditions for which the commutator $[b, I_\alpha]$ is bounded on Morrey spaces for some α. Later, in **[48]**, Komori and Mizuhara refined the results contained in **[23]** by using the duality argument and the factorization theorem for $H^1(\mathbb{R}^n)$.

Precisely, the result on $[b, I_\alpha]$ stated in **[23]** is the following.

Theorem 5.4 (**[23]**)**.** *Let* $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

If $b \in BMO(\mathbb{R}^n)$, *then* $[b, I_\alpha]$ *is a bounded operator from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Conversely, if $n - \alpha$ *is an even integer and* $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$ *for some* p, q, λ *as above, then* $b \in BMO(\mathbb{R}^n)$.

As we can see easily, the conditions for the converse part of Theorem 5.4 are very strong. In fact, when $n = 1, 2$ there does not exist α satisfying the conditions. When $n = 3$, the assumptions are satisfied only for $\alpha = 1$. When $n = 4$, the assumptions are satisfied for $\alpha = 1, 2$.

Komori and Mizuhara in **[48]** weakened the strong condition of Theorem 5.4.

CHAPTER 2

Integral operators on generalized Morrey spaces

This chapter is based on the following publications:

V.S. Guliyev, M.N. Omarova, M.A. Ragusa, A. Scapellato,
Commutators and generalized local Morrey spaces,
J. Math. Anal. Appl. **457** (2018), 1388-1402, http://dx.doi.org/10.1016/j.jmaa.2016.09.070

A. Scapellato,
Some properties of integral operators on generalized Morrey spaces,
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1. Introduction

In this chapter, we study in generalized local Morrey spaces $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ and generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ the boundedness of Hardy-Littlewood maximal operator in terms of sharp maximal function and, as consequence, the boundedness of Commutators of the type

$$[a,K](f) = a\,(K,f) - K\,(a,f),$$

where K is a Calderón-Zygmund singular integral operator, f is in a Generalized Local Morrey Space $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ and the function a belongs to the Bounded Mean Oscillation class (B.M.O.) at first defined by John-Nirenberg.

The Generalized Morrey Spaces $M^{p,\varphi}(\mathbb{R}^n)$ are obtained by replacing in the classical Morrey Space $L^{p,\lambda}(\mathbb{R}^n)$, r^λ by a function φ.

The classical Morrey spaces were introduced by Morrey [**54**] to study the local behavior of solutions to second order elliptic partial differential equations (see e.g.[**49**], [**62**]). For the properties and applications of classical Morrey spaces, we refer the readers to [**24, 29, 37, 54**]. Mizuhara [**53**] and Nakai [**57**] introduced generalized Morrey spaces. Later, Guliyev [**37**] defined the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ with normalized norm.

We point out that φ is a measurable non-negative function and no monotonicity type condition is imposed on it.

We observe that in this chapter we extend results contained in [**23**], basic tool in the subsequent study of regularity results of solutions of partial differential equations of elliptic and parabolic type and systems (see e.g. [**24**], [**25**], [**64**], [**65**] and others). Also, Remark 4.3 can be view as a generalization of a well known inequality by Fefferman and Stein, see [**31**] p.153, and Theorem 4.5, is true under more general hypotheses that can be found in literature, see [**79**] pp.417-418.

2. Definitions and useful tools

We set, throughout the chapter,

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$$

a generic ball in $\mathbb{R}^n$ centered at x with radius r.

We find it convenient to define the generalized Morrey spaces in the following form.

Definition 2.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M^{p,\varphi} \equiv M^{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))}.$$

Also by $WM^{p,\varphi} \equiv WM^{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^p\mathrm{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))} < \infty,$$

where $WL^p(B(x,r))$ denotes the weak L^p-space consisting of all measurable functions f for which

$$\|f\|_{WL^p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL^p(\mathbb{R}^n)} < \infty.$$

According to this definition we recover, for $0 \leq \lambda < n$, the Morrey space $M^{p,\lambda}$ and the weak Morrey space $WM^{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M^{p,\lambda} = M^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM^{p,\lambda} = WM^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

The vanishing Morrey space $VL^{p,\lambda}(\mathbb{R}^n)$ in its classical version was introduced in **[82]**, where applications to PDE were considered. We also refer to **[17]**, **[78]** for some properties of such spaces.

We are ready to give the following definition of Vanishing generalized Morrey spaces, inspired by the classical one of Vanishing Morrey spaces gives by Vitanza and deeply treated in **[82]** and **[83]**.

Definition 2.2. *(Vanishing generalized Morrey space) The vanishing generalized Morrey space $VM^{p,\varphi}(\mathbb{R}^n)$ is defined as the space of functions $f \in M^{p,\varphi}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))} = 0.$$

Definition 2.3. *(Vanishing weak generalized Morrey space) The vanishing weak generalized Morrey space $VWM^{p,\varphi}(\mathbb{R}^n)$ is defined as the space of functions $f \in WM^{p,\varphi}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \tag{2.19}$$

and

$$\sup_{0<r<\infty} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty,$$

which makes the spaces $VM^{p,\varphi}(\mathbb{R}^n)$ and $VWM^{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong then to this space.

The spaces $VM^{p,\varphi}(\mathbb{R}^n)$ and $WVM^{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{p,\varphi}} \equiv \|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))},$$

$$\|f\|_{VWM^{p,\varphi}} \equiv \|f\|_{WM^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))},$$

respectively.

We also use the notation

$$\mathfrak{M}^{p,\varphi}(f;x,r) := \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))}$$

and

$$\mathfrak{M}^{W}_{p,\varphi}(f;x,r) := \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))}$$

for brevity, so that

$$VM^{p,\varphi}(\mathbb{R}^n) = \left\{ f \in M^{p,\varphi}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{p,\varphi}(f;x,r) = 0 \right\}$$

and similarly for $VWM^{p,\varphi}(\mathbb{R}^n)$.

Besides the modular $\mathfrak{M}^{p,\varphi}(f;x,r)$ we also use its least non-decreasing dominant

$$\widetilde{\mathfrak{M}}^{p,\varphi}(f;x,r) = \sup_{0<t<r} \mathfrak{M}^{p,\varphi}(f;x,t), \tag{2.20}$$

which may be equivalently used in the definition of the vanishing spaces, since

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{p,\varphi}(f;x,r) = 0 \not\Leftrightarrow \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \widetilde{\mathfrak{M}}^{p,\varphi}(f;x,r) = 0.$$

Let us consider, for $f \in L^1_{loc}(\mathbb{R}^n)$, the *Hardy-Littlewood maximal function* M as

$$Mf(x) = \sup_{B(x,r)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy,$$

where $B(x,r)$ is the ball centered at x of radius r (see **[76]**, pp. 8-9).

Remark 2.4. *We observe that the properties stated for M hold for the larger "uncentred" maximal function $\widetilde{M}f$ defined by*

$$\widetilde{M}f(x) = \sup_{x\in B} \frac{1}{|B|} \int_{B} |f(y)|\,dy,$$

where the supremum is taken, not just over all balls B centred in x but to all balls B containing x.

It is true because, for every x, we can write

$$(Mf)(x) \le (\widetilde{M}f)(x)$$

and also exists a constant c greater than 1 such that

$$(\widetilde{M}f)(x) \le c\,(Mf)(x).$$

For this observations see **[76]** *p.13 (also* **[79]** *p.80).*

Two variants of Hardy-Littlewood Maximal function M, are the following *sharp maximal function*

$$f^{\#}(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy,$$

where the supremum is taken over the balls B containing x (see **[76]**, p.146) and the *fractional maximal function* $M_\eta f$ used, for instance, by Muchkenhoupt and Wheeden in their relevant results contained in **[55]**:

$$M_\eta f(x) = \sup_{x \in B} \frac{1}{|B|^{1-\eta}} \int_B |f(y)| \, dy,$$

where $f \in L^1_{loc}(\mathbb{R}^n)$, $0 < \eta < 1$ and the supremum is taken over the balls B containing x.

Let K be a Calderón-Zygmund singular integral operator (see e.g. **[62]**). Useful in the sequel is the following *commutator* between the operator K and the multiplication operator by a locally integrable function a on $\mathbb{R}^n$:

$$[a, K](f)\, x = a(x) \left(K f \right)(x) - K \left(a f \right)(x),$$

for suitable functions f. Later, is useful to consider the function a in the space BMO of Bounded Mean Oscillation functions (see **[45]**).

Lemma 2.5. *(see* **[23]**, *Lemma 1). Let K be a Calderón-Zygmund singular integral operator, $1 < q < s < p < +\infty$, $0 < \lambda < n$ and $a \in BMO(\mathbb{R}^n)$.*

Then, there exists a constant $c \geq 0$ independent of a and f such that

$$\left([a, K](f) \right)^{\#}(x) \leq c \, \|a\|_* \left\{ \left(M |Kf|^q \right)^{\frac{1}{q}}(x) + \left(M |f|^s \right)^{\frac{1}{s}}(x) \right\}$$

for a. a. $x \in \mathbb{R}^n$ and every $f \in M^{p,\lambda}(\mathbb{R}^n)$.

The proof of this Lemma is similar to that one contained in **[79]**, pg.418–419, due to J.-O. Strömberg, it could be generalized for functions $f \in M^{p,\varphi}(\mathbb{R}^n)$.

In the sequel we need the following supremal inequalities.

Let v be a weight. We denote by $L^\infty_v(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L^\infty_v(0,\infty)} = \sup_{t>0} v(t)|g(t)|$$

and $L^\infty(0,\infty) \equiv L^\infty_1(0,\infty)$.

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in **[9]**.

Theorem 2.6. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{ß,v_1}(0,\infty)$ to $L_{ß,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty.$$

3. Generalized Local Morrey spaces and Vanishing Generalized Local Morrey spaces

Definition 3.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM^{p,\varphi} \equiv LM^{p,\varphi}(\mathbb{R}^n)$ the local generalized Morrey space, the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM^{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1}\, |B(0,r)|^{-\frac{1}{p}}\, \|f\|_{L^p(B(0,r))}.$$

Also, by $WLM^{p,\varphi} \equiv WLM^{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1}\, |B(0,r)|^{-\frac{1}{p}}\, \|f\|_{WL^p(B(0,r))} < \infty.$$

Definition 3.2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{p,\varphi}_{\{x_0\}} \equiv LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ the local generalized Morrey space as the class of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM^{p,\varphi}_{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM^{p,\varphi}}.$$

Also, by $WLM^{p,\varphi}_{\{x_0\}} \equiv WLM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ *we denote the weak generalized Morrey space of all functions* $f \in WL^p_{\rm loc}(\mathbb{R}^n)$ *for which*

$$\|f\|_{WLM^{p,\varphi}_{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM^{p,\varphi}} < \infty.$$

According to this definition we recover, for $0 \le \lambda < n$, the local Morrey space $LM^{p,\lambda}_{\{x_0\}}$ and weak local Morrey space $WLM^{p,\lambda}_{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM^{p,\lambda}_{\{x_0\}} = LM^{p,\varphi}_{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLM^{p,\lambda}_{\{x_0\}} = WLM^{p,\varphi}_{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [**85, 86**] lookes for a way to describe the behavior of a function at the infinity. The conditions he considers are related to appropriate weighted L^q spaces. Beurling [**7**] extends this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L^q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [**32**] observes that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\ge 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L^q(\mathbb{R}^n)}, \tag{3.21}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$, $k = 1,2,\ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L^q(\mathbb{R}^n)}. \tag{3.22}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (3.21) and (3.22) instead of $k \ge 0$ there.

If $\lambda < 0$, then $LM^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $LM^{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$.

Alvarez, Guzman-Partida and Lakey [**6**] in order to study the relationship between central BMO spaces and Morrey spaces, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\lambda}(\mathbb{R}^n)$.

The following lemma, useful in itself, shows that the quasi-norm of the local Morrey space $LM^{p,\lambda}(\mathbb{R}^n)$, $\lambda \ge 0$ is equivalent to the quasi-norm $\dot{B}_{p,\lambda}(\mathbb{R}^n)$:

$$\|f\|_{\dot{B}_{p,\lambda}} = \sup_{k\in\mathbb{Z}} 2^{-\frac{k\lambda}{p}} \|f\chi_k\|_{L_p},$$

where χ_k is the characteristic function of the annulus $B(0,2^k) \setminus B(0,2^{k-1})$, $k \in \mathbb{Z}$.

Lemma 3.3. *For $0 < p \le \infty$, $\lambda \ge 0$, the quasi-norm $\|f\|_{LM^{p,\lambda}}$ is equivalent to the quasi-norm $\|f\|_{\dot{B}_{p,\lambda}}$.*

Proof. Let $0 < p \le \infty$, $\lambda \ge 0$ and $f \in LM^{p,\lambda}(\mathbb{R}^n)$. Then

$$\|f\|_{\dot{B}_{p,\lambda}} \le \sup_{k\in\mathbb{Z}} (2^k)^{-\frac{\lambda}{p}} \|f\|_{L_p(B(0,2^k))} \le \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(0,r))} = \|f\|_{LM^{p,\lambda}}.$$

On the other hand, for $0 < p < \infty$,

$$\begin{aligned}
\|f\|^p_{LM^{p,\lambda}} &= \sup_{k\in\mathbb{Z}} \sup_{2^{k-1}<r\le 2^k} r^{-\lambda} \int\limits_{B(0,r)} |f(y)|^p dy \\
&\le 2^\lambda \sup_{k\in\mathbb{Z}} (2^k)^{-\lambda} \int\limits_{B(0,2^k)} |f(y)|^p dy \\
&= 2^\lambda \sup_{k\in\mathbb{Z}} 2^{-k\lambda} \sum_{m=-\infty}^{k} 2^{m\lambda} 2^{-m\lambda} \int\limits_{B(0,2^m)\setminus B(0,2^{m-1})} |f(y)|^p dy \\
&\le 2^\lambda \Big(\sup_{m\in\mathbb{Z}} 2^{-m\lambda} \int\limits_{B(0,2^m)\setminus B(0,2^{m-1})} |f(y)|^p dy \Big) \sup_{k\in\mathbb{Z}} \Big(2^{-k\lambda} \sum_{m=-\infty}^{k} 2^{m\lambda} \Big) \\
&= \frac{2^\lambda}{1-2^{-\lambda}} \|f\|^p_{\dot{B}_{p,\lambda}}.
\end{aligned}$$

So for $0 < p < \infty$

$$\|f\|_{LM^{p,\lambda}} \le 2^{\frac{\lambda}{p}} \left(1-2^{-\lambda}\right)^{-\frac{1}{p}} \|f\|_{\dot{B}_{p,\lambda}}.$$

A similar argument shows that

$$\|f\|_{LM_{ß,\lambda}} \le \|f\|_{\dot{B}_{ß,\lambda}}.$$

$\square$

The quasi-norms $\|f\|_{\dot{B}_{p,\lambda}}$ in the case $\lambda = n$ were investigated by Beurling **[7]**, Feichtinger **[32]** and others.

The following statement is proved in **[35]** (see also **[36, 37, 38]**).

Theorem 3.4. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int\limits_r^\infty \varphi_1(x_0,t) \frac{dt}{t} \le C\, \varphi_2(x_0,r), \tag{3.23}$$

where C does not depend on r. Let also K be a Calderón-Zygmund singular integral operator. Then the operator K is bounded from $LM^{p,\varphi_1}_{\{x_0\}}$ to $LM^{p,\varphi_2}_{\{x_0\}}$ for $p > 1$ and from $LM^{1,\varphi_1}_{\{x_0\}}$ to $WLM^{1,\varphi_2}_{\{x_0\}}$ for $p = 1$.

The following statement, containing results obtained in [**53**], [**57**] is proved in [**35**] (see also [**36, 37**]).

Corollary 3.5. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(x,t)\, \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.24}$$

where C does not depend on x and r. Let also K be a Calderón-Zygmund singular integral operator. Then, the operator K is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$.

4. Results

Theorem 4.1. *For any fixed $x_0 \in \mathbb{R}^n$, $r > 0$, $f \in L^q_{\rm loc}(\mathbb{R}^n)$ and $1 < q < +\infty$*

$$\begin{aligned} \|Mf\|_{L^q(B(x_0,r))} &\le c\, r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))} \\ &\le c\, r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f^\sharp\|_{L^q(B(x_0,t))}, \end{aligned} \tag{4.25}$$

and for all $x_0 \in \mathbb{R}^n$, $r > 0$ and $f \in L^1_{\rm loc}(\mathbb{R}^n)$

$$\begin{aligned} \|Mf\|_{WL^1(B(x_0,r))} &\le c\, r^n \sup_{t>2r} t^{-n}\, \|f\|_{L^1(B(x_0,t))} \\ &\le c\, r^n \sup_{t>2r} t^{-n}\, \|f^\sharp\|_{L^1(B(x_0,t))}, \end{aligned} \tag{4.26}$$

where c is independent of f, x_0 and r.

Proof. Inequalities (4.25) and (4.26) are consequence of Lemma 3.3 in [**5**] and the following inequality

$$\|f\|_{L^q(B(x_0,t))} \le \|f^\sharp\|_{L^q(B(x_0,t))}$$

which is contained in [**31**].

□

Theorem 4.2. *Let $x_0 \in \mathbb{R}^n$, $1 \le q < \infty$ and the functions φ_1, φ_2 satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{q}}} \le C\, \varphi_2(x_0,r), \tag{4.27}$$

where C does not depend on r. Then, for $1 < q < \infty$ the maximal operator M is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $LM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator M is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $WLM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Mf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c\, \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\, \|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f and for $1 \le q < \infty$

$$\|Mf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} \le c\, \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\, \|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f.

Proof. By Theorems 4.1 and 2.6 we get

$$\begin{aligned}\|Mf\|_{LM^{q,\varphi_2}_{\{x_0\}}} &\le c \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))} \\ &\le c \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))} \\ &= c\, \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\, \|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},\end{aligned}$$

where c does not depend on x_0 and f, if $1 \le q < \infty$ and

$$\begin{aligned}\|Mf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} &\le c \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))} \\ &= \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\, \|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},\end{aligned}$$

where c does not depend on x_0 and f, if $1 \le q < \infty$. □

Remark 4.3. *Let $1 \le q < \infty$ and the functions φ_1, φ_2 satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{q}}} \le C\, \varphi_2(x,r), \tag{4.28}$$

where C does not depend on x and r. Then, for $1 < q < \infty$ the maximal operator M is bounded from $M^{q,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator M is bounded from $M^{q,\varphi_1}(\mathbb{R}^n)$ to $WM^{q,\varphi_2}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Mf\|_{M^{q,\varphi_2}} \le c\, \|f\|_{M^{q,\varphi_1}} \le c\, \|f^\sharp\|_{M^{q,\varphi_1}},$$

where c does not depend on f and for $1 \le q < \infty$

$$\|Mf\|_{WM^{q,\varphi_2}} \le c\, \|f\|_{M^{q,\varphi_1}} \le c\, \|f^\sharp\|_{M^{q,\varphi_1}},$$

where c does not depend on f.

Remark 4.4. *Let us consider $x_0 \in \mathbb{R}^n$, $1 < p < +\infty$, $0 < \lambda < n$.*

Then, there exists a nonnegative constant c independent of x_0 and f such that

$$\|M f\|_{LM^{p,\lambda}_{\{x_0\}}} \le c\, \|f\|_{LM^{p,\lambda}_{\{x_0\}}} \le c\, \|f^\sharp\|_{LM^{p,\lambda}_{\{x_0\}}}$$

for every $f \in LM^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$.

An improvement of the above theorem it the next result in the Vanishing Generalized Morrey Spaces.

Theorem 4.5. *Let us consider* $1 \leq q < +\infty$, φ_2 *satisfy the condition* (2.19), *the functions* φ_1, φ_2 *satisfy the conditions*

$$c_\delta := \sup_{\delta<t<\infty} \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) < \infty \tag{4.29}$$

for every $\delta > 0$ *and*

$$\frac{\sup\limits_{r<t<\infty} \varphi_1(x,t)}{\varphi_2(x,r)} \leq C_0, \tag{4.30}$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then, for* $1 < q < \infty$ *the maximal operator* M *is bounded from* $VM^{q,\varphi_1}(\mathbb{R}^n)$ *to* $VM^{q,\varphi_2}(\mathbb{R}^n)$ *and, for* $1 \leq q < \infty$, *from* $VM^{q,\varphi_1}(\mathbb{R}^n)$ *to* $VWM^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. The norm inequalities follow from Remark 4.3, so we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_1}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_2}(Mf;x,r) = 0, \tag{4.31}$$

when $1 < q < \infty$, and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_1}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_W^{q,\varphi_2}(Mf;x,r) = 0, \tag{4.32}$$

when $1 \leq q < \infty$. In this estimation we follow some ideas of **[69]**, but base ourselves on Theorem 4.1.

We start with (4.31). We rewrite the inequality (4.25) in the form

$$\mathfrak{M}^{q,\varphi_2}(Mf;x,r) \leq C\frac{\sup_{t>r} t^{-\frac{n}{q}} \|f\|_{L^q(B(x,t))}}{\varphi_2(x,r)}. \tag{4.33}$$

To show that $\sup\limits_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_2}(Mf;x,r) < \varepsilon$ for small r, we split the right-hand side of (4.33):

$$\mathfrak{M}^{q,\varphi_2}(Mf;x,r) \leq C\,[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{4.34}$$

where $\delta_0 > 0$ will be chosen as shown below (we may take $\delta_0 < 1$) and

$$I_{\delta_0}(x,r) := \frac{\sup_{r<t<\delta_0} t^{-\frac{n}{q}} \|f\|_{L^q(B(x,t))}}{\varphi_2(x,r)},$$

$$J_{\delta_0}(x,r) := \frac{\sup_{t>\delta_0} t^{-\frac{n}{q}} \|f\|_{L^q(B(x,t))}}{\varphi_2(x,r)}$$

and it is supposed that $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_1}(f;x,t) < \frac{\varepsilon}{2CC_0}, \quad \text{for all} \quad 0 < t < \delta_0,$$

where C and C_0 are constants from (4.34) and (4.30), which is possible since $f \in VM^{q,\varphi_1}(\mathbb{R}^n)$. Then $\|f\|_{L^q(B(x,t))} < \frac{\varepsilon}{2CC_0}\,\varphi_1(x,t)$ and we obtain the estimate of the first term uniform in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0$$

by (4.30).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.19). We have

$$J_\delta(x,r) \le \frac{c_{\delta_0}\|f\|_{M^{q,\varphi_1}}}{\varphi_2(x,r)},$$

where c_{δ_0} is the constant of (4.29) for $\delta = \delta_0$.

Then, by (2.19) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_1(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{M^{q,\varphi_1}}},$$

which completes the proof of (4.31).

The proof of (4.32) is, line by line, similar to the proof of (4.31).

□

The following theorem was proved by Guliyev in [**38**].

Theorem 4.6. *Let $x_0 \in \mathbb{R}^n$, $1 \le q < \infty$, K be a Calderón-Zygmund singular integral operator and the functions φ_1, φ_2 satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{q}+1}}\,dt \le C\,\varphi_2(x_0,r), \tag{4.35}$$

where C does not depend on r. Then, for $1 < q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $LM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $WLM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Kf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f and for $1 \le q < \infty$

$$\|Kf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f.

The following theorem is valid.

Theorem 4.7. *Let $x_0 \in \mathbb{R}^n$, $1 < q < s < p < +\infty$, K be a Calderón-Zygmund singular integral operator and the function φ satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \le C\, \varphi(x_0,r), \tag{4.36}$$

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \le C\, \varphi(x_0,r) \tag{4.37}$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi(x_0,r), \tag{4.38}$$

where C does not depend on r.

If $a \in BMO(\mathbb{R}^n)$ then, the commutator

$$[a,K](f) \;=\; a\,Kf \;-\; K\,(af)$$

is a bounded operator from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself. Precisely, $\forall f \in LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} \;\le\; c\,\|a\|_*\,\|f\|_{LM^{p,\varphi}_{\{x_0\}}} \le c\;\|a\|_*\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

for some constant $c \ge 0$ independent on a and f.

Proof. Using Lemma 1 in **[23]** and Theorem 4.2 we get, for $1 < q < s < p < \infty$,

$$\begin{aligned}
\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} &\le c \cdot \|M([a,K])\|_{LM^{p,\varphi}_{\{x_0\}}} \\
&\le c \cdot \|\; [a,\,K]^\sharp \;\|_{LM^{p,\varphi}_{\{x_0\}}} \\
&\le c \cdot \|a\|_* \cdot \|\Big(\,M\,|Kf|^q\,\Big)^{\frac{1}{q}} \;+\; \Big(\,M\,|f|^s\,\Big)^{\frac{1}{s}}\,\|_{LM^{p,\varphi}_{\{x_0\}}}.
\end{aligned}$$

Note that from the boundedness of the maximal operator M from $LM^{\frac{p}{q},\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself and from $LM^{\frac{p}{s},\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself, $1 < q < s < p < \infty$ the sufficient conditions are (4.36) and (4.37), consequently (see, Theorem 4.2).

Also, from the boundedness of the Calderón-Zygmund singular integral operator K from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself the sufficient condition is (3.15) (see, Theorem 4.6).

Then, we have

$$\begin{aligned} \|\Big(M\,|Kf|^q \Big)^{\frac{1}{q}}\|_{LM^{p,\varphi}_{\{x_0\}}} &\leq \left(\|M(|Kf|^q)\|_{LM^{\frac{p}{q},\varphi}_{\{x_0\}}} \right)^{\frac{1}{q}} \\ &\leq c\cdot \left(\||Kf|^q\|_{LM^{\frac{p}{q},\varphi}_{\{x_0\}}} \right)^{\frac{1}{q}} \\ &\leq c\||Kf|\|_{LM^{p,\varphi}_{\{x_0\}}} \\ &\leq c\|f\|_{LM^{p,\varphi}_{\{x_0\}}} \end{aligned}$$

and

$$\|(M(|Kf|^q)^{\frac{1}{q}}\|_{LM^{p,\varphi}_{\{x_0\}}} \leq c\ \|f\|_{LM^{p,\varphi}_{\{x_0\}}}.$$

In the same way one can easily see that

$$\|(M(|f|^s)^{\frac{1}{s}}\|_{LM^{p,\varphi}_{\{x_0\}}} \leq c\ \|f\|_{LM^{p,\varphi}_{\{x_0\}}},$$

we get

$$\|[a,K](f)\|_{LM^{q,\varphi}_{\{x_0\}}} \leq c\ \|a\|_*\ \|f\|_{LM^{q,\varphi}_{\{x_0\}}}.$$

So, the theorem was proved. □

Corollary 4.8. *Let $x_0 \in \mathbb{R}^n$, $1 < p < +\infty$, K be a Calderón-Zygmund singular integral operator and the function $\varphi(x_0,\cdot) : (0,\infty) \to (0,\infty)$ be an decreasing function. Assume that the mapping $r \mapsto \varphi(x_0,r)\, r^{\frac{n}{p}}$ is almost increasing (there exists a constant c such that for $s < r$ we have $\varphi(x_0,s)\, s^{\frac{n}{p}} \leq c\varphi(x_0,r)\, r^{\frac{n}{p}}$). Let also*

$$\int_r^\infty \varphi(x_0,t)\,\frac{dt}{t} \leq C\,\varphi(x_0,r), \tag{4.39}$$

where C does not depend on r.

If $a \in BMO(\mathbb{R}^n)$, then the commutator $[a,K]$ is a bounded operator from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself.

From Theorem 4.7 we get the following corollary.

Corollary 4.9. *Let $1 < q < s < p < +\infty$, K be a Calderón-Zygmund singular integral operator and the function φ satisfy the condition .*

$$\sup_{r<t<\infty} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi(x,\tau)\,\tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \leq C\,\varphi(x,r),$$

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty} \varphi(x,\tau)\, \tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \le C\, \varphi(x,r)$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty} \varphi(x,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi(x,r),$$

where C does not depend on x and r.

If $a \in BMO(\mathbb{R}^n)$, then the commutator $[a,K]$ is a bounded operator from $M^{p,\varphi}(\mathbb{R}^n)$ in itself. Precisely, $\forall f \in M^{p,\varphi}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{M^{p,\varphi}} \le c\, \|a\|_*\, \|f\|_{M^{p,\varphi}} \le c\, \|a\|_*\, \|f^\sharp\|_{M^{p,\varphi}},$$

for some constant $c \ge 0$ independent on a and f.

Corollary 4.10. *Let $1 < p < +\infty$, K be a Calderón-Zygmund singular integral operator and the function $\varphi(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be an decreasing function on r. Assume that the mapping $r \mapsto \varphi(x,r)\, r^{\frac{n}{p}}$ is almost increasing on r (there exists a constant c such that for $s < r$ we have $\varphi(x,s)\, s^{\frac{n}{p}} \le c\varphi(x,r)\, r^{\frac{n}{p}}$). Let also*

$$\int_r^\infty \varphi(x,t)\, \frac{dt}{t} \le C\, \varphi(x,r), \tag{4.40}$$

where C does not depend on x and r.

If $a \in BMO(\mathbb{R}^n)$, then the commutator $[a,K]$ is a bounded operator from $M^{p,\varphi}(\mathbb{R}^n)$ in itself.

Remark 4.11. *Note that, Corollaries 4.8, 4.9 and 4.10 are news.*

Remark 4.12. *Note that the condition* (4.28) *in Theorem 4.3 is weaker than the condition* (3.15) *in Theorem 4.6 and the condition* (3.15) *in Theorem 4.6 is weaker than the condition* (4.39) *in Corollary 4.8. Indeed, if condition* (4.39) *holds, then*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le \int_r^\infty \varphi_1(x_0,t) \frac{dt}{t},$$

so conditions (3.15) *holds.*

Also, if condition (3.15) *holds, then for any* $\tau \in (r, \infty)$

$$C\varphi_2(x_0, r) \geq \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \geq \int_\tau^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt$$

$$\geq \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \int_\tau^\infty \frac{dt}{t^{\frac{n}{p}+1}} \approx \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{\tau^{\frac{n}{p}}},$$

so that

$$\sup_{r<\tau<\infty} \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(x_0, s)\, s^{\frac{n}{p}}}{\tau^{\frac{n}{p}}} \leq C \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C\, \varphi_2(x_0, r),$$

so conditions (4.28) *holds.*

On the other hand , the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p}-\beta}}, \quad \varphi_2(r) = r^{-\frac{n}{p}}\left(1 + r^\beta\right) \tag{4.41}$$

for $0 < \beta \leq \frac{n}{p}$ *satisfy condition* (4.28)*, for* $0 < \beta < \frac{n}{p}$ *satisfy condition* (3.15)*, but for* $0 < \beta < \frac{n}{p}$ *do not satisfy condition* (4.39)*. Also, for* $\beta = \frac{n}{p}$ *the pair function* (4.41) *satisfy condition* (4.28)*, but do not satisfy condition* (3.15)*.*

5. Applications to partial differential equations

In the last thirty years a number of papers have been devoted to the study of local and global regularity properties of strong solutions to elliptic equations with discontinuous coefficients. To be more precise, let us consider the second order equation

$$\mathcal{L}u := \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} = f(x) \quad \text{for almost all } x \in \Omega, \tag{5.42}$$

where $\mathcal{L}$ is a uniformly elliptic operator over the bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Regularizing properties of $\mathcal{L}$ in Hölder spaces (i.e. $\mathcal{L}u \in C^\alpha(\overline{\Omega})$ implies $u \in C^{2+\alpha}(\overline{\Omega})$) have been well studied in the case of Hölder continuous coefficients $a_{ij}(x)$. Also, unique classical solvability of the Dirichlet problem for (5.42) has been derived in this case (we refer to **[34]** and the references therein). In the case of uniformly continuous coefficients a_{ij}, an L^p-Schauder theory has been elaborated for the operator $\mathcal{L}$ (**[3, 4, 34]**). In particular, $\mathcal{L}u \in L^p(\Omega)$ always implies that the strong solution to (5.42) belongs to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1, \infty)$.

However, the situation becomes rather difficults if one tries to allow discontinuity at the principal coefficients of $\mathcal{L}$. In general, it well known ([**51**]) that arbitrary discontinuity of a_{ij} implies that the L^p-theory of $\mathcal{L}$ and the strong solvability of the Dirichlet problem for (5.42) break down.

A notable exception of that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). It was shown by G. Talenti ([**77**]) that the solely condition on measurability and boundedness of a_{ij}'s ensures isomorphic properties of $\mathcal{L}$ considered as a mapping from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ into $L^2(\Omega)$.

To handle with the multidimensional case ($n \geq 3$) requires that additional properties on $a_{ij}(x)$ should be added to the uniform ellipticity in order to guarantee that $\mathcal{L}$ possesses the regularizing property in Sobolev functional scales.

In particular, if $a_{ij}(x) \in W^{1,n}(\Omega)$ ([**52**]), or if the difference between the largest and the smallest eigenvalues of $\{a_{ij}(x)\}$ is small enough (the Cordes condition, [**12**]), then $\mathcal{L}u \in L^2(\Omega)$ yields that $u \in W^{2,2}(\Omega)$ and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2-\epsilon, 2+\epsilon)$ with sufficiently small ϵ.

Later (see e.g. [**15**] for an exhaustive presentation) the Sarason class VMO of functions with vanishing mean oscillation was used in the study of local and global Sobolev regularity of the strong solutions to (5.42).

This class of functions was considered by many others. At first, we recall the paper by F. Chiarenza, M. Frasca and P. Longo [**18**], where the authors answer a question raised thirty years before by C. Miranda in [**52**]. In his note he considers a linear elliptic equation where the coefficients a_{ij} of the higher order derivatives are in the class $W^{1,n}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > n$. In [**18**] the authors suppose that $a_{ij} \in VMO$ and prove that Du is Hölder continuous for all $p \in (1, +\infty)$.

In this section we consider the equation (5.42), where f is assumed to be in some Generalized Morrey space $LM^{p,\varphi}_{\{x_0\}}(\Omega)$ and $a_{ij} \in L^\infty(\Omega) \cap VMO$. Are known some regularity results on Morrey spaces $L^{p,\lambda}$ (see [**24**]) of the second derivatives of a solution of the previous equation. In order to obtain local regularity results, we use the boundedness of some integral operators on generalized local Morrey spaces.

We assume the following regularity and ellipticity assumptions on the coefficients of the partial differential equation under consideration:

$$\begin{cases} a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \quad \forall i,j = 1,...,n \\ a_{ij}(x) = a_{ji}(x), \quad \forall i,j = 1,...,n \\ \exists\, \lambda > 0 : \ \lambda^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \end{cases} \tag{5.43}$$

Set η_{ij} for the VMO-modulus of the function $a_{ij}(x)$ and let $\eta(r) = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{1/2}$. Denote by $\tilde{B}$ the subset of B where the second and the third conditions in (5.43) hold. We set

$$\Gamma(x,t) = \frac{1}{(n-2)\omega_n(\det a_{ij}(x))^{\frac{1}{2}}} \left(\sum_{i,j=1}^n A_{ij}(x)t_it_j\right)^{\frac{2-n}{2}}$$

for a.a. $x \in B$, and all $t \in \mathbb{R}^n \setminus \{0\}$, where we denote by A_{ij} the entries of the inverse matrix of the matrix $(a_{ij}(x))_{i,j=1,...,n}$.

Observe that, for any fixed $x_0 \in \tilde{B}$, $\Gamma(x_0, t)$ is a fundamental solution for the operator

$$\mathcal{L}_0 u(x) := \sum_{i,j=1}^n a_{ij}(x_0)u_{x_ix_j}(x),$$

obtained from $\mathcal{L}$ freezing the coefficients in x_0.

Also we set

$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t), \qquad \Gamma_{ij}(x,t) = \frac{\partial}{\partial t_i \partial t_j}\Gamma(x,t).$$

It is well known that $\Gamma_{ij}(x,t)$ are Calderón–Zygmund kernels in the t variable. In fact, they are the first derivatives of a homogeneous function of degree $1-n$.

Hearth of the main results of this section is the following representation formula that, combined with the boundedness result for Calderón-Zygmund singular integral operator and commutators, allows us to obtain a Morrey-type regularity result.

Lemma 5.1 ([18]). *Let $n \ge 3$, B and $(a_{ij})_{i,j=1,...,n}$ as above and $u \in W_0^{2,p}(B)$. Also set*

$$Lu := \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}(x).$$

Then, for a.a. $x \in B$,

$$\begin{aligned} u_{x_ix_j} &= P.V. \int_B \Gamma_{ij}(x, x-y) \left[\sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y))u_{x_hx_k}(y) + \mathcal{L}u(y)\right] \mathrm{d}y \\ &\quad + \mathcal{L}u(x) \int_{|t|=1} \Gamma_i(x,t)t_j \, \mathrm{d}\sigma_t. \end{aligned}$$

The following theorem is a local Morrey type regularity result for solutions of the differential equation under consideration.

Theorem 5.2 ([72]). *Let $x_0 \in \mathbb{R}^n$, the ellipticity assumptions* (5.43) *be true, $1 < q < s < p < \infty$, K be a Calderón-Zygmund singular integral operator and we assume that the function $\varphi(x,r)$, defined on $\mathbb{R}^n \times (0,\infty)$, is positive and measurable and such that the following conditions are fulfilled:*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \leq C\, \varphi(x_0,r), \quad \sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \leq C\, \varphi(x_0,r)$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \leq C\, \varphi(x_0,r),$$

where C does not depend on r. Then, there exists a constant γ independent of u and f and there exists a number σ, also independent of u and f, such that for every ball $B_R \Subset \Omega$ having radius $R < \sigma$ and every $u \in W^{2,p}(B_R)$ satisfying (5.42) *such that $\partial_{ij}u \in LM^{p,\varphi}_{\{x_0\}}(B_R)$, we have*

$$\|\partial_{ij}u\|_{LM^{p,\varphi}_{\{x_0\}}(B_R)} \leq \gamma \|\mathcal{L}u\|_{LM^{p,\varphi}_{\{x_0\}}(B_R)}, \qquad \forall i,j = 1,\ldots,n.$$

Proof. The proof is a straightforward consequence of Theorems 4.6, 4.7, and the representation formula stated in Lemma 5.1. $\square$

CHAPTER 3

Mixed Morrey spaces

This chapter is based on the following publication:

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In this chapter, new classes of functions are defined. These spaces generalize Morrey spaces and give a refinement of Lebesgue spaces. Some embeddings between these new classes are also proved. Finally, as an application, these functional classes are used to obtain regularity results for solutions of partial differential equations of parabolic type.

1. Introduction

Aim of this chapter is to define new spaces and study some embeddings between them. We will refer to them with the symbol $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. As applications we obtain some estimates, in these classes of functions, for the solutions of partial differential equations of parabolic type in nondivergence form. Preparatory to achieving these results is the study of the behaviour of Hardy-Littlewood Maximal function, Riesz potential, Sharp and Fractional maximal functions, Singular integral operators with Calderón-Zygmund kernel and Commutators (see e.g. **[64]**, **[66]**).

We stress that are obtained results, known in L^p, in a new class of functions that can be view as an extension of the Morrey class (**[54]**), and used by a lot of authors, see e.g. in **[11]**, recently in **[68]**, **[59]**, **[40]**, **[41]**, **[42]** and others.

Let us point out that in doing this we need an extension to $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ of a celebrated inequality of Fefferman and Stein (see **[30]**) concerning the Sharp and the Maximal function (Theorem 4.8) and, also, we study the behavior of Riesz potential in the new class of functions, obtaining an extension of both a known estimate originally proved by Adams in **[1]** as well as of a result announced by Peetre in **[58]**.

2. Definitions and Preliminary Tools

In the sequel let $T > 0$ and let Ω be a bounded open set of $\mathbb{R}^n$ such that $\exists A > 0$: $\forall x \in \Omega$ and $0 \le \rho \le \operatorname{diam}(\Omega)$, $|Q(x,\rho) \cap \Omega| \ge A\,\rho^n$, being $Q(x,\rho)$ a cube centered in x, having edges parallel to the coordinate axes and lenght 2ρ.

Classical Morrey spaces are used, among others, in the theory of regular solutions to nonlinear partial differential equations and for the study of local behavior of solutions to nonlinear equations and systems (see e.g. **[54]**, **[56]**).

In the sequel $B_\rho(x)$ stands for the open ball $B(x,\rho) = \{y \in \mathbb{R}^n : |x-y| < \rho\}$.

Definition 2.1. *Let $1 < p,q < +\infty$, $0 < \lambda,\mu < n$. We define the set $L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))$ as the class of functions f such that is finite:*

$$\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} := \left(\sup_{\substack{t_0,t\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{\Omega\cap B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

with obvious modifications if $\Omega = \mathbb{R}^n$.

Definition 2.2. *Let Σ be the unit sphere: $\Sigma = \{x \in \mathbb{R}^{n+1}, |x| = 1\}$. We say that the function $k : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}$ is the classical Calderón-Zygmund kernel if:*

(1) $k \in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$;

(2) $k(\mu x_1, \mu x_2, \dots, \mu x_n, \mu^2 t) = \mu^{-(n+2)}\,k(x)$, *for each* $\mu > 0$;

(3) $\int\limits_\Sigma |k(x)|\,d\sigma_x < \infty$ *and* $\int\limits_\Sigma k(x)\,d\sigma_x = 0$.

The above definition, in particular condition (2), suggest to endow $\mathbb{R}^{n+1}$ with a metric, different to the standard Euclidean one. Thus let us consider, as Fabes and

Riviére in the celebrated paper [28], the following distance $d(x,y)=\rho(x-y)$ between two generic points $x,y\in\mathbb{R}^{n+1}$ (used e.g. in [8]),

$$\rho(x) \;=\; \sqrt{\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}}, \qquad x=(x',t) \;=\; (x'_1,\ldots,x'_n,t)\in\mathbb{R}^{n+1}. \tag{2.1}$$

Then $\mathbb{R}^{n+1}$, endowed with this metric, is a metric space.

Definition 2.3. *We say that the function* $k(x,y)\,:\,\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\backslash\{0\}\;\to\;\mathbb{R}$ *is a variable Calderón-Zygmund kernel if:*

(1) $k(x,\cdot)$ *is a kernel in the sense of the above Definition 2.2, for a.e.* $x\in\mathbb{R}^{n+1}$

(2) $\sup_{\rho(y)=1}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}k(x,y)\right|\le c(\beta)$*, for every multi-index* β*, independently of* x*.*

Next Proposition is proved in [60] (see also [13] or [49]), it is useful to recall the statement and the technique used in the proof, because will inspire us to techniques contained therein, for subsequent results.

Proposition 2.4. *If* $1<q<p<\infty$, $0<\lambda<\mu<n$, $q=\frac{(n-\mu)\,p}{(n-\lambda)}$*. The following embedding is true*

$$L^{p,\lambda}(\Omega)\subset L^{q,\mu}(\Omega).$$

Proof. Applying Hölder inequality, we have

$$\begin{aligned}
\int\limits_{\Omega\cap B_\rho(x)}|f|^q(y)dy \;&\le\; \left(\int\limits_{\Omega\cap B_\rho(x)}|f|^{q\cdot\frac{p}{q}}(y)dy\right)^{\frac{q}{p}}\cdot|B_\rho|^{1-\frac{q}{p}}\\
&=\; C\left(\int\limits_{\Omega\cap B_\rho(x)}|f|^p(y)dy\right)^{\frac{q}{p}}\cdot\rho^{n\cdot(1-\frac{q}{p})}\\
&=\; C\rho^{n\cdot(1-\frac{q}{p})}\cdot\left(\frac{1}{\rho^\lambda}\int\limits_{\Omega\cap B_\rho(x)}|f|^p(y)dy\right)^{\frac{q}{p}}\cdot\rho^{\lambda\cdot\frac{q}{p}}\\
&\le\; C\rho^{n-n\cdot\frac{q}{p}+\lambda\cdot\frac{q}{p}}\cdot\|f\|_{L^{p,\lambda}(\Omega)}=C\rho^\mu\cdot\|f\|^q_{L^{p,\lambda}(\Omega)};
\end{aligned}$$

then we obtain

$$\frac{1}{\rho^\mu}\int\limits_{\Omega\cap B_\rho(x)}|f|^q(y)dy\le C\cdot\|f\|^q_{L^{p,\lambda}(\Omega)},$$

where

$$\mu = n - n \cdot \frac{q}{p} + \lambda \cdot \frac{q}{p}$$

and, obviously, we have

$$\frac{n-\mu}{n-\lambda} = \frac{q}{p}$$

and the conclusion follows. $\square$

Remark 2.5. *It is possible to extend the previous result considering* $1 \leq q \leq p < \infty$ *and* $0 \leq \lambda, \mu < n$ *such that* $\frac{n-\mu}{q} \geq \frac{n-\lambda}{p}$.

3. Embedding Results

Theorem 3.1. *Let* $1 < p < +\infty$, $0 < \lambda < n$, $1 < q < q_1 < \infty$, $0 < \mu_1 < \mu < 1$ *and* $q = \frac{(1-\mu)\,q_1}{(1-\mu_1)}$, *we have*

$$L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega)) \subset L^{q,\mu}(0,T,L^{p,\lambda}(\Omega)).$$

Proof. Let us suppose that $f \in L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))$, then is finite

$$\left(\sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\lambda}} \int\limits_{\Omega\cap B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q_1}{p}} dt \right)^{\frac{1}{q_1}}.$$

Let us set $t \in (0,T)$ and apply Hölder inequality

$$\int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\lambda}} \int\limits_{\Omega\cap B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q}{p}} dt \leq$$

$$\leq \left(\int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\lambda}} \int\limits_{\Omega\cap B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q}{p}\cdot\frac{q_1}{q}} dt \right)^{\frac{q}{q_1}} \left|(0,T)\cap(t_0-\rho;t_0+\rho)\right|^{1-\frac{q}{q_1}} =$$

$$= C \left(\int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\lambda}} \int\limits_{\Omega\cap B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q}{p}\cdot\frac{q_1}{q}} dt \right)^{\frac{q}{q_1}} \cdot \rho^{(1-\frac{q}{q_1})} =$$

$$= C\rho^{(1-\frac{q}{q_1})} \left(\frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\lambda}} \int\limits_{\Omega\cap B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q_1}{p}} dt \right)^{\frac{q}{q_1}} \cdot \rho^{\mu_1\cdot\frac{q}{q_1}} =$$

$$= C\|f\|^{q}_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))} \cdot \rho^{1-\frac{q}{q_1}+\mu_1\cdot\frac{q}{q_1}}.$$

Let

$$\mu = 1 - \frac{q}{q_1} + \mu_1 \cdot \frac{q}{q_1} = 1 - (1 - \mu_1)\frac{q}{q_1},$$

$$\frac{1-\mu}{1-\mu_1} = \frac{q}{q_1};$$

it follows, as request, that

$$q = \frac{(1-\mu)q_1}{1-\mu_1},$$

and the proof is complete. □

Remark 3.2.. *It is possible to extend the previous result considering* $1 < q \leq q_1 < \infty$, $0 < \mu_1 \leq \mu < 1$ *or* $1 < \mu_1 \leq \mu < n$ *and*

$$\frac{1-\mu}{q} \geq \frac{1-\mu_1}{q_1}.$$

Theorem 3.3. *Let* $1 < q < p < \infty$, $0 < \lambda < \mu < n$, $q = \frac{(n-\mu)\,p}{(n-\lambda)}$, $1 < q_2 < q_1 < \infty$, $0 < \mu_1 < \mu_2 < 1$ *or* $1 < \mu_1 < \mu_2 < n$ *and* $q_2 = \frac{(1-\mu_2)q_1}{(1-\mu_1)}$, *we have*

$$L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega)) \subset L^{q_2,\mu_2}(0,T,L^{q,\mu}(\Omega)).$$

Proof. Let us set $t \in (0,T)$. If $1 < q < p < \infty$, $0 < \lambda < \mu < n$ and $q = \frac{(n-\mu)\,p}{(n-\lambda)}$, we have, from Proposition 2.4,

$$\frac{1}{\rho^\mu}\int_{\Omega\cap B_\rho(x)} |f|^q(y,t)dy \leq C\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)}|f|^{q\cdot\frac{p}{q}}(y,t)dy\right)^{\frac{q}{p}}.$$

Let us fix $t_0 \in (0,T)$, then, integrating in $(0,T)\cap(t_0-\rho,t_0+\rho)$, we have

$$\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^\mu}\int_{\Omega\cap B_\rho(x)}|f|^q(y,t)dy\right)^{\frac{1}{q}\cdot q_2} dt \leq$$

$$\leq C\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)}|f|^{q\cdot\frac{p}{q}}(y,t)dy\right)^{\frac{1}{p}\cdot q_2} dt \leq$$

applying Hölder inequality, we have

$$\leq C\left(\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)}|f|^p(y,t)dy\right)^{\frac{q_2}{p}\cdot\frac{q_1}{q_2}} dt\right)^{\frac{q_2}{q_1}}\cdot\rho^{1-\frac{q_2}{q_1}} =$$

$$= C\left(\frac{1}{\rho^{\mu_1}}\int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\lambda}}\int\limits_{\Omega\cap B_\rho(x)}|f|^p(y,t)dy\right)^{\frac{q_1}{p}}dt\right)^{\frac{q_2}{q_1}}\cdot\rho^{1-\frac{q_2}{q_1}+\mu_1\cdot\frac{q_2}{q_1}} =$$

$$= C\|f\|^{q_2}_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))}\cdot\rho^{\mu_2}$$

where

$$\mu_2 = 1-(1-\mu_1)\cdot\frac{q_2}{q_1},$$

then

$$\frac{1-\mu_2}{1-\mu_1} = \frac{q_2}{q_1};$$

it follows

$$q_2 = \frac{(1-\mu_2)q_1}{(1-\mu_1)}.$$

Then, we obtain

$$\left(\frac{1}{\rho^{\mu_2}}\int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\mu}}\int\limits_{\Omega\cap B_\rho(x)}|f|^q(y,t)dy\right)^{\frac{q_2}{q}}dt\right)^{\frac{1}{q_2}}\leq C\|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))}$$

and, finally

$$\|f\|_{L^{q_2,\mu_2}(0,T,L^{q,\mu}(\Omega))}\leq C\,\|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))}.$$

□

Remark 3.4. *It is possible to extend the previous result considering* $1<q\leq p<\infty$, $0<\lambda\leq\mu<n$, $1<q_2\leq q_1<\infty$, $0<\mu_1\leq\mu_2<1$ *or* $1<\mu_2\leq\mu_1<n$ *and*

$$\frac{n-\mu}{q}\geq\frac{n-\lambda}{p};\qquad\frac{1-\mu_2}{q_2}\geq\frac{1-\mu_1}{q_1}.$$

4. Main Results

4.1. Estimates of some integral operators. Let $f\in L^1_{loc}(\mathbb{R}^n)$ and recall the following Hardy-Littlewood maximal function

$$Mf(x) = \sup_{\rho>0}\frac{1}{|B_\rho(x)|}\int\limits_{B_\rho(x)}|f(y)|\,dy$$

where $B_\rho(x)$ is a ball centered at x and with radius ρ.

Proposition 4.1.. *Let* $1<p<+\infty$, $0<\lambda<n$. *Then*

$$\|Mf\|_{L^{p,\lambda}(\mathbb{R}^n)}\leq C\,\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$$

where C *is independent of* f.

Let us now extend the previous result as follows.

Theorem 4.2. *Let* $1 < p < +\infty$, $0 < \lambda < n$, $1 < q' < +\infty$, $0 < \mu < 1$ *or* $1 < \mu < n$ *and* $f \in L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. *Then,*

$$\|Mf\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \leq C\ \|f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Proof. Let $t \in (0,T)$. From **[30]** (Lemma 1, pg.111), we have

$$\int_{\mathbb{R}^n} |Mf(y,t)|^p \chi(y)\,dy \leq c \int_{\mathbb{R}^n} |f(y,t)|^p (M\chi)(y)\,dy$$

for any function f and χ the characteristic function of a ball $B_\rho(x) \subset \mathbb{R}^n$, being the constant c independent of f. Then

$$\int_{B_\rho(x)} |Mf(y,t)|^p\,dy \leq \int_{B_{2\rho}(x)} |f(y,t)|^p (M\chi(y))dy + \sum_{k=1}^{+\infty} \int_{B_{2^{k+1}\rho}\setminus B_{2^k\rho}(x)} |f(y,t)|^p (M\chi(y))\,dy$$

it follows

$$\frac{1}{\rho^\lambda} \int_{B_\rho(x)} |Mf(y,t)|^p\,dy \leq$$

$$\leq C\frac{1}{(2\rho)^\lambda} \int_{B_{2\rho}(x)} |f(y,t)|^p (M\chi(y))\,dy + C\sum_{k=1}^{+\infty} \frac{1}{(2^{k+1}\rho)^\lambda} \int_{B_{2^{k+1}\rho}(x)} |f(y,t)|^p (M\chi(y))\,dy,$$

using the method applied in **[16]** and considering the supremum for $x \in \mathbb{R}^n$ and $\rho > 0$. Let us fix $t_0 \in (0,T)$, then, elevating to $\frac{q'}{p}$, integrating in $(0,T)\cap(t_0-\rho,t_0+\rho)$ and multiplying for $\rho^{-\mu}$, we obtain

$$\frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |Mf(y,t)|^p\,dy \right)^{\frac{q'}{p}} dt \leq$$

$$\leq C\frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y,t)|^p\,dy \right)^{\frac{q'}{p}} dt$$

taking the supremum, in both sides, for $t_0 \in (0,T)$ and $\rho > 0$, we obtain

$$\left[\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |Mf(y,t)|^p\,dy \right)^{\frac{q'}{p}} dt \right]^{\frac{1}{q'}} \leq$$

$$\leq \left[\sup_{\substack{t_0, t \in (0,T) \\ \rho > 0}} \frac{1}{\rho^{\mu}} \int\limits_{(0,T) \cap (t_0 - \rho, t_0 + \rho)} \left(\sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \frac{1}{\rho^{\lambda}} \int\limits_{B_\rho(x)} |f(y,t)|^p \, dy \right)^{\frac{q'}{p}} dt \right]^{\frac{1}{q'}}$$

or, equivalently

$$\|Mf\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

□

As application of this result we prove some estimates of the Riesz potential in $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ spaces.

Let us set $t \in (0,T)$ and consider, for $0 < \alpha < n$, the fractional integral operator of order α,

$$I_\alpha f(x,t) = \int\limits_{\mathbb{R}^n} \frac{f(y,t)}{|x-y|^{n-\alpha}} dy, \quad a.e. \, in \, \mathbb{R}^n.$$

Theorem 4.3. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, $1 < q' < +\infty$, $0 < \mu' < 1$ and $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then,*

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Proof. Let us fix $x \in \mathbb{R}^n$, $t_0 \in (0,T)$ and $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then, set $t \in (0,T)$,

$$(I_\alpha f)(x,t) = \int\limits_{|x-y| \leq \epsilon} \frac{f(y,t)}{|x-y|^{n-\alpha}} dy + \int\limits_{|x-y| > \epsilon} \frac{f(y,t)}{|x-y|^{n-\alpha}} dy = I_1 + I_2,$$

estimating separately each integral I_1 and I_2, as in **[1]** (Theorem 3.1) or **[16]** (Theorem 2), we obtain

$$|I_\alpha f|(x,t) \leq C (Mf)^{\frac{n-\lambda-\alpha p}{n-\lambda}}(x) \cdot \left(\sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \frac{1}{\rho^{\lambda}} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{1}{p} \cdot \frac{\alpha p}{n-\lambda}},$$

recalling that $\frac{n-\lambda-\alpha p}{n-\lambda} = \frac{p}{q}$, elevating to the power q, integrating in $B_\rho(x)$ and multiplying to $\rho^{-\lambda}$, we have

$$\frac{1}{\rho^{\lambda}} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \leq \frac{1}{\rho^{\lambda}} \int\limits_{B_\rho(x)} (Mf)^p(y,t) \cdot \left(\sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \frac{1}{\rho^{\lambda}} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{\alpha q}{n-\lambda}} dy \leq$$

applying Theorem 4.2, and observing that $\frac{\alpha q}{n-\lambda} + 1 = \frac{q}{p}$,

$$\leq C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{\alpha q}{n-\lambda}} \cdot \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right) =$$

$$= C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q}{p}},$$

then

$$\frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \leq C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q}{p}},$$

considering the supremum for $x \in \mathbb{R}^n$ and $\rho > 0$ and elevating both member to $\frac{1}{q}$

$$\left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{1}{q}} \leq C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{1}{p}}.$$

Now, elevating to q', integrating in $(0,T) \cap (t_0 - \rho, t_0 + \rho)$ and multiplying to $\rho^{-\mu}$, we have

$$\frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{q'}{q}} dt \leq$$

$$\leq C \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q'}{p}} dt,$$

taking the supremum for $t_0 \in (0,T), \rho > 0$, we have

$$\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{q'}{q}} dt \leq$$

$$\leq C \sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q'}{p}} dt.$$

Finally, elevating to $\frac{1}{q'}$ we have

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

□

Corollary 4.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha,}$, $0 < \lambda < n - \alpha p$. Let us also set $1 < q < p$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\lambda < \mu < n$ such that $\mu = \frac{n\lambda}{(n-\alpha p)}$, $1 < q' < +\infty$, $0 < \mu' < 1$ and $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then,*

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\mu}(\mathbb{R}^n))} \leq C\ \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

where C is independent of f.

Proof. Let us fix $x \in \mathbb{R}^n$ and $t \in (0,T)$. From Corollary in **[16]**, we have

$$\left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\mu} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{1}{q}} \leq C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{1}{p}},$$

elevating to q', fixing $t_0 \in (0,T)$, integrating in $(0,T) \cap (t_0 - \rho, t_0 + \rho)$ and multiplying for $\rho^{-\mu'}$,

$$\left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu'}} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{q'}{q}} dt \right)^{\frac{1}{q'}} \leq$$

$$\leq C \left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu'}} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q'}{p}} dt \right)^{\frac{1}{q'}}$$

that is

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\mu}(\mathbb{R}^n))} \leq C\ \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

□

One more application of the technique used in the proof of Theorem 4.2 is the following result, where we set T a convolution singular integral operator $T = k * f$, where k is an usual Calderón-Zygmund kernel, studied by Coifman and Fefferman in **[19]**.

Theorem 4.5.. *Let $1 < p < \infty$, $0 < \lambda < n$ $1 < q' < +\infty$, $0 < \mu' < 1$ and $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then,*

$$\|Tf\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C\ \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Proof. Let us fix $x \in \mathbb{R}^n$, $t \in (0,T)$, $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ and χ the characteristic function of a ball $B_\rho(x)$. Then, from a result by Coifman and Rochberg (see **[20]** pg. 251), $M(M\chi)^\gamma \le c\,(M\chi)^\gamma$, then $(M\chi)^\gamma$ is a A_1 weight.

It follows, from a result contained in **[19]**, that

$$\int_{B_\rho(x)} |Tf(y,t)|^p\,dy \le \int_{\mathbb{R}^n} |Tf(y,t)|^p (M\chi(y))^\gamma dy \le C \int_{\mathbb{R}^n} |f(y,t)|^p (M\chi(y))^\gamma\,dy,$$

estimating the last term following the lines of the proof of Theorem 4.2, we get the conclusion. □

Before we prove the next results we need to consider two variants of the Hardy-Littlewood maximal operator, that are the sharp maximal function and the fractional maximal function (see e.g. Chapter 1 and **[23]**).

Definition 4.6. *Given $f \in L^1_{loc}(\mathbb{R}^n)$ let us define the following Sharp Maximal function*

$$f^\sharp(x) = \sup_{B \supset \{x\}} \frac{1}{|B|} \int_B |f(y) - f_B|\,dy,$$

for a.e. $x \in \mathbb{R}^n$, where B is a generic ball in $\mathbb{R}^n$.

Definition 4.7. *Set $t \in (0,T)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < \eta < 1$. Let us define the Fractional Maximal function*

$$(M_\eta f)(x) = \sup_{B \supset \{x\}} \frac{1}{|B|^{1-\eta}} \int_B |f(y,t) - f_B|\,dy,$$

for a.e. $x \in \mathbb{R}^n$, where B is a generic ball in $\mathbb{R}^n$.

The next Theorem is a generalization of a well known inequality by Fefferman and Stein, see **[30]**, pg. 153, or Chapter 1.

Theorem 4.8. *Let $1 < p,q < \infty$, $0 < \lambda,\mu < n$ and $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$.*

Then, there exists a constant $C \ge 0$ independent of f such that

$$\|Mf\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \le C\,\|f^\sharp\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Proof. Let us fix $x \in \mathbb{R}^n, t \in (0,T)$. Let us also consider $\rho > 0, \gamma \in]\frac{\lambda}{n};1[$, $\chi = \chi_{B_\rho(x)}$, $\forall x \in \mathbb{R}^n$ the characteristic function of a ball $B_\rho(x)$. We know that $(M\chi)^\gamma \in A_1$ and, from **[33]** pg. 410, we have

$$\int_{\mathbb{R}^n} (Mf)^p(y,t)\omega(y)dy \le C \int_{\mathbb{R}^n} |f^\sharp(y,t)|^p \omega(y)\,dy, \quad \forall \omega \in A_\infty, \forall f \in L^p_\omega(\mathbb{R}^n)$$

where $L^p_\omega(\mathbb{R}^n)$ is the L^p space with respect to the measure $d\mu = \omega\, dx$. We can use this inequality because $f \in L^{p,\lambda}(\mathbb{R}^n)$ implies $f \in L^p_{(M\chi)^\gamma}(\mathbb{R}^n)$ (see the calculation in **[16]** pg. 275).

Choosing $\omega(y) = (M\chi)^\gamma(y)$, we have, from **[23]** pg.327,

$$\int_{B_\rho(x)} (Mf)^p(y,t)dy \leq \int_{\mathbb{R}^n} (Mf)^p(y,t)(M\chi)^\gamma(y)dy \leq$$

$$\leq C \cdot \int_{\mathbb{R}^n} |f^\sharp(y,t)|^p (M\chi)^\gamma(y)\, dy \leq C\rho^\lambda \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y,t)|^p\, dy, \quad \forall f \in L^p_\omega(\mathbb{R}^n)$$

then

$$\frac{1}{\rho^\lambda} \int_{B_\rho(x)} (Mf)^p(y,t)dy \leq C \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y,t)|^p\, dy, \quad \forall f \in L^p_\omega(\mathbb{R}^n)$$

and, taking the supremum for $x \in \mathbb{R}^n$ and $\rho > 0$ we have

$$\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} (Mf)^p(y,t)dy \leq C \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y,t)|^p\, dy,$$

set $t_0 \in (0,T)$, elevating to $\frac{q}{p}$, integrating in $(0,T) \cap (t_0-\rho, t_0+\rho)$ and multiplying for $\rho^{-\mu}$, we have

$$\frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} (Mf(y,t))^p(y,t)dy \right)^{\frac{q}{p}} dt \leq$$

$$\leq C \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y,t)|^p\, dy \right)^{\frac{q}{p}} dt$$

then, we obtain

$$\left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} (Mf(y,t))^p(y,t)dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \leq$$

$$\leq C \left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y,t)|^p\, dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

and we get the conclusion. □

Theorem 4.9. *Let* $1 < p, q, q_1 < \infty$, $0 < \lambda, \mu_1 < n$ *and* $f \in L^{q_1,\mu_1}(0, T, L^{p,\lambda}(\mathbb{R}^n))$.

Then, for every $\eta \in]0, (1 - \frac{\lambda}{n})\frac{1}{p}[$, *there exists a constant* $C \geq 0$ *independent of* f *such that*

$$\|M_\eta f\|_{L^{q_1,\mu_1}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\mathbb{R}^n))}$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{n\,\eta}{n - \lambda}. \tag{4.1}$$

Proof. Let $x \in \mathbb{R}^n$ and $t_0 \in (0, T)$.

Let us fix $1 < r < p$ and

$$\varepsilon = \frac{\left(1 - \frac{\lambda}{n}\right) \cdot \frac{p}{n} - \eta}{\left(1 - \frac{\lambda}{n}\right) \frac{1}{p}}. \tag{4.2}$$

Set $t \in (0, T)$, for a generic ball B of $\mathbb{R}^n$, we have

$$\frac{1}{|B|^{1-\eta}} \int\limits_B |f(y,t)|dy \leq$$

$$\leq \left(\frac{1}{|B|} \int\limits_B |f(y,t)|^r dy \right)^{\frac{\varepsilon}{r}} \cdot \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int\limits_B |f(y,t)|^p dy \right)^{\frac{(1-\varepsilon)}{p}}$$

then

$$\frac{1}{|B|^{1-\eta}} \int\limits_B |f(y,t)|dy \leq [M(|f|^r)]^{\frac{\varepsilon}{r}}(y,t) \cdot \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_B |f(y,t)|^p dy \right)^{1-\varepsilon}$$

from which it follows

$$\left(M_\eta(f)\right)^{\frac{p}{\varepsilon}}(y,t) \leq \left(M(|f|^r)\right)^{\frac{p}{r}}(y,t) \cdot \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{(1-\varepsilon)}{\varepsilon}\cdot p} \qquad a.\ e.\ y \in \mathbb{R}^n,\ t \in (0,T).$$

Denoting by $\chi(y) = \chi_{B_\rho(x)}(y)$ we have

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \left(M_\eta(f)\right)^{\frac{p}{\varepsilon}}(y,t) \cdot \chi(y)\, dy &\leq \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{(1-\varepsilon)}{\varepsilon}\cdot p} \int\limits_{\mathbb{R}^n} \left(M(|f|^r)\right)^{\frac{p}{r}}(y,t) \cdot \chi(y)\, dy \\ &\leq \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{(1-\varepsilon)}{\varepsilon}\cdot p} \int\limits_{\mathbb{R}^n} |f|^p(y,t) \cdot (M\chi(y))\, dy. \end{aligned}$$

Then, we obtain

$$\int\limits_{B_\rho(x)} \left(M_\eta(f)\right)^{\frac{p}{\varepsilon}}(y,t)\, dy \leq C\, \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{p}{\varepsilon}} \cdot \rho^\lambda.$$

Let us observe that

$$\frac{p}{\varepsilon} = q$$

indeed, using (4.1), we have

$$\varepsilon = \frac{n - \lambda - n\,\eta\, p}{n - \lambda},$$

dividing by $n \cdot p$, we deduce exactly (4.2).

Then, we obtain

$$\left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} \left(M_\eta(f)\right)^q (y,t)\, dy \right)^{\frac{1}{q}} \leq C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f|^p(y,t)\, dy \right)^{\frac{1}{p}},$$

elevating to q_1 integrating both sides in $(0,T)\cap(t_0-\rho;t_0+\rho)$ and multiplying for $\frac{1}{\rho^{\mu_1}}$, we have

$$\frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho;t_0+\rho)} \left[\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} \left(M_\eta(f)\right)^{\frac{p}{\varepsilon}} (y,t)\, dy \right]^{\frac{q_1}{q}} dt \leq$$

$$\leq C\, \frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho;t_0+\rho)} \left[\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f|^p(y,t)\, dy \right]^{\frac{q_1}{p}} dt$$

the last term is less or equal than

$$C \sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho;t_0+\rho)} \left[\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f|^p(y,t)\, dy \right]^{\frac{q_1}{p}} dt.$$

Finally, we have

$$\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho;t_0+\rho)} \left[\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} (M_\eta f)^q(y,t)\, dy \right]^{\frac{q_1}{q}} dt \leq$$

$$\leq C \sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu_1}} \int\limits_{(0,T)\cap(t_0-\rho;t_0+\rho)} \left[\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f|^p(y,t)\, dy \right]^{\frac{q_1}{p}} dt.$$

Elevating both sides to $\frac{1}{q_1}$, we have

$$\|M_\eta f\|_{L^{q_1,\mu_1}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C\, \|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

$\square$

4.2. Estimates of singular integral operators and commutators. Let $k(x,y)$ be a variable Calderón-Zygmund kernel for a.e. $x \in \mathbb{R}^{n+1}$, $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ with $1 < p,q < \infty$, $0 < \lambda,\mu < n$, $a \in BMO(\mathbb{R}^{n+1})$. For $\varepsilon > 0$ let us define the operator K_ε and the commutator $C_\varepsilon[a,f]$, as follows

$$K_\varepsilon f(x) = \int\limits_{\rho(x-y)>\varepsilon} k(x,x-y)f(y)dy$$

$$C_\varepsilon[a,f] = K_\varepsilon(af)(x) - a(x)K_\varepsilon f(x) = \int\limits_{\rho(x-y)>\varepsilon} k(x,x-y)[a(x)-a(y)]f(y)dy.$$

In the next theorem we prove that $K_\varepsilon f$ and $C_\varepsilon[a,f]$ are, uniformly in ε, bounded from $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ into itself. This fact allows us to let $\varepsilon \to 0$ obtaining as limits in $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ the following singular integral and commutator

$$Kf(x) = P.V.\int\limits_{\mathbb{R}^n} k(x,x-y)f(y)dy = \lim_{\varepsilon\to 0} K_\varepsilon f(x)$$

$$C[a,f](x) = P.V.\int\limits_{\mathbb{R}^n} k(x,x-y)[a(x)-a(y)]f(y)dy = \lim_{\varepsilon\to 0} C_\varepsilon[a,f](x)$$

These operators are bounded in the class $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$.

Theorem 4.10. *Let $k(x,y)$ be a variable Calderón-Zygmund kernel, for a.e. $x \in \mathbb{R}^{n+1}$, $1 < p,q < \infty$, $0 < \lambda,\mu < n$ and $a \in VMO(\mathbb{R}^{n+1})$.*

For any $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ the singular integrals Kf, $C[a,f] \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. exist as limits in $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$, for $\varepsilon \to 0$, of $K_\varepsilon f$ and $C_\varepsilon[a,f]$, respectively. Then, the operators Kf, $C[a,f] : L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n)) \to L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ are bounded and satisfy the following inequalities

$$\|Kf\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \le c\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \tag{4.3}$$

$$\|C[a,f]\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \le c\|a\|_*\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \tag{4.4}$$

where $c = c(n,p,\lambda,\alpha,K)$, the dependence on K is through the constant $c(\beta)$ in Definition 2.3 part 2), for suitable β.

Moreover, for every $\epsilon > 0$ there exists $\rho_0 > 0$ such that, if B_r is a ball with radius r such that $0 < r < \rho_0$, $k(x,y)$ satisfies the above assumptions and $f \in L^{q,\mu}(0,T,L^{p,\lambda}(B_r))$, we have

$$\|C[a,f]\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \le c\,\epsilon\,\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \tag{4.5}$$

for some constant c independent of f.

Proof. For every $t \in (0,T),$ from the known inequality (see e.g. **[16]**)

$$\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |(Kf)(y,t)|^p dy \le c \sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p\, dy,$$

fixing $t_0 \in (0,T),$ elevating to $\frac{q}{p}$, integrating in $(0,T)\cap(t_0-\rho,t_0+\rho)$, multiplying for $\rho^{-\mu}$,we have

$$\frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |(Kf)(y,t)|^p dy \right)^{\frac{q}{p}} dt \le$$

$$\le c \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p\, dy \right)^{\frac{q}{p}} dt$$

then, we have

$$\sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |(Kf)(y,t)|^p dy \right)^{\frac{q}{p}} dt \le$$

$$\le c \sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q}{p}} dt$$

elevating to $\frac{1}{q}$, we get the conclusion for Kf. The proof of (4.4) is similar, starting from the inequality

$$\|C[a,f]\|_{L^{p,\lambda}(\mathbb{R}^n)} \le c\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\,.$$

Finally, using the VMO assumption, if we fix ρ_0 such that $\eta(\rho_0) < \epsilon$, we get the conclusion. Let us remark that the result is also true if we assume a defined only in some ball with $\|a\|_* < \epsilon$.

□

5. Applications to Partial Differential Equations

As application of the previous results we obtain a regularity result for strong solutions to the nondivergence form parabolic equations.

Precisely, let $n \geq 3$, $Q_T = \Omega' \times (0,T)$ be a cylinder of $\mathbb{R}^{n+1}$ of base $\Omega' \subset \mathbb{R}^n$. In the sequel let us set $x = (x',t) = (x'_1, x_2, \dots, x'_n, t)$ a generic point in Q_T, $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\Omega'))$, $1 < p,q < \infty$, $0 < \lambda, \mu < n$ and

$$L\,u = u_t - \sum_{i,j=1}^{n} a_{ij}(x',t)\frac{\partial^2 u}{\partial x'_i \partial x'_j} \tag{5.1}$$

where

$$a_{ij}(x',t) = a_{ji}(x',t), \qquad \forall i,j = 1,\dots,n, \qquad a.e.\, x \in Q_T \tag{5.2}$$

$$\exists \nu > 0 \,:\, \nu^{-1}|\xi|^2 \leq \Sigma_{i,j=1}^{n} a_{ij}(x',t)\,\xi_i\xi_j \leq \nu|\xi|^2, \qquad a.e.\ in\ Q_T, \forall \xi \in \mathbb{R}^n \tag{5.3}$$

$$a_{ij}(x',t) \in VMO(Q_T) \cap L^\infty(Q_T), \qquad \forall i,j = 1,\dots,n,. \tag{5.4}$$

Let us consider

$$Lu(x',t) = f(x',t). \tag{5.5}$$

A strong solution to (5.5) is a function $u(x) \in L^{q,\mu}(0,T,L^{p,\lambda}(\Omega'))$ with all its weak derivatives $D_{x'_i}u$, $D_{x'_i x'_j}u$, $i,j = 1,\dots,n$ and $D_t u$, satisfying (5.5) , $\forall x \in Q_T$.

Let us now fix the coefficient $x_0 = (x'_0, t_0) \in Q_T$ and consider the fundamental solution of $L_0 = L(x_0)$, is given, for $\tau > 0$, by

$$\Gamma(x_0;\theta) = \Gamma(x'_0, t_0; \zeta, \tau) = \frac{(4\pi\tau)^{\frac{1-n}{2}}}{\sqrt{a^{ij}(x_0)}} \exp\left(-\frac{A^{ij}(x_0)\zeta_i\zeta_j}{4\,\tau}\right) \tag{5.6}$$

that is equals to zero if $\tau \leq 0$, being $A^{ij}(x_0)$ the entries of the inverse matrix $\{a^{ij}(x_0)\}^{-1}$.

The second order derivatives with respect to ζ_i and ζ_j, denoted by $\Gamma_{ij}(x_0, t_0; \zeta, \tau)$, $i,j = 1,\dots,n$, and $\Gamma_{ij}(x;\theta)$, are kernels of mixed homogeneity.

Theorem 5.1. *Let $n \geq 3, a_{ij} \in VMO(Q_T) \cap L^\infty(Q_T)$, $B_r \subset\subset \Omega'$ a ball in $\mathbb{R}^n$*

Then, for every u having compact support in $B_r \times (0,T)$, solution of $Lu = f$ such that $D_{x'_i x'_j}u \in L^{q,\mu}(0,T,L^{p,\lambda}(B_r))$ $\forall i,j = 1,\dots,n$, there exists $r_0 = r_0(n.p,\nu,\eta)$ such that, if $r < r_0$, then

$$\|D_{x'_i x'_j}\, u\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \leq C\|Lu\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))}, \qquad i,j = 1,\dots,n \tag{5.7}$$

$$\|\, u_t\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \leq C\|Lu\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))}, \tag{5.8}$$

Proof. Let $C_t = \{v \in C_0^\infty(\mathcal{A}) : v(x',0) = 0, \mathcal{A} = \mathbb{R}^{n+1} \cap \{t \geq 0\}\}$ and $u \in C_t$. The local representation formula for the second order spatial derivatives of u (see **[8]**), is the following

$$\begin{aligned} D_{x_i'x_j'}u(x) &= \lim_{\varepsilon\to 0} \int\limits_{\rho(x-y)>\varepsilon} \Gamma_{ij}(x; x-y) Lu(y)dy \\ &+ \lim_{\varepsilon\to 0} \int\limits_{\rho(x-y)>\varepsilon} \Gamma_{ij}(x; x-y) \sum_{h,k=1}^{n} [a^{hk}(y) - a^{hk}(x)] \cdot D_{y_h' y_k'} u(y)dy \\ &+ Lu(x) \int\limits_{\Sigma_{n+1}} \nu_i(y)\Gamma_j(x; y)d\sigma, \end{aligned}$$

for $i, j = 1, \ldots, n$, and for x in the support of u, being $\Sigma_{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ and $\nu_i(y)$ the i-th component of the unit outward normal to Σ_{n+1} at $y \in \Sigma_{n+1}$.

From (4.3) and (4.4) we get the first inequality (5.7). Let us now observe that

$$u_t = Lu + \sum_{i,j=1}^{n} a_{ij}(x', t) \frac{\partial^2 u}{\partial x_i' \partial x_j'}$$

and the second inequality (5.8) is proved.

□

CHAPTER 4

Integral operators on modified local generalized Morrey spaces

This chapter is based on the following publication:

V.S. Guliyev, M.N. Omarova, M. A. Ragusa, A. Scapellato,
Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces,
to appear.

In this chapter we prove regularity results, in some Modified Local Generalized Morrey Spaces, for the first derivatives of the solutions of a divergence elliptic second order equation of the form

$$\mathscr{L}u := \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} = \nabla \cdot f, \qquad \text{for almost all } x \in \Omega$$

where the coefficients a_{ij} belong to the Sarason class VMO and f is assumed to be in some Modified Local Generalized Morrey Spaces $\widetilde{LM}^{p,\varphi}_{\{x_0\}}$. Hearth of this chapter is to use an explicit representation formula for the first derivatives of the solutions of the elliptic equation in divergence form, in terms of singular integral operators and commutators with Calderón-Zygmund kernels. Combining the representation formula with some Morrey-estimates type for each operator that appears in it, we derive a regularity result.

1. Introduction

In this chapter we consider the following divergence form elliptic equation

$$\mathscr{L}u := \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} = \nabla \cdot f, \qquad \text{for almost all } x \in \Omega \tag{1.9}$$

in a bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

We assume that $\mathscr{L}$ is a linear elliptic operator and its coefficients belong to the space VMO and the vectorial field $f = (f_1, f_2, ..., f_n)$ is such that $f_i \in LM^{p,\varphi}$ for $i = 1, ..., n$, with $1 < p < \infty$ and φ positive and measurable function. The space VMO was introduced by Sarason and it is the proper subspace of the John-Nirenberg space BMO whose BMO norm over a ball vanishes as the radius of the ball approaches zero.

In the last few years have been studied several differential problems on Generalized Morrey Spaces (see, for instance, [**43**]).

Recently, in [**71**] and [**72**] the author studied some regularity results for solutions of linear partial differential equations with discontinuous coefficients in nondivergence form.

The main result in this chapter is the study of local regularity in the Generalized Morrey Spaces $LM^{p,\varphi}$ of the first derivatives of the solutions of the equation under consideration as in the past has been done in L^p-spaces and in $L^{p,\lambda}-$spaces.

See, for instance, [**25**] and [**65**] where the authors obtain local regularity in the classical Lebesgue spaces L^p for the first derivatives of solutions of the solutions of the equation with discontinuous coefficients. See, also, [**63**] in which has been done the same in the Morrey spaces $L^{p,\lambda}$. Hearth of the technique is the use of an integral representation formula for the first derivatives of the solutions of equation (2.10) and the boundedness in $L^{p,\varphi}$ of some integral operators and commutators appearing in this formula.

Precisely, in the sequel we apply the boundedness on Generalized *local* Morrey Spaces of singular integral operators and its commutators obtained in [**44**]. We would like to point out that in the last decades a lot of authors studied the boundedness of such operators in several functional spaces (see e.g. [**21**], [**27**]).

2. Useful definitions

Let Ω be an open bounded subset of $\mathbb{R}^n$, with $n \geq 3$, and f be a locally integrable function on Ω.

In 1991, Mizuhara in [53] extended the previous definition of Morrey Space, introducing the Generalized Morrey Spaces.

Definition 2.1. *Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M^{p,\varphi}(\Omega)$ $(WM^{p,\varphi}(\Omega))$ the Generalized Morrey space (the weak Generalized Morrey space), the space of all functions $f \in L^p_{\mathrm{loc}}(\Omega)$ with finite quasinorm*

$$\|f\|_{M^{p,\varphi}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<r<d}} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{L^p(\Omega(x,r))}$$

$$\left(\|f\|_{WM^{p,\varphi}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<r<d}} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{WL^p(\Omega(x,r))}\right).$$

According to this definition we obtain, for $0 \le \lambda < n$, the Morrey space $L^{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$L^{p,\lambda} = M^{p,\varphi}\big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In this note we are interested in the study of regularity properties of solutions to elliptic equations in the local version of Generalized Morrey Spaces. In order to achieve this purpose we need of the following definitions.

Definition 2.2. *Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \le p < \infty$. Fixed $x_0 \in \Omega$, we denote by $LM^{p,\varphi}_{\{x_0\}}(\Omega)$ $(WLM^{p,\varphi}_{\{x_0\}}(\Omega))$ the local Generalized Morrey space (the weak local Generalized Morrey space), the space of all functions $f \in L^p_{\mathrm{loc}}(\Omega)$ with finite quasinorm*

$$\|f\|_{LM^{p,\varphi}_{\{x_0\}}(\Omega)} = \sup_{0<r<d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{p}}} \|f\|_{L^p(\Omega(x_0,r))}$$

$$\left(\|f\|_{WLM^{p,\varphi}_{\{x_0\}}(\Omega)} = \sup_{0<r<d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{p}}} \|f\|_{WL^p(\Omega(x_0,r))}\right).$$

Definition 2.3. *Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \le p < \infty$. We denote by $\widetilde{M}^{p,\varphi}(\Omega)$ $\left(W\widetilde{M}^{p,\varphi}(\Omega)\right)$ the modified Generalized Morrey space (the modified weak Generalized Morrey space), the space of all functions $f \in L^p(\Omega)$ with finite norm*

$$\|f\|_{\widetilde{M}^{p,\varphi}(\Omega)} = \|f\|_{M^{p,\varphi}(\Omega)} + \|f\|_{L^p(\Omega)}$$

$$\left(\|f\|_{W\widetilde{M}^{p,\varphi}(\Omega)} = \|f\|_{WM^{p,\varphi}(\Omega)} + \|f\|_{WL^p(\Omega)}\right).$$

According to this definition we obtain, for $\lambda \ge 0$, the local Morrey Space $LM^{p,\lambda}_{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM^{p,\lambda}_{\{x_0\}}(\Omega) = LM^{p,\varphi}_{\{x_0\}}(\Omega)\big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.4. *Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,\infty)$ and $1 \le p < \infty$. Fixed $x_0 \in \Omega$, we denote by $\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)$ $\big(\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)\big)$ the modified local Generalized Morrey space (the modified weak local Generalized Morrey space), the space of all functions $f \in L^p(\Omega)$ with finite norm*

$$\|f\|_{\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)} = \|f\|_{LM^{p,\varphi}_{\{x_0\}}(\Omega)} + \|f\|_{L^p(\Omega)}$$

$$\Big(\|f\|_{W\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)} = \|f\|_{WLM^{p,\varphi}_{\{x_0\}}(\Omega)} + \|f\|_{WL^p(\Omega)}\Big).$$

Remark 2.5. *For further details on Local Generalized Morrey Spaces, see for instance* **[38, 39, 47]**.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \ge 3$, let us consider

$$\mathscr{L}u \equiv -\sum_{i,j=1}^{n} \big(a_{ij}(x)u_{x_i}\big)_{x_j} = \nabla \cdot f, \qquad \text{a.a. } x \in \Omega, \tag{2.10}$$

and, fixed $x_0 \in \mathbb{R}^n$, we suppose that exists $p \in]1,+\infty[$ and a positive measurable function φ defined on $\mathbb{R}^n \times (0,\infty)$ such that:

$$f = (f_1, ..., f_n) \in [LM^{p,\varphi}_{\{x_0\}}(\Omega)]^n; \tag{2.11}$$

$$a_{ij}(x) \in L^\infty \cap VMO, \forall i,j = 1,...,n; \tag{2.12}$$

$$a_{ij}(x) = a_{ji}(x), \quad \forall i,j = 1,...,n, \text{ a.a. } x \in \Omega; \tag{2.13}$$

$$\exists \kappa > 0 : \kappa^{-1}|\xi|^2 \le a_{ij}\xi_i\xi_j \le \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \tag{2.14}$$

We say that a function u is a solution to (2.10) if $u, \partial_{x_i}u \in L^p(\Omega)$, $\forall i = 1,...,n$ and for some $1 < p < \infty$

$$\int_\Omega a_{ij}u_{x_i}\varphi_{x_j}\,\mathrm{d}x = -\int_\Omega f_i\varphi_{x_i}\,\mathrm{d}x, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

3. Calderón-Zygmund kernel and preliminary results

In order to present the representation formula for the first derivatives of a solution of 2.10, we find it convenient to present the definition of Calderón-Zygmund kernel:

Definition 3.1. *Let $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. We say that $k(x)$ is a Calderón-Zygmund kernel (C-Z kernel) if: .*

(1) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
(2) $k(x)$ is homogeneous of degree $-n$;
(3) $\int_\Sigma k(x)\,\mathrm{d}x = 0$, where $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$.

Many authors obtained several boundedness results for integral operators involving Calderón-Zygmund kernels. For instance, in [**24**] the authors studied the boundedness of Calderón-Zygmund singular integral operators and commutators on Morrey Spaces. Recently, in [**44**] the authors extended the previous results in Generalized Local Morrey Spaces. Precisely, using the boundedness of the Calderón-Zygmund singular integral operators from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself (see [**38**]), the authors in [**44**] obtained the following theorem (see Chapter 2 for further details).

Theorem 3.2. *Let $x_0 \in \mathbb{R}^n$, $1 < q < s < p < +\infty$, K be a Calderón-Zygmund singular integral operator and the measurable function $\varphi : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^+$ satisfy the conditions*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \le C\varphi(x_0,r),$$

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\tau^{\frac{n}{p}}}{\tau^{\frac{n}{p}+1}} \le C\varphi(x_0,r),$$

where C does not depend on r.

If $a \in BMO(\mathbb{R}^n)$, the commutator

$$[a,K](f) = aKf - K(af)$$

is a bounded operator from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself.

Precisely, for every $f \in LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} \le c\|a\|_*\|f\|_{LM^{p,\varphi}_{\{x_0\}}}.$$

The previous theorem was proved using the following important result contained in [**38**].

Theorem 3.3. *Let $x_0 \in \mathbb{R}^n$, $1 \le q < \infty$, K be a Calderon-Zygmund singular integral operator and the functions φ_1, φ_2 satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{q}+1}}\,dt \le C\,\varphi_2(x_0,r), \tag{3.15}$$

where C does not depend on r. Then for $1 < q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $LM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $WLM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Kf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f and for $1 \le q < \infty$

$$\|Kf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f.

4. Hardy operators and boundedness results

In order to achieve the regularity results, we must to prove the following theorem.

Theorem 4.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $d = \sup_{x,y\in\Omega}|x-y| < \infty$, $\Omega(x_0,r) = \Omega \cap B(x_0,r)$, $x_0 \in \Omega$, $0 < r \le d$, $1 \le q < p < \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$ and*

$$Tg(x) = \int_\Omega \frac{g(y)}{|x-y|^{n-1}}dy.$$

(i) If $g \in L^q(\Omega)$ such that

$$\int_r^d t^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,t))}\,dt < \infty \quad \text{for all} \quad r \in (0,d), \tag{4.16}$$

then for any $r \in (0,d)$ the inequality

$$\|Tg\|_{WL^p(\Omega(x_0,r))} \le cr^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,t))}\,dt + cr^{\frac{n}{p}}\|g\|_{L^q(\Omega)} \tag{4.17}$$

holds with constant $c > 0$ independent of g, x_0 and r.
(ii) Let $1 < q < \infty$. If $g \in L^q(\Omega)$ satisfies condition (4.16)*, then for any $r \in (0,d)$ the inequality*

$$\|Tg\|_{L^p(\Omega(x_0,r))} \le cr^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,t))}\,dt + cr^{\frac{n}{p}}\|g\|_{L^q(\Omega)} \tag{4.18}$$

holds with constant $c > 0$ independent of g, x_0 and r.

Proof. Let $1 \le q < p < \infty$. Since

$$\begin{aligned} r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt &\ge r^{\frac{n}{p}} \|g\|_{L^q(\Omega(x_0,r))} \int_r^d t^{-\frac{n}{p}-1}\, dt \\ &\approx \|g\|_{L^q(\Omega(x_0,r))} (d^{\frac{n}{p}} - r^{\frac{n}{p}}), \quad r \in (0,d), \end{aligned}$$

we get that

$$\|g\|_{L^q(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)}, \quad r \in (0,d). \tag{4.19}$$

(i). Assume that $1 \le q < \infty$. Let $r \in (0, d/2)$. We write $g = g_1 + g_2$ with $g_1 = g\chi_{\Omega(x_0,2r)}$ and $g_2 = g\chi_{\Omega \setminus \Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tg\|_{WL^q(\Omega(x_0,r))} \le \|Tg_1\|_{WL^q(\Omega(x_0,r))} + \|Tg_2\|_{WL^q(\Omega(x_0,r))}. \tag{4.20}$$

Since $g_1 \in L^q(\Omega)$, in view of (4.19), the boundedness of T from $L^q(\Omega)$ to $WL^p(\Omega)$ implies that

$$\begin{aligned} \|Tg_1\|_{WL^q(\Omega(x_0,r))} &\le \|Tg_1\|_{WL^q(\Omega)} \lesssim \|g_1\|_{L^q(\Omega)} \approx \|g\|_{L^q(\Omega(x_0,2r))} \\ &\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)}, \end{aligned} \tag{4.21}$$

where the constant is independent of g, x_0 and r.

We have

$$|Tg_2(x)| \lesssim \int_{\Omega \setminus \Omega(x_0,2r)} \frac{|g(y)|}{|x-y|^{n-1}}\, dy, \qquad x \in \Omega(x_0,r).$$

It's clear that $x \in \Omega(x_0,r)$, $y \in \Omega \setminus (\Omega(x_0,2r))$ implies $\frac{1}{2}|x_0 - y| \le |x-y| < \frac{3}{2}|x_0 - y|$. Therefore we obtain that

$$\|Tg_2\|_{L^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{\Omega \setminus (\Omega(x_0,2r))} \frac{|g(y)|}{|x_0 - y|^{n-1}}\, dy.$$

By Fubini's theorem, we get that

$$\int_{\Omega\setminus\Omega(x_0,2r)} \frac{|g(y)|}{|x_0-y|^{n-1}}\,dy \approx \int_{\Omega\setminus\Omega(x_0,2r)} |g(y)| \left(1+\int_{|x_0-y|}^{d} \frac{ds}{s^n}\right) dy$$
$$= \int_{\Omega\setminus\Omega(x_0,2r)} |g(y)|\,dy + \int_{\Omega\setminus\Omega(x_0,2r)} |g(y)| \left(\int_{|x_0-y|}^{d} \frac{ds}{s^n}\right) dy$$
$$= \int_{\Omega\setminus\Omega(x_0,2r)} |g(y)|\,dy + \int_{2r}^{d} \left(\int_{2r\le|x_0-y|\le s} |g(y)|\,dy\right) \frac{ds}{s^n}$$
$$\le \int_{\Omega} |g(y)|\,dy + \int_{2r}^{d} \left(\int_{\Omega(x_0,s)} |g(y)|\,dy\right) \frac{ds}{s^n}.$$

Applying Hölder's inequality, we arrive at

$$\int_{\Omega\setminus\Omega(x_0,2r)} \frac{|g(y)|}{|x_0-y|^{n}}\,dy \lesssim \|g\|_{L^q(\Omega)} + \int_{2r}^{d} s^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,s))}\,ds.$$

Thus the inequality

$$\|Tg_2\|_{L^p(\Omega(x_0,r))} \lesssim \tau^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,s))}\,ds + r^{\frac{n}{p}}\|g\|_{L^q(\Omega)} \tag{4.22}$$

holds for all $r \in (0, d/2)$.

On the other hand, since

$$\|Tg_2\|_{WL^p(\Omega(x_0,r))} \le \|Tg_2\|_{L^p(\Omega(x_0,r))}$$

using (4.27), we get that

$$\|Tg_2\|_{WL^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,s))}\,ds + r^{\frac{n}{p}}\|g\|_{L^q(\Omega)} \tag{4.23}$$

holds true for all $r \in (0, d/2)$.

Finally, combining (4.20) and (4.21), we obtain that

$$\|Tg\|_{L^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1}\|g\|_{L^q(\Omega(x_0,s))}\,ds + r^{\frac{n}{p}}\|g\|_{L^q(\Omega)} \tag{4.24}$$

holds for all $r \in (0, d/2)$ with a constant independent of f, x_0 and r.

Let now $r \in [d/2, d)$. Then, using $(L^q(\Omega), L^p(\Omega))$-boundedness of T, we obtain

$$\|Tg\|_{L^p(\Omega(x_0,r))} \leq \|Tg\|_{L^p(\Omega)} \lesssim \|g\|_{L^q(\Omega)} \approx r^{\frac{n}{p}} \|g\|_{L^q(\Omega)},$$

and, inequality (4.17) holds.

(ii). Assume that $1 < q < \infty$. Let again $r \in (0, d/2)$. We write $g = g_1 + g_2$ with $g_1 = g\chi_{\Omega(x_0,2r)}$ and $g_2 = g\chi_{\Omega\setminus\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tg\|_{L^q(\Omega(x_0,r))} \leq \|Tg_1\|_{L^q(\Omega(x_0,r))} + \|Tf_2\|_{L^q(\Omega(x_0,r))}. \tag{4.25}$$

Since $g_1 \in L^q(\Omega)$, in view of (4.19), the boundedness of T from $L^q(\Omega)$ to $L^p(\Omega)$ implies that

$$\begin{aligned}\|Tg_1\|_{L^q(\Omega(x_0,r))} &\leq \|Tg_1\|_{L^q(\Omega)} \lesssim \|g_1\|_{L^q(\Omega)} \approx \|g\|_{L^q(\Omega(x_0,2r))} \\ &\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)},\end{aligned} \tag{4.26}$$

where the constant is independent of f, x_0 and r.

Using (4.24), we get that

$$\|Tg_2\|_{L^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d s^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)} \tag{4.27}$$

holds true for all $r \in (0, d/2)$.

Combining (4.25), (4.26) and (4.27), we see that inequality (4.18) holds true for all $r \in (0, d/2)$ with a constant independent of g, x_0 and r.

If $r \in [d/2, d)$, then, using the boundedness of T from $L^q(\Omega)$ to $L^p(\Omega)$, we obtain that

$$\|Tg\|_{L^p(\Omega(x_0,r))} \leq \|Tg\|_{L^p(\Omega)} \lesssim \|g\|_{L^q(\Omega)} \approx r^{\frac{n}{p}} \|g\|_{L^q(\Omega)},$$

and, inequality (4.18) holds. □

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^d g(s)w(s)ds,\ 0 < t < d < \infty,$$

where w is a fixed function non-negative and measurable on $(0, d)$.

The following theorem was proved in **[38, 39]** and in the case $w = 1$ in **[10]**.

Theorem 4.2. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,d)$. The inequality*

$$\operatorname*{ess\,sup}_{0<t<d} v_2(t) H_w^* g(t) \leq C \operatorname*{ess\,sup}_{0<t<d} v_1(t) g(t) \tag{4.28}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,d)$ if and only if

$$B := \operatorname*{ess\,sup}_{0<t<d} v_2(t) \int_t^d \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<d} v_1(\tau)} < \infty. \tag{4.29}$$

Moreover, if C^* is the minimal value of C in (4.28)*, then $C^* = B$.*

Remark 4.3. *In* (4.28) *and* (4.29) *it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.*

In order to achieve the regularity results, we must to prove the following theorem.

Theorem 4.4. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $1 \leq q < p < \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$. Let also $\varphi_1(x,r)$ and $\varphi_2(x,r)$ two positive measurable functions defined on $\Omega \times (0,d)$ such that the following condition is fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_2(x_0,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}} dt \leq C\, \varphi_1(x_0,r), \tag{4.30}$$

where C does not depend on r. Then, for every $g \in \widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)$, the function $Tg(x)$ is a.e. defined, Tg belongs to the space $W\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)$ and there exists $c = c(q,\varphi_1,\varphi_2,n) > 0$ such that

$$\|Tg\|_{W\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} \leq c\|g\|_{LM^{q,\varphi_2}_{\{x_0\}}(\Omega)}.$$

In the case $q > 1$ the function Tg belongs to the space $\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)$ and there exists $c = c(q,\varphi_1,\varphi_2,n) > 0$ such that

$$\|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} \leq c\|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)}.$$

Proof. By Theorem 4.1 and Theorem 4.2 with

$$v_2(r) = \varphi_1(x_0,r)^{-1}, \qquad v_1(r) = \varphi_2(x_0,r)^{-1} r^{-\frac{n}{q}}$$

and $w(r) = r^{-\frac{n}{p}}$ we have

$$
\begin{aligned}
\|Tg\|_{W\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} \int_r^d \|f\|_{WL^q(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tg\|_{WL^p(\Omega)} \\
&\lesssim \sup_{0<r<d} \varphi_2(x_0,r)^{-1}\, r^{-\frac{n}{q}}\, \|g\|_{L^q(\Omega(x_0,r))} + \|g\|_{L^q(\Omega)} \\
&= \|g\|_{LM^{q,\varphi_2}_{\{x_0\}}(\Omega)} + \|g\|_{L^q(\Omega)} \\
&= \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)}
\end{aligned}
$$

and for $q > 1$

$$
\begin{aligned}
\|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} \int_r^d \|f\|_{L^q(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tg\|_{L^p(\Omega)} \\
&\lesssim \sup_{0<r<d} \varphi_2(x_0,r)^{-1}\, r^{-\frac{n}{q}}\, \|g\|_{L^q(\Omega(x_0,r))} + \|g\|_{L^q(\Omega)} \\
&= \|g\|_{LM^{q,\varphi_2}_{\{x_0\}}(\Omega)} + \|g\|_{L^q(\Omega)} \\
&= \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)}.
\end{aligned}
$$

□

From Theorem 4.4 we get the following corollary.

Corollary 4.5. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $1 \le q < p < \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$. Let also $\varphi_1(x,r)$ and $\varphi_2(x,r)$ two positive measurable functions defined on $\Omega \times (0,d)$ such that the following condition is fulfilled:*

$$
\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<d} \varphi_2(x,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi_1(x,r), \tag{4.31}
$$

where C does not depend on x and r. Then, for every $g \in \widetilde{M}^{q,\varphi_2}(\Omega)$, the function $Tg(x)$ is a.e. defined, Tg belongs to the space $W\widetilde{M}^{p,\varphi_1}(\Omega)$ and there exists $c = c(q,\varphi_1,\varphi_2,n) > 0$ such that

$$
\|Tg\|_{W\widetilde{M}^{p,\varphi_1}(\Omega)} \le c\|g\|_{\widetilde{M}^{q,\varphi_2}(\Omega)}.
$$

In the case $q > 1$ the function Tg belongs to the space $\widetilde{M}^{p,\varphi_1}(\Omega)$ and there exists $c = c(q,\varphi_1,\varphi_2,n) > 0$ such that

$$
\|Tg\|_{\widetilde{M}^{p,\varphi_1}(\Omega)} \le c\|g\|_{\widetilde{M}^{q,\varphi_2}(\Omega)}.
$$

5. Application to partial differential equations

Let us consider the divergence form elliptic equation (2.10), in a bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 3$. We set

$$\Gamma(x,t) = \frac{1}{n(2-n)\omega_n\sqrt{\det\{a_{ij}(x)\}}}\left(\sum_{i,j=1}^{n} A_{ij}(x)t_i t_j\right)^{\frac{2-n}{2}},$$

$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t), \qquad \Gamma_{ij}(x,t) = \frac{\partial}{\partial t_i \partial t_j}\Gamma(x,t),$$

$$M = \max_{i,j=1,\dots,n} \max_{|\alpha|\leq 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x,t)}{\partial t^\alpha} \right\|_{L^\infty(\Omega\times\Sigma)},$$

for a.a. $x \in B$ and $\forall t \in \mathbb{R}^n \setminus \{0\}$, where A_{ij} denote the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

It is well known that $\Gamma_{ij}(x,t)$ are Calderón-Zygmund kernels in the t variable.

Let $r, R \in \mathbb{R}^+$, $r < R$ and $\varphi \in C^\infty(\Omega)$ be a standard cut-off function such that for every ball $B_R \subset \Omega$,

$$\varphi(x) = 1 \quad \text{in } B_r, \qquad \varphi(x) = 0, \quad \text{in } \Omega \setminus B_R.$$

Then if u is a solution of (2.10) and $v = \varphi u$ we have

$$L(v) = \nabla \cdot G + g$$

where

$$G = \varphi f + uA\nabla\varphi,$$
$$g = \langle A\nabla u, \nabla\varphi\rangle - \langle f, \nabla\varphi\rangle.$$

Using the notations above, we are able to recall an integral representation formula for the first derivatives of a solution u of (2.10).

Lemma 5.1 ([25]). *For every $i = 1, \dots, n$, let $a_{ij} \in L^\infty(\mathbb{R}^n) \cap VMO$ satisfy* (2.13) *and* (2.14), *let u be a solution of* (2.10) *and let φ, g and G defined as above. Then, for every $i = 1, \dots, n$, we have*

$$\begin{aligned}\partial_{x_i}(\varphi u) \;&=\; \sum_{h,j=1}^{n} P.V. \int_{B_R} \Gamma_{ij}(x, x-y)\{(a_{jh}(x) - a_{jh}(y))\partial x_h(\varphi u)(y) - G_j(y)\}\, \mathrm{d}y \\ &\quad - \int_{B_R} \Gamma_i(x, x-y) g(y)\, \mathrm{d}y + \sum_{h=1}^{n} c_{ih}(x) G_h(x), \qquad \forall x \in B_R,\end{aligned}$$

setting $c_{ih} = \int_{|t|=1} \Gamma_i(x,t) t_h \, \mathrm{d}\sigma_t$.

Using the representation formula stated in Lemma 5.1, we can obtain a regularity result for the solutions to (2.10).

Theorem 5.2. *Let a_{ij} be such that* (2.12), (2.13), (2.14) *are true, we assume that the condition* (4.30) *is fulfilled and that $\varphi_2 \gtrsim \varphi_1$. Let also suppose that u is a solution of* (2.10) *such that $\partial_{x_i} u \in \widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)$, for all $i = 1, ..., n$, $f \in [\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(\Omega)]^n$, $x_0 \in \Omega$. Let $\varphi \in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K \subset \Omega$ there exists a constant $c(n, p, \varphi_1, \varphi_2, dist(K, \partial\Omega))$ such that*

(1) $\partial_{x_i} u \in \widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K), \quad \forall i = 1, ..., n,$

(2) $\|\partial_{x_i} u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \lesssim \|u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} + \|\partial_{x_i} u\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)} + \|f\|_{\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(\Omega)}, \quad \forall i = 1, ..., n,$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.

Proof. Let $K \subset \Omega$ be a compact set. Using Lemma and the boundedness of the commutator proved in **[44]**, we obtain the following estimate:

$$
\begin{aligned}
\|\partial_{x_i}(\varphi u)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} &\leq \|C[a_{ij}, \varphi]\partial_{x_h}(u\varphi)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|KG\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \\
&\quad + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \\
&\leq c\|a\|_* \|\partial_{x_h}(u\varphi)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} \\
&\quad + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)},
\end{aligned}
$$

where the norm $\|a\|_*$ is taken in the set B_R.
Taking into account that $a \in VMO$, we can choose the radius R of the ball B_R such that $c\|a\|_* < \frac{1}{2}$. This remark allow us to write

$$
\begin{aligned}
\|\partial_{x_i}(\varphi u)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} &\leq \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \\
&\approx \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} \\
&= \|\varphi f + uA\nabla\varphi\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|\langle A\nabla u, \nabla\varphi\rangle - \langle f, \nabla\varphi\rangle\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} \\
&\leq \|f\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|\partial_{x_i} u\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} + \|f\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)}.
\end{aligned}
$$

Now we apply the hypothesis $\varphi_2 \gtrsim \varphi_1$, obtaining the following estimate for the norm $\|f\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}}$:

$$\begin{aligned} \|f\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} &\leq \sup_{0<r<d} \frac{1}{\varphi_2(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{q}}} \|f\|_{L^q(|B(x_0,r)\cap K)} + \|f\|_{L^q(K)} \\ &\lesssim \sup_{0<r<d} \frac{1}{\varphi_1(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{q}}} \|f\|_{L^q(|B(x_0,r)\cap K)} + \|f\|_{L^q(K)} \\ &= \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}(K)} + \|f\|_{L^q(K)} = \|f\|_{\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(K)}. \end{aligned}$$

Using the previous estimate we finally obtain that

$$\|\partial_{x_i} u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \leq C\left(\|u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} + \|\partial_{x_i} u\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)} + \|f\|_{\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(\Omega)} \right), \quad \forall i = 1, ..., n,$$

□

From Theorem 5.2 we get the following corollary.

Corollary 5.3. *Let a_{ij} be such that* (2.12), (2.13), (2.14) *are true, we assume that the condition* (4.31) *is fulfilled and that $\varphi_2 \gtrsim \varphi_1$. Let also suppose that u is a solution of* (2.10) *such that $\partial_{x_i} u \in \widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)$, for all $i = 1, ..., n$, $f \in [\widetilde{M}^{p,\varphi_1}(\Omega)]^n$. Let $\varphi \in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K \subset \Omega$ there exists a constant $c(n, p, \varphi_1, \varphi_2, dist(K, \partial\Omega))$ such that*

(1) $\partial_{x_i} u \in \widetilde{M}^{p,\varphi_1}(K), \quad \forall i = 1, ..., n,$

(2) $\|\partial_{x_i} u\|_{\widetilde{M}^{p,\varphi_1}(K)} \lesssim \|u\|_{\widetilde{M}^{p,\varphi_1}(\Omega)} + \|\partial_{x_i} u\|_{\widetilde{M}^{q,\varphi_2}(\Omega)} + \|f\|_{\widetilde{M}^{q,\varphi_1}(\Omega)}, \quad \forall i = 1, ..., n,$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.

In the case $\varphi_1(x,r) = \varphi_2(x,r)$ we get the following corollaries.

Corollary 5.4. *Let a_{ij} be such that* (2.12), (2.13), (2.14) *are true, we assume that $\varphi(x,r)$ positive measurable function defined on $\Omega \times (0,d)$ and the following condition is fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}} dt \leq C\, \varphi(x_0,r),$$

where C does not depend on r.

Let also suppose that u is a solution of (2.10) *such that $\partial_{x_i} u \in \widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)$, for all $i = 1, ..., n$, $f \in [\widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)]^n$, $x_0 \in \Omega$. Let $\varphi \in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K \subset \Omega$ there exists a constant $c(n, p, \varphi, dist(K, \partial\Omega))$ such that*

(1) $\partial_{x_i} u \in \widetilde{LM}^{p,\varphi}_{\{x_0\}}(K), \quad \forall i = 1, ..., n,$

(2) $\|\partial_{x_i} u\|_{\widetilde{LM}^{p,\varphi}_{\{x_0\}}(K)} \lesssim \|u\|_{\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)} + \|\partial_{x_i} u\|_{\widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)} + \|f\|_{\widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)}, \quad \forall i = 1, ..., n,$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.

Corollary 5.5. *Let* a_{ij} *be such that* (2.12), (2.13), (2.14) *are true, we assume that* $\varphi(x,r)$ *positive measurable function defined on* $\Omega \times (0,d)$ *and the following condition is fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi(x,r),$$

where C *does not depend on* x, r.

Let also suppose that u *is a solution of* (2.10) *such that* $\partial_{x_i} u \in \widetilde{M}^{q,\varphi}(\Omega)$, *for all* $i = 1, ..., n$, $f \in [\widetilde{M}^{q,\varphi}(\Omega)]^n$. *Let* $\varphi \in C^\infty(\Omega)$ *a standard cut-off function. Then, for any* $K \subset \Omega$ *there exists a constant* $c(n, p, \varphi, dist(K, \partial\Omega))$ *such that*

(1) $\partial_{x_i} u \in \widetilde{M}^{p,\varphi}(K), \quad \forall i = 1, ..., n,$

(2) $\|\partial_{x_i} u\|_{\widetilde{M}^{p,\varphi}(K)} \lesssim \|u\|_{\widetilde{M}^{p,\varphi}(\Omega)} + \|\partial_{x_i} u\|_{\widetilde{M}^{q,\varphi}(\Omega)} + \|f\|_{\widetilde{M}^{q,\varphi}(\Omega)}, \quad \forall i = 1, ..., n,$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.

Conclusions

We wish to continue the research started during the PhD. Specially, we would to investigate the behavior of other integral operators both on mixed Morrey spaces and modified generalized local Morrey spaces.

In line with the two sides studied in this thesis, the aim of the future research should be the applications of the estimates for integral operators to the field of partial differential equations and systems of various type.

Taking into account the recent development of real and harmonic analysis related to Morrey-type spaces, it seems that the parallel study of the theory of integral operators and the regularity of solutions to partial differential equations is very fruitful. For this reason, we hope to contribute to the development of new issues and new useful techniques.

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